Conic lifts of polytopes

João Gouveia

University of Coimbra

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1. Linear Representations and Yannakakis Theorem

Polytopes

The usual way to describe a polytope P is by listing the vertices or giving an inequality description

$$P = \{x \in \mathbb{R}^n : a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq b\},$$

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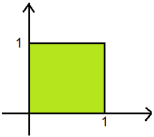

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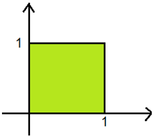

$$C = \left\{ (x, y) : \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} y \geq \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

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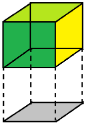

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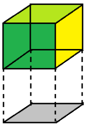

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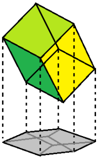

Linear representation of an hexagon.

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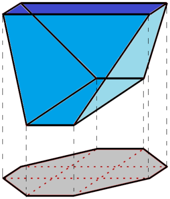

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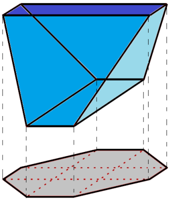

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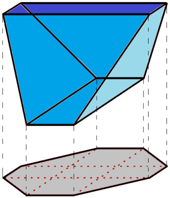

To do linear optimization on the projection we can optimize on the “upper” polytope.

Given a polytope P we are interested in finding how small can its linear representation be. This tells us how hard it is to optimize over P using LP.

Slack Matrix

Let P be a polytope with facets given by
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	0 0 0	1 0 0	0 1 0	0 0 1	1 1 0	0 1 1	1 0 1	1 1 1
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$x \geq 0$	0	1	0	0	1	0	1	1
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Example:

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

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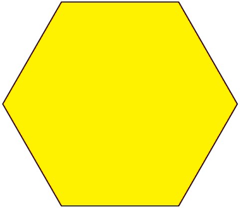

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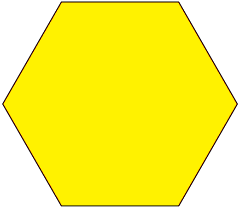

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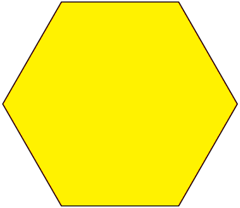

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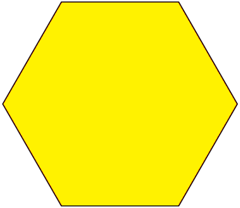

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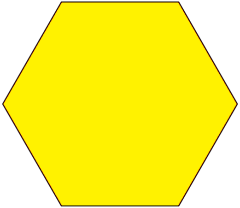

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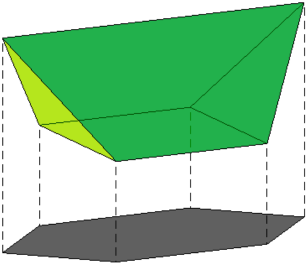

2. Other Representations: General Yannakakis Theorem

Semidefinite Representations

A semidefinite representation of size k of a polytope P is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

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The Square

The $0/1$ square is the projection onto x_1 and x_2 of

$$\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1 & y \\ x_2 & y & x_2 \end{bmatrix} \succeq 0.$$

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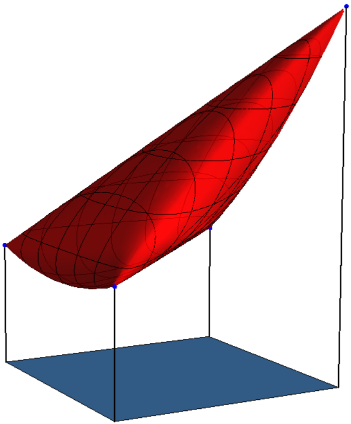

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$$\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \end{matrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

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The psd rank of a polytope P is defined as

$$\mathrm{rank}_{psd}(P) := \mathrm{rank}_{psd}(S_P).$$

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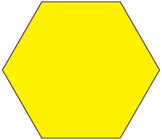

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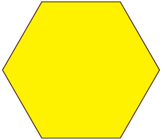

Its 6×6 slack matrix.

$$\begin{bmatrix} 0 & 0 & 2 & 4 & 4 & 2 \\ 2 & 0 & 0 & 2 & 4 & 4 \\ 4 & 2 & 0 & 0 & 2 & 4 \\ 4 & 4 & 2 & 0 & 0 & 2 \\ 2 & 4 & 4 & 2 & 0 & 0 \\ 0 & 2 & 4 & 4 & 2 & 0 \end{bmatrix}$$

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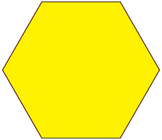

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$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

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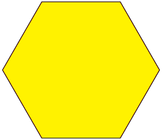

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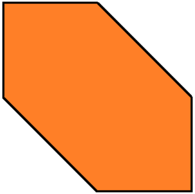
Consider the affinely equivalent hexagon H with vertices $(\pm 1, 0), (0, \pm 1), (1, -1)$ and $(-1, 1)$.



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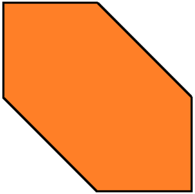

$$H = \left\{ (x_1, x_2) : \begin{bmatrix} 1 & x_1 & x_2 & x_1 + x_2 \\ x_1 & 1 & y_1 & y_2 \\ x_2 & y_1 & 1 & y_3 \\ x_1 + x_2 & y_2 & y_3 & 1 \end{bmatrix} \succeq 0 \right\}$$

Conic Representations

In general given any closed cone K, a K-lift or K-representation of a polytope P is a representation

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If $K = \mathbb{R}^k_+$ or $K = \text{PSD}_k$ we recover the linear and semidefinite representations respectively. Other possible choices for K would be SOCP, CoP, CP ...

Conic Factorizations

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Note that since both $\mathbb{R}^k_+$ and PSD_k are self-dual, this notion generalizes both the notions of nonnegative and semidefinite factorization.

Conic Yannakakis Theorem

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A polytope P has a K-representation if and only if its slack matrix has a K-factorization.

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Several further generalizations are possible (convex bodies, symmetric lifts).

3. Nonnegative and Semidefinite ranks

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Many other complexity questions are open.

Rectangle covering bound

The nonnegative rank of a matrix M is larger than the size of its smallest rectangle cover.

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The rectangle bound corresponds to the boolean rank and also relates to the minimum communication complexity of a 2-party protocol to compute the support of M.

Hadamard Square Roots

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$\sqrt[H+]{M}$ is the nonnegative Hadamard square root of M.

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Proposition (G.-Robinson-Thomas 2012)

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Corollary

For $0/1$ matrices

$$\text{rank}_{\text{psd}}(M) \leq \text{rank}_H(M) \leq \text{rank}(M).$$

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Consider the matrix $A \in \mathbb{R}^{n\times n}$ defined by $a_{i,j} = (i-j)^2$.

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rank_{+} can be arbitrarily larger than rank and rank_{psd}.

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An linear representation of P of size k induces an embedding from the facial lattice of P, $L(P)$, to the boolean lattice $2^{[k]}$. In particular:

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- $n_P = 28 \Rightarrow \text{rank}_+(P) \geq \log_2(28) \approx 4.807$.
- $n_{\text{edges}} = 12$, $\binom{5}{2} = 10$, $\binom{6}{3} = 20$, hence $\text{rank}_+(P) \geq 6$.

Bounds for polytopes - SDP

Theorem (G.-Parrilo-Thomas 2011)

If a polytope P in $\mathbb{R}^n$ has $\text{rank}_{psd} = k$ than it has at most $k^{O(k^2 n)}$ facets.

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Big open question:

- Can we find a separation between rank_{psd} and rank_+ for polytopes?

Polytopes with minimal representations

Lemma

A polytope of dimension d does not have a semidefinite representation of size smaller than $d + 1$.

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But we can say much more.

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Let P have dimension d. Then

$$\text{rank}_{\text{psd}}(P) = d+1 \Leftrightarrow \text{rank}_H(S_P) = d+1.$$

Properties of SDP-minimal Polytopes

We will say a dimension d polytope P is SDP-minimal if it has a semidefinite representation of size $d+1$.

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- Pyramids over SDP-minimal polytopes are SDP-minimal.

Results in $\mathbb{R}^2$

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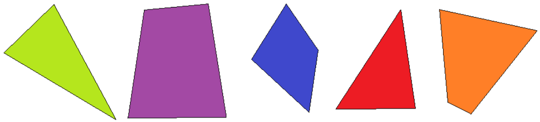
Proposition

A convex polygon is SDP-minimal if and only if it is a triangle or a quadrilateral.



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Proposition

If P is combinatorially equivalent to an octahedron then it is SDP-minimal if and only if there are two distinct sets of four coplanar vertices of P.

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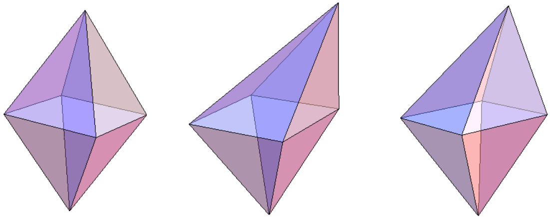

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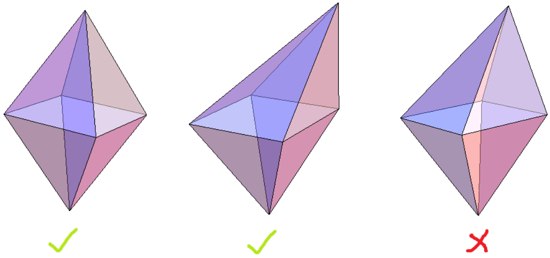

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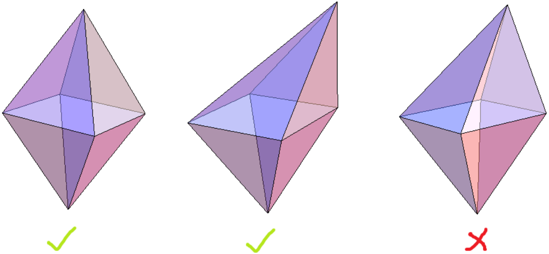

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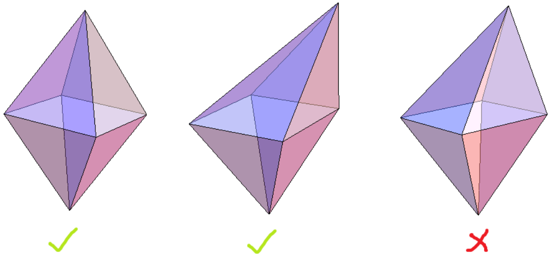

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It also suggests some underlying matroid characterization.

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- Any insight on matching polytope.

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Still many unanswered questions.

- Complexity or rank calculations.
- No polynomial psd lift of TSP.
- Any insight on matching polytope.
- Better understanding of Hadamard ranks.

Conclusion

Conic Lifts/Factorizations is an exciting area of research with many recent breakthroughs.

- Symmetry matters for linear lifts (Kaibel-Pashkovich-Theis 2010)
- No polynomial size lift for TSP (Fiorini et al 2012)
- Connections to communication complexity (Faenza et al 2011, Fiorini et al 2012)...

Still many unanswered questions.

- Complexity or rank calculations.
- No polynomial psd lift of TSP.
- Any insight on matching polytope.
- Better understanding of Hadamard ranks.
- psd/lp separation...

For more information

Polytopes of Minimum Positive Semidefinite Rank - Gouveia, Robinson and Thomas - arXiv:1205.5306

Lifts of convex sets and cone factorizations - Gouveia, Parrilo and Thomas - arXiv:1111.3164

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Thank you