Positive Semidefinite Rank

João Gouveia

University of Coimbra

CMUC - 1st September 2014

jointly with Pablo Parrilo [MIT], Rekha Thomas [UW], Hamza Fawzi [MIT] and Richard Robinson [UW]

Section 1

What is it

Definition

Let M be a m by n nonnegative matrix.

Definition

Let M be a m by n nonnegative matrix. A semidefinite factorization of M of size k is a set of $k \times k$ positive semidefinite matrices $A_1, \cdots, A_m$ and $B_1, \cdots B_n$ such that $M_{i,j} = \langle A_i, B_j \rangle$.

Definition

Let M be a m by n nonnegative matrix. A semidefinite factorization of M of size k is a set of $k \times k$ positive semidefinite matrices $A_1, \cdots, A_m$ and $B_1, \cdots B_n$ such that $M_{i,j} = \langle A_i, B_j \rangle$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Definition

Let M be a m by n nonnegative matrix. A semidefinite factorization of M of size k is a set of $k \times k$ positive semidefinite matrices $A_1, \cdots, A_m$ and $B_1, \cdots B_n$ such that $M_{i,j} = \langle A_i, B_j \rangle$.

$$\begin{array}{c|ccc} & \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} & \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \hline \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} & 1 & 1 & 0 \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} & 1 & 0 & 1 \\ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} & 1 & 1 & 1 \end{array}$$

Definition

Let M be a m by n nonnegative matrix. A semidefinite factorization of M of size k is a set of $k \times k$ positive semidefinite matrices $A_1, \cdots, A_m$ and $B_1, \cdots B_n$ such that $M_{i,j} = \langle A_i, B_j \rangle$.

$$\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \end{matrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

The smallest size of a semidefinite factorization is defined to be the positive semidefinite rank of M, $\text{rank}_{\text{psd}}(M)$

Open questions

(i) Computing psd rank is hard, even for very small problems. How hard is still open.

Open questions

(i) Computing psd rank is hard, even for very small problems. How hard is still open.

(ii) Efficient upper and lower bounds are still scarce.

Open questions

(i) Computing psd rank is hard, even for very small problems. How hard is still open.

(ii) Efficient upper and lower bounds are still scarce.

(iii) Very interesting connections to quantum information theory and combinatorial optimization complexity theory among others.

Open questions

(i) Computing psd rank is hard, even for very small problems. How hard is still open.

(ii) Efficient upper and lower bounds are still scarce.

(iii) Very interesting connections to quantum information theory and combinatorial optimization complexity theory among others.

(iv) It is a natural generalization of the nonnegative rank.

Open questions

(i) Computing psd rank is hard, even for very small problems. How hard is still open.

(ii) Efficient upper and lower bounds are still scarce.

(iii) Very interesting connections to quantum information theory and combinatorial optimization complexity theory among others.

(iv) It is a natural generalization of the nonnegative rank.

Section 2

Why I do I care

Semidefinite Representations

A semidefinite representation of size k of a polytope P is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where A_i and B_i are $k \times k$ real symmetric matrices.

Semidefinite Representations

A semidefinite representation of size k of a polytope P is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where A_i and B_i are $k \times k$ real symmetric matrices.

Given a polytope P we are interested in finding how small can such a description be.

Semidefinite Representations

A semidefinite representation of size k of a polytope P is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where A_i and B_i are $k \times k$ real symmetric matrices.

Given a polytope P we are interested in finding how small can such a description be.

This tells us how hard it is to optimize over P using semidefinite programming.

The Square

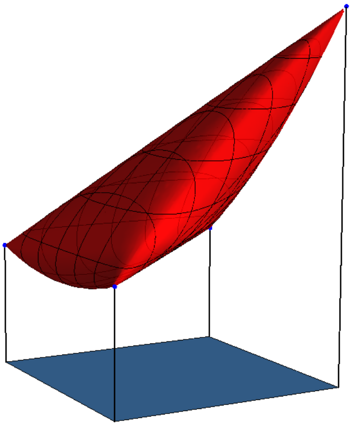
The $0/1$ square is the projection onto x_1 and x_2 of

$$\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1 & y \\ x_2 & y & x_2 \end{bmatrix} \succeq 0.$$

The Square

The $0/1$ square is the projection onto x_1 and x_2 of

$$\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1 & y \\ x_2 & y & x_2 \end{bmatrix} \succeq 0.$$



Slack Matrix

Let P be a polytope with facets given by
$h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

Slack Matrix

Let P be a polytope with facets given by
$h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Slack Matrix

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Example: For the unit cube.

Slack Matrix

Let P be a polytope with facets given by
$h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Example: For the unit cube.

$$\begin{array}{c|c|c|c|c|c|c|c} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$\begin{array}{r} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ 1 - x \geq 0 \\ 1 - y \geq 0 \\ 1 - z \geq 0 \end{array} \quad \left[\begin{array}{cccccccc} \\ \\ \\ \\ \\ \\ \end{array} \right]$$

Slack Matrix

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Example: For the unit cube.

	0 0 0	1 0 0	0 1 0	0 0 1	1 1 0	0 1 1	1 0 1	1 1 1
$x \geq 0$	0	1	0	0	1	0	1	1
$y \geq 0$								
$z \geq 0$								
$1 - x \geq 0$								
$1 - y \geq 0$								
$1 - z \geq 0$								

Slack Matrix

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Example: For the unit cube.

$$\begin{array}{c|c|c|c|c|c|c|c} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{array}$$

$$\begin{array}{r} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ 1 - x \geq 0 \\ 1 - y \geq 0 \\ 1 - z \geq 0 \end{array} \quad \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \\ & & & & & & & \end{bmatrix}$$

Slack Matrix

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Example: For the unit cube.

	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	$(1,1,0)$	$(0,1,1)$	$(1,0,1)$	$(1,1,1)$
$x \geq 0$	0	1	0	0	1	0	1	1
$y \geq 0$	0	0	1	0	1	1	0	1
$z \geq 0$	0	0	0	1	0	1	1	1
$1 - x \geq 0$	1	0	1	1	0	1	0	0
$1 - y \geq 0$	1	1	0	1	0	0	1	0
$1 - z \geq 0$	1	1	1	0	1	0	0	0

Semidefinite Yannakakis Theorem

Theorem (G.-Parrilo-Thomas 2011)

A polytope P has a semidefinite representation of size k if and only if $\text{rank}_{\text{psd}}(S_P) \leq k$.

Semidefinite Yannakakis Theorem

Theorem (G.-Parrilo-Thomas 2011)

A polytope P *has a semidefinite representation of size* k *if and only if* $\text{rank}_{\text{psd}}(S_P) \leq k$.

Many interesting open questions are raised. The two most important:

Semidefinite Yannakakis Theorem

Theorem (G.-Parrilo-Thomas 2011)

A polytope P has a semidefinite representation of size k if and only if $\mathrm{rank}_{\mathrm{psd}}(S_P) \leq k$.

Many interesting open questions are raised. The two most important:

- What is the psd rank of the travelling salesman polytope?

Semidefinite Yannakakis Theorem

Theorem (G.-Parrilo-Thomas 2011)

A polytope P has a semidefinite representation of size k if and only if $\operatorname{rank}_{\text{psd}}(S_P) \leq k$.

Many interesting open questions are raised. The two most important:

- What is the psd rank of the travelling salesman polytope?
- Can psd lifts do much better than linear lifts?

The end

H. Fawzi, J. Gouveia, P.A. Parrilo, R. Z. Robinson and R.R. Thomas.
Positive semidefinite rank.
arXiv preprint arXiv:1407.4095, 2014.

H. Fawzi, J. Gouveia, and R. Z. Robinson.
Rational and real positive semidefinite rank can be different.
arXiv preprint arXiv:1404.4864, 2014.

J. Gouveia, P.A. Parrilo, and R.R. Thomas.
Lifts of convex sets and cone factorizations.
Mathematics of Operations Research, 38(2):248–264, 2013.

J. Gouveia, R. Z. Robinson, and R. R. Thomas.
Worst-case results for positive semidefinite rank.
arXiv preprint arXiv:1305.4600, 2013.

J. Gouveia, R.Z. Robinson, and R.R. Thomas.
Polytopes of minimum positive semidefinite rank.
Discrete & Computational Geometry, 50(3):679–699, 2013.

Thank you