# Geometry of Sums of Squares Relaxations

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University of Washington

12th May - Final Exam

# Convex Hulls of Algebraic Sets

**Problem**

Given an algebraic set

$$\{\mathbf{x} \in \mathbb{R}^n : g_1(\mathbf{x}) = \ldots = g_m(\mathbf{x}) = 0\},$$

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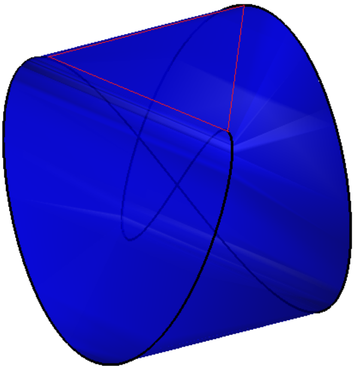
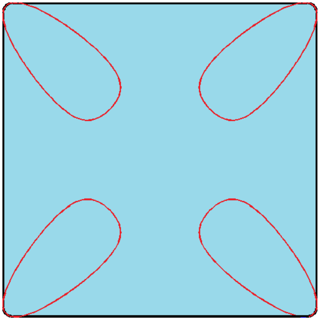
- $I = \langle g_1, \ldots, g_m \rangle$,
- $\mathcal{V}_{\mathbb{R}}(I) = \{\text{Real zeros of } I\}$.

# Examples



$I = \left\langle x^2 - y^2 - xz, z - 4x^3 + 3x \right\rangle$



$I = \left\langle 25(x^4 + y^4 + 1) - 34(x^2y^2 + x^2 + y^2) \right\rangle$

# Theta body

## Convex Hull

$$\mathrm{cl}(\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I))) = \bigcap_{\ell \text{ linear }, \ell|_{\mathcal{V}_{\mathbb{R}}(I)} \geq 0} \{x \in \mathbb{R}^n : \ell(x) \geq 0\}$$

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## Definition

$$\mathrm{TH}_k(I) := \bigcap_{\ell \text{ linear }, \ell \text{ } k\text{-sos modulo } I} \{x \in \mathbb{R}^n : \ell(x) \geq 0\}$$

# Theta body - Example

(Loading...)

$\mathrm{TH}_2(I)$ for $I = \langle x(x^2+y^2) - x^4 - x^2y^2 - y^4 \rangle$.

# PSD cone and spectrahedra

$\text{PSD}_n$ - cone of all $n \times n$ positive semidefinite matrices.

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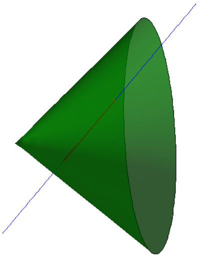

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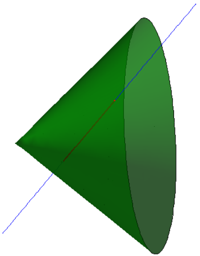

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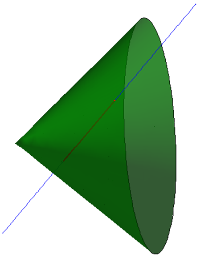

**G-Parrilo-Thomas**

Theta Bodies are projections of spectrahedra [moment theory]

# Convergence

$$\mathrm{TH}_1(I) \supseteq \mathrm{TH}_2(I) \supseteq \ldots \supseteq \mathrm{TH}_k(I) \supseteq \mathrm{cl}(\mathrm{conv}(\mathcal{V}_{\mathbb{R}}(I)))$$

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**G-Netzer**

If $\mathcal{V}_{\mathbb{R}}(I)$ has “bad” singularities, that convergence is not finite.

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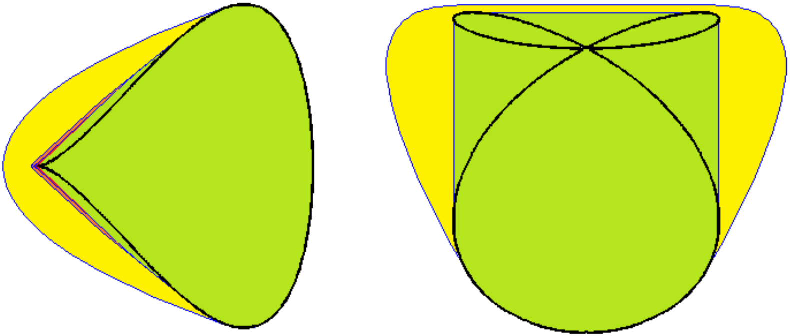

Two quartics and their theta body sequence.

# Finite sets

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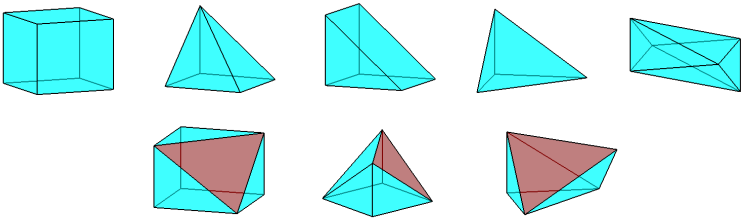
**G-Parrilo-Thomas**

If $S \subseteq \mathbb{R}^n$ is finite, $I(S)$ is $\mathrm{TH}_1$-exact if and only if $S$ is the set of vertices of a 2-level polytope.

# 2-level polytopes



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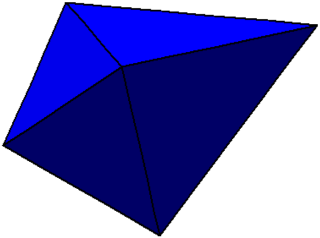

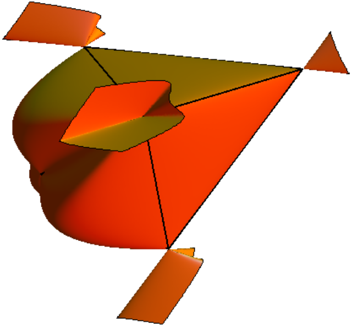

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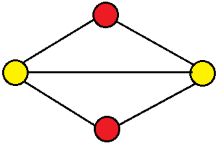

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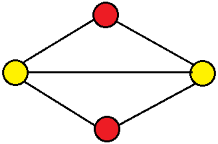
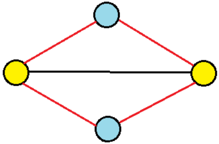


**G-Laurent-Parrilo-Thomas**

The max-cut problem.



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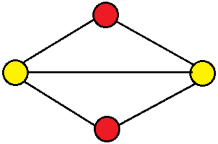

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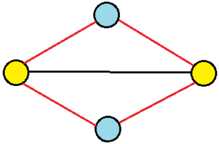
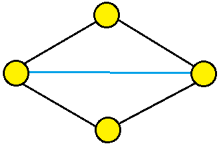


**G-Thomas**

The max triangle-free subgraph / min $K_3$-cover problem.



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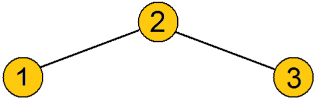
STAB($G$) - stable set polytope of $G$.

# Example



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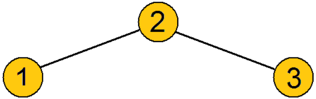


$$S_G = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,0,1)\}$$

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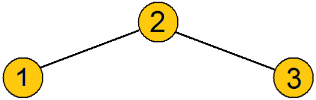


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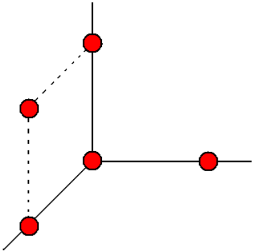

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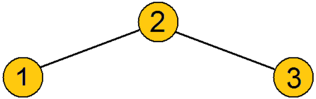


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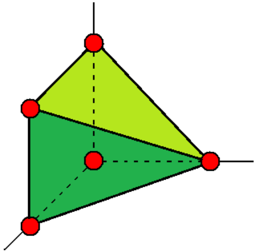

# Theta body for stable set

Given a graph $G$ with $n$ nodes, $\mathrm{TH}_1$ is the set of all vectors $x \in \mathbb{R}^n$ such that

$$\begin{bmatrix} 1 & x^t \\ x & U \end{bmatrix} \succeq 0$$

for some symmetric $U \in \mathbb{R}^{n \times n}$ with $\mathrm{diag}(U) = x$ and $U_{ij} = 0$ for all edges $(i, j)$.

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It is a projected spectrahedron.

**Theorem (Lovász)**

$\mathrm{TH}_1 = \mathrm{STAB}(G)$ *if and only if $G$ is perfect.*

# Lifts of stable sets

If $G$ is perfect STAB($G$) is a projection of a slice of the cone $\text{PSD}_{n+1}$. $\binom{n+1}{2}$ variables

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We want to frame all these approaches and their limits in one single theory

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For every even set $A \subseteq \{1, \ldots, n\}$,

$$\sum_{i \in A} x_i - \sum_{i \notin A} x_i \leq |A| - 1$$

is a facet, so we have at least $2^{n-1}$ facets.

# Parity Polytope

There is a much shorter description.

$\mathrm{PP}_n$ is the set of $\mathbf{x} \in \mathbb{R}^n$ such that there exists for every odd $1 \leq k \leq n$ a vector $\mathbf{z}_k \in \mathbb{R}^n$ and a real number $\alpha_k$ such that

- $\sum_k \mathbf{z}_k = \mathbf{x}$;
- $\sum_k \alpha_k = 1$;
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$O(n^2)$ variables and $O(n^2)$ constraints.

# Complexity of a Polytope

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## Canonical LP Lift

Given a polytope $P$, a canonical LP lift is a description

$$P = \Phi(\mathbb{R}^k_+ \cap L)$$

for some affine space $L$ and affine map $\Phi$. We say it is a $\mathbb{R}^k_+$-lift.

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We are interested in the smallest $k$ such that $P$ has a $\mathbb{R}^k_+$-lift, a much better measure of "LP-complexity" .

# Two definitions

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The slack matrix of $P$ is the matrix $S_P \in \mathbb{R}^{v \times f}$ defined by

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## Nonnegative Factorization

Given a nonnegative matrix $M \in \mathbb{R}_+^{n \times m}$ we say that it has a $k$-nonnegative factorization, or a $\mathbb{R}_+^k$-factorization if there exist matrices $A \in \mathbb{R}_+^{n \times k}$ and $B \in \mathbb{R}_+^{k \times m}$ such that

$$M = A \cdot B.$$

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*A polytope $P$ has a $\mathbb{R}^k_+$-lift if and only if $S_P$ has a $\mathbb{R}^k_+$-factorization.*

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- Can we compare the power of different lifts?
- Does LP solve all polynomial combinatorial problems?

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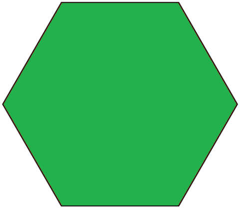

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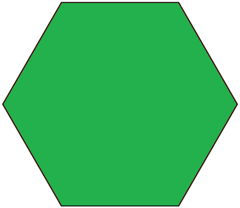

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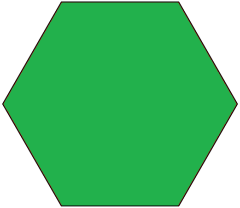

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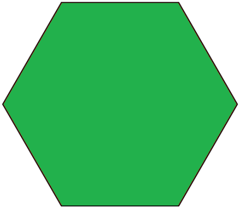

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# Hexagon - continued

It is the projection of the slice of $\mathbb{R}^5_+$ cut out by

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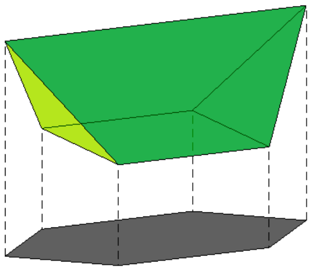

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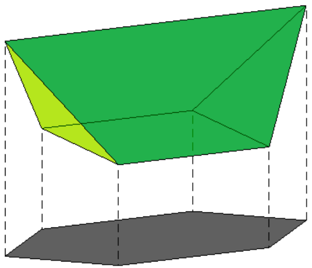

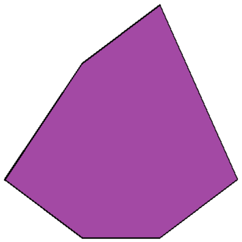

For irregular hexagons a $\mathbb{R}^6_+$-lift is the only we can have.

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Note that if the theta body is exact, it is a PSD-lift.

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## $K$-Factorization

Given a nonnegative matrix $M \in \mathbb{R}_+^{n\times m}$ we say that it has a $K$-factorization if there exist $a_1, \ldots a_n \in K$ and $b_1, \ldots, b_m \in K^*$ such that

$$M_{i,j} = \langle a_i, b_j \rangle .$$

We can now generalize Yannakakis.

# Generalized Yannakakis for polytopes

**Theorem (G-Parrilo-Thomas)**

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- That is actually the best possible PSD-lift.
- **[Burer]** In general STAB($G$) has a CP$_{n+1}$-lift.
- We can generalize Yannakakis further to other convex sets by introducing a slack operator.

# The Square

The $0/1$ square is the projection onto $x$ and $y$ of the slice of $\mathrm{PSD}_3$ given by

$$\begin{bmatrix} 1 & x & y \\ x & x & z \\ y & z & y \end{bmatrix} \succeq 0.$$

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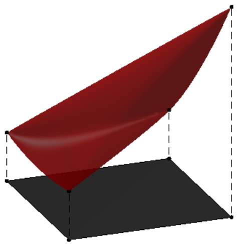

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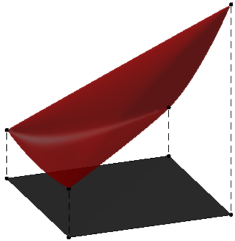

Its slack matrix is given by

$$S_P = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

and should factorize in $PSD_3$.

# Square - continued

$$S_P = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

is factorized by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

for the rows and

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- The role of symmetry.
- Are there polynomial sized [symmetric] SDP-lifts for the matching polytope? What about LP?
- Are there polynomial sized LP-lifts for the stable set polytope of a perfect graph?
- Which sets are SDP-representable, i.e., which sets have SDP-lifts?

# The end

**Thank You**