Sums of Squares with Multipliers: Advantages and Limitations

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Demanding a polynomial to be sos is a semidefinite constrain in the coefficients.

Sums of Squares in global polynomial optimization

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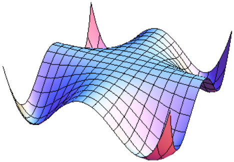
It does not always work.

Hierarchy of Sums of Squares



$$M(x,y) = 1 + x^4y^2 + y^4x^2 - 3x^2y^2$$

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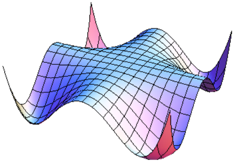

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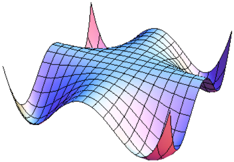

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This motivates a new hierarchy:

Global Polynomial Optimization Relaxation Hierarchy

$p_{\text{sos},k} = \max \lambda$ s.t. $\|x\|^{2k}(p(x) - \lambda)$ is sos.

Properties of this hierarchy

Good News

For all polynomials, $p_{\text{sos},k} \to p_{\text{min}}$.

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$$D(x, y, z, w) = x^4y^2w^2 + y^4z^2w^2 + x^2z^4w^2 - 3x^2y^2z^2w^2 + z^8$$

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However $(x^2 + y^2 + z^2)D(x, y, z, w)$ is sos, which is enough.

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Not so Bad News

It is however a quasi-convex problem, hence still doable. It is also OK for fixed λ.

Copositive Matrices

Cone of Copositive Matrices

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Parrilo's Hierarchy

For a symmetric matrix M consider the polynomial

$$p_M(x) = \begin{bmatrix} {x_1}^2 \\ {x_2}^2 \\ \vdots \\ {x_n}^2 \end{bmatrix}^t M \begin{bmatrix} {x_1}^2 \\ {x_2}^2 \\ \vdots \\ {x_n}^2 \end{bmatrix}.$$

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$$\mathrm{Par}_n^1 \subseteq \mathrm{Par}_n^2 \subseteq \cdots \subseteq \mathrm{CoP}_n$$

There is assimptotic convergence, but $\mathrm{Par}_5^r \neq \mathrm{CoP}_5$ for any r.

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$\text{Par}_n^{*,r} = \{M \in \mathbb{R}^{n\times n} \ : \ M = M^t, \ q(x)p_M(x) \text{ is sos}, q(x) \neq 0 \text{ is sos}\}.$

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Finite Convergence

For all n there exists r such that $\text{Par}_n^{*,r} = \text{CoP}_n$. In particular $\text{Par}_5^{*,1} = \text{CoP}_5$.

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$p_{\min} = \max_\lambda \lambda \text{ s.t. } p(x) - \lambda \geq 0 \text{ for all } x \text{ s.t. } g_i(x) = 0, \; i = 1, \cdots, r.$

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We can now apply sums of squares

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$p_{sos} = \max_\lambda \lambda \text{ s.t. } p(x) - \lambda + \sum q_i(x) g_i(x)$ is sos, for some q_i.

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Degree bounds are needed.

Costrained polynomial optimization (continued)

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$p_{\text{sos}}^k = \max_\lambda \lambda$ s.t. $p(x) - \lambda + \sum q_i(x)g_i(x)$ is sos, for some polynomials q_i, with degree of p and $q_i g_i$ at most $2k$.

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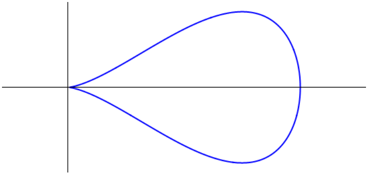
Same advantages and disadvantage as before.

Example

Consider the teardrop curve given by $x^4 - x^3 + y^2 = 0$.



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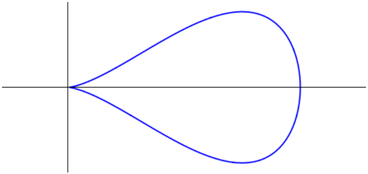

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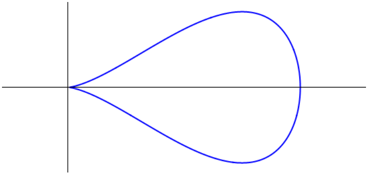

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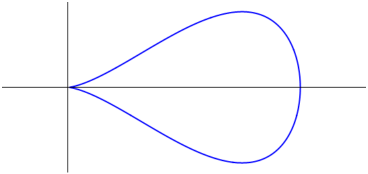

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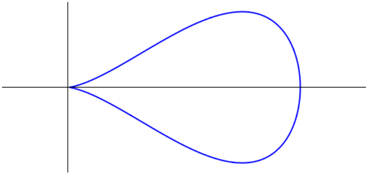

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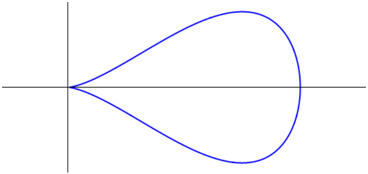

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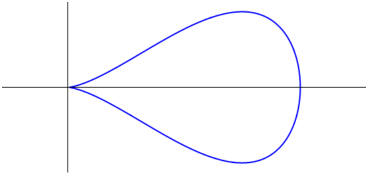

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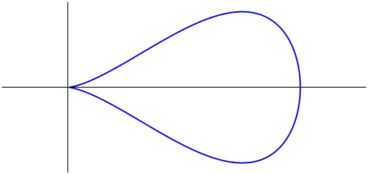

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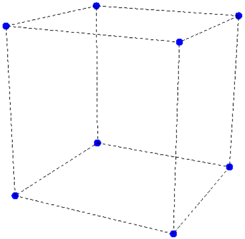
Multipliers make the relaxations less sensitive to singularities.

The n-cube

We are interested in the n-cube:

$$C_n = \{0,1\}^n = \{x \in \mathbb{R}^n \ : \ x_i^2 - x_i = 0, \ i = 1, \cdots, n\}.$$



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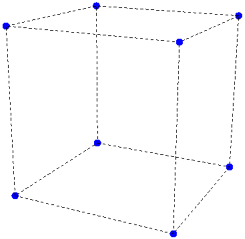

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S_n acts on C_n by permuting coordinates, and if p is symmetric, it will be completely characterized by its evaluation at the levels T_k of the cube:

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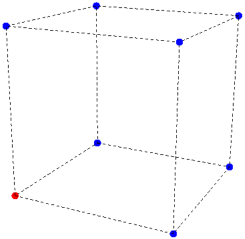

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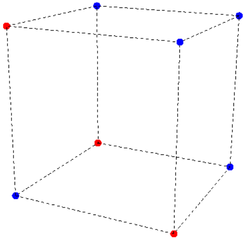

Level T_1

S_n acts on C_n by permuting coordinates, and if p is symmetric, it will be completely characterized by its evaluation at the levels T_k of the cube:

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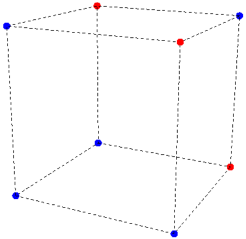

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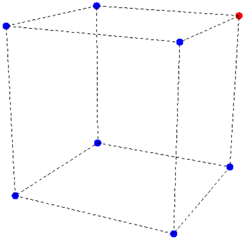

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Main Result 1 - Bad news

Let p be a symmetric square-free polynomial attaining its minimum over C_n at level T_k, with $\deg p \leq k \leq n/2$.

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If T_k is not a local extreme of p over $\mathbb{R}^n$ (seen as a polynomial in $\sum x_i$) then $p_{\min} > p_{\text{sos}}^{k-r,k}$, where $r = \lceil (\deg p)/2 \rceil$.

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This means that if the minimizer of p is "simple enough" and is close to the central levels of the cube, we need high level sos relaxations.

Notes on the result

The proof reduces to this lemma.

Lemma

If p has degree d and vanishes at T_k with $d \leq k \leq n - d$ then

$$p = (k - \sum x_i)q \mod I_n,$$

with $\deg q < \deg p$.

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Lemma

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with $\deg q < \deg p$.

This is a divisibility result. Surprisingly, the only proof we know uses representation theory.

Sketch of Proof:

Consider the action of S_n in $\mathbb{R}[I]_k$.

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Consider the action of S_n in $\mathbb{R}[I]_k$. It decomposes:

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& & & \oplus & & \oplus & & & & \oplus \\
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and is enough to check that M_j does not vanish at T_k.

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Recall that the maxcut problem over K_n can be reduced to

Binary polynomial formulation of MaxCut

$$\max p(x) = \sum_{i \neq j} (1 - x_i) x_j \text{ s.t. } x \in C_n$$

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For $n = 2k + 1$, $p_{\text{sos}}^k > p_{\text{max}}$.

Note that p attains its maximum in C_n at T_k and T_{k+1}, which are not local maxima of p over $\mathbb{R}^n$.

First corollary of main result 1

For $n = 2k + 1$, ${p_{\text{sos}}}^{k-1,k} > p_{\text{max}}$.

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For any k there is a degree 4 polynomial in $\mathbb{R}^{2k+1}$ for which $p_{\min} \neq p_{\text{sos}}^{k-2,k}$.

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Main Result 2 - Not so bad news

We have showed lower bounds to the effectiveness of sos for binary polynomial programming. Luckily we also can show some upper bounds.

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The proof is based in dimension counting.

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Consider the weighted maxcut formulation.

Binary polynomial formulation of MaxCut

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A weaker version can now be proved.

Corollary of main result 2

If $n = 2k + 1$, $(p_\omega)_{\min} = (p_\omega)^{k+1,k+2}_{\text{sos}}$ for all weights.

Open Questions

- Show that for every r there exists n such that $\mathrm{Par}_n^{*,r} \neq \mathrm{CoP}_n$. (Adapt the polynomial we have?)

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- Any progress on sos/sdp hardness of matching.

The End

Thank You