# Geometry of Sums of Squares Relaxations

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14th July '10 / SIAM Annual Meeting

# Optimization over semialgebraic sets

## Goal

Optimize a linear objective function over

$$S = \{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) \geq 0, \ldots, g_k(\mathbf{x}) \geq 0\}$$

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- It is NP-hard
- We use a hierarchy of relaxations instead

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where the $\sigma_i$ are sums of squares of polynomials.

We will denote by $\Sigma_d^S$ the set of all polynomials that have such a representation with $\deg(\sigma_i g_i) \leq 2d$ for all $i$.

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We can therefore relax it by

## Lasserre Bodies

$$\mathcal{L}_d(S) = \bigcap_{\ell \text{ linear }, \ell \in \Sigma_d^S} \{\mathbf{x} \in \mathbb{R}^n : \ell(\mathbf{x}) \geq 0\}$$

which we call the $d$-th Lasserre Relaxation of $S$.

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- From the definition $\overline{\mathrm{conv}(S)} \subseteq \mathcal{L}_d(S) \subseteq \mathcal{L}_{d-1}(S)$ for all $d$.
- If $S$ is compact we have assymptotic convergence.
- When is the convergence (not) finite?

# Technical Lemma

To verify that we don't have finite convergence in a particular case we will use a well-known result:

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In geometrical terms this implies the following lemma:

**Lemma**

*If $S$ has non-empty interior and $\ell$ is a linear polynomial that is non-negative over $\mathcal{L}_d(S)$, then $\ell \in \Sigma_d^S$.*

which we can use to give a certificate of $\mathcal{L}_d(S) \neq \overline{\text{conv}(S)}$.

# Obstruction Lemma

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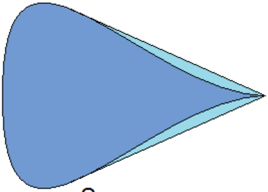

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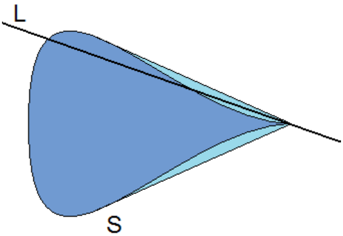

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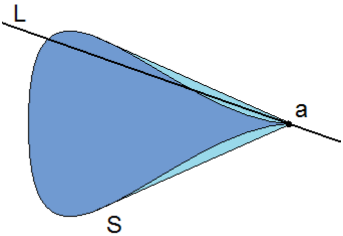

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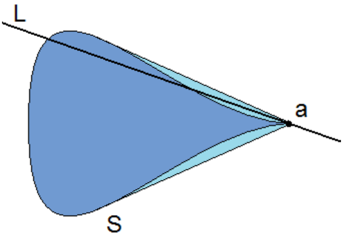
*If for all* $g_i$ *s.t.* $\underline{g_i(a) = 0}$ *we have* $\triangledown g_i(a) \perp L$ *then, for all* $d$*, we have* $\mathcal{L}_d(S) \neq \overline{\text{conv}(S)}$*.*



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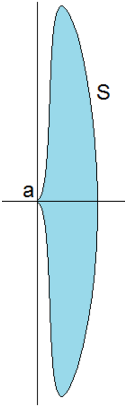

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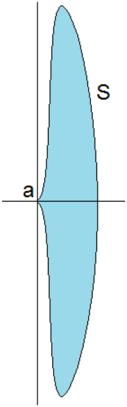


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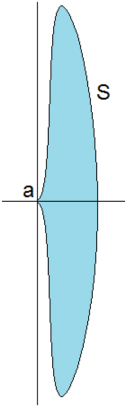


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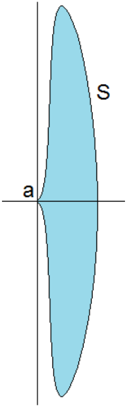


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- Let $i \in I$ if $g_i(0) = 0$ and $i \in J$ otherwise. By hypothesis $(g_i^*)'(0) = g_i^*(0) = 0$ for $i \in I$.

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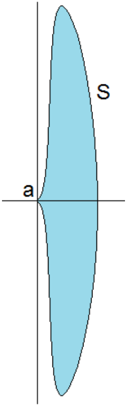


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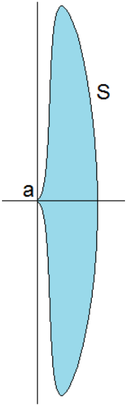


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by evaluating at 0, $\sigma_0(0) = \sigma_j(0) = 0$.

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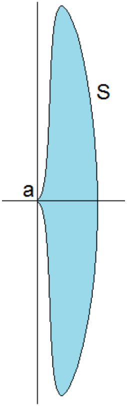


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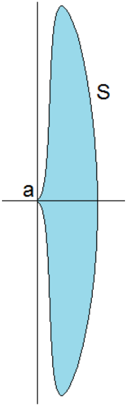


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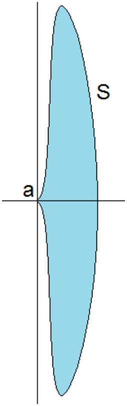


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by evaluating at 0, $1 = 0$. $\square$

# Singularities

**Corollary**

*If $S$ has non-empty interior and there exists a point $a \in S$ that is on the boundary of $\text{conv}(S)$ s.t. all $g_i$ verifying $g_i(a) = 0$ are singular at $a$, then we have $\mathcal{L}_d(S) \neq \overline{\text{conv}(S)}$ for all $d$.*

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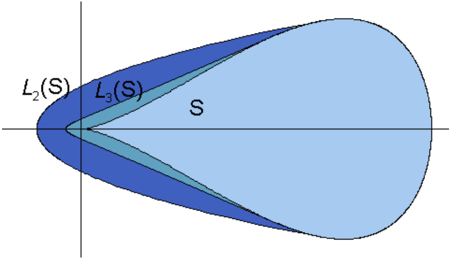


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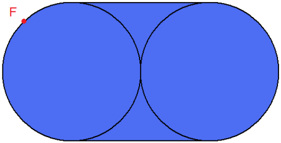

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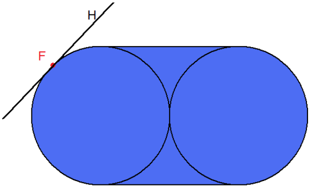

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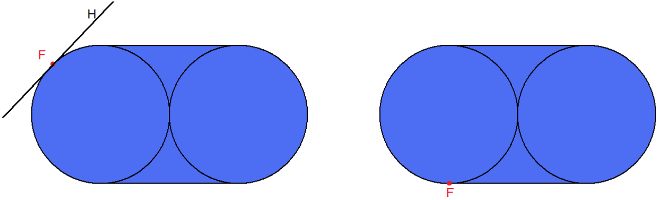

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**Corollary (Netzer-Plaumann-Schweighofer)**

*Suppose S is convex and has non-empty interior. If S has a non-exposed face then* $\mathcal{L}_d(S) \neq \overline{\mathrm{conv}(S)}$ *for all* $d$.

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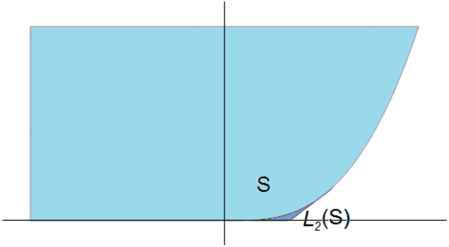


$$g_1(x,y) = y - x^3,\ g_2(x,y) = y,\ g_3(x,y) = x+1,\ g_4(x,y) = 1-y$$

# Questions from NPS

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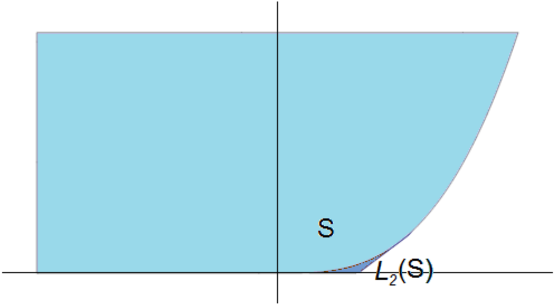

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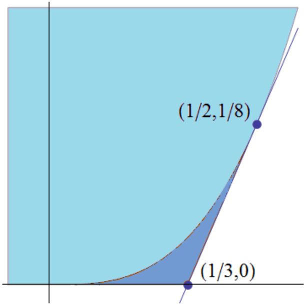

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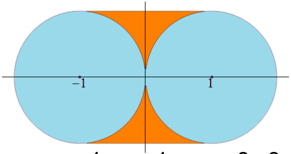


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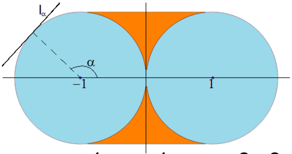


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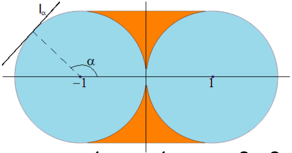


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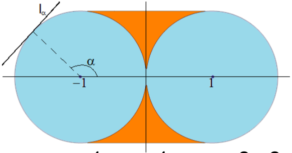


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$$(8-8\cos(\alpha))l_\alpha = g(x,y) + (x^2+y^2-2+2\cos(\alpha))^2 +$$
$$+\left(2\sqrt{1-\cos(\alpha)}(y-\sin(\alpha))\right)^2 + \left(2\sqrt{-\cos(\alpha)}(x-\cos(\alpha)+1)\right)^2$$

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### Question 1

What other obstructions exist to the finite convergence of the Lasserre hierarchy?

### Question 2

Give an example of a smooth, convex, basic semialgebraic set with no non-exposed faces such that the Lasserre hierarchy does not converge finitely. Is there such an example?

# Reference

**Positive Polynomials and Projections of Spectrahedra**,
João Gouveia and Tim Netzer, *arXiv:0911.2750*

# The End

Thank You