PSD-minimality and slack ideals

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with Kanstantsin Pashkovich (U. Waterloo), Richard Z. Robinson and Rekha Thomas (U.Washington)

Semidefinite Representations

A semidefinite representation of size k of a d-polytope P is a description

$$P = \left\{ x \in \mathbb{R}^d \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

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Psd-minimal polytopes

The size of any semidefinite representation of a d-polytope P cannot be smaller than $d+1$. If it equals $d+1$ we call the polytope **psd-minimal**.

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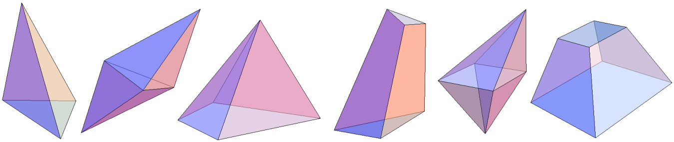
- In $\mathbb{R}^3$ there are six classes of psd-minimal polytopes: simplices, triangular bipyramids, quadrilateral pyramids, (combinatorial) triangular prisms, biplanar octahedra and biplanar cubes .



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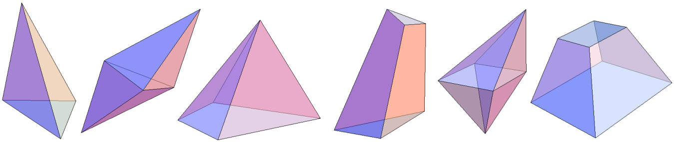

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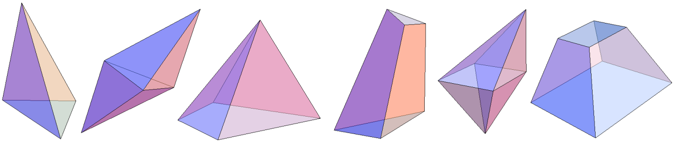

- What happens in $\mathbb{R}^4$? **[GPRT15]** There are precisely 31 combinatorial classes of psd-minimal 4-polytopes.

Slack matrices

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

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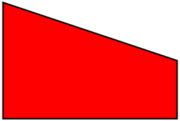

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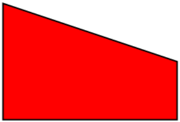

$$S_P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 2 \end{pmatrix}$$

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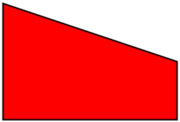

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Let P be a d-polytope and $S_P(x)$ a symbolic matrix with the same support as S_P. Then the slack ideal of P is

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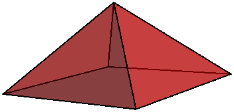

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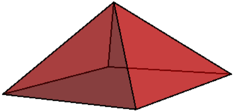

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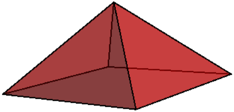

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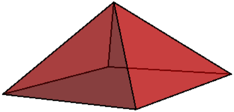

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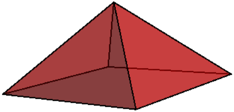

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Trinomial obstructions

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If I_P has a trinomial of the form $x^a + x^b - x^c$ then P is not psd-minimal.

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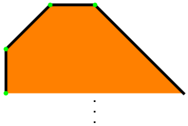

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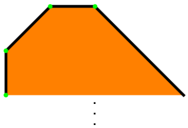

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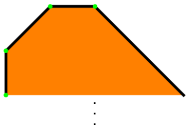

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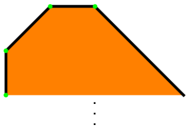

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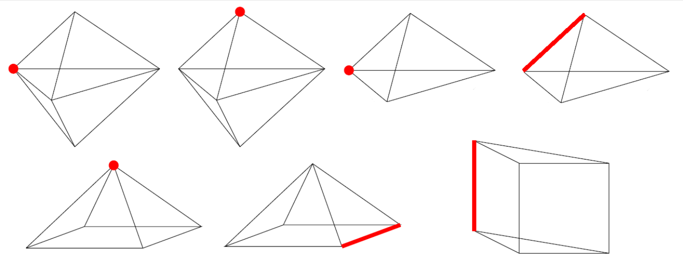
Only triangles and quadrilaterals can be psd-minimal in $\mathbb{R}^2$.

Combinatorial obstructions

From these we can derive combinatorial obstructions.

Proposition

Non-trivial intersections of facets of psd-minimal 4-polytopes:



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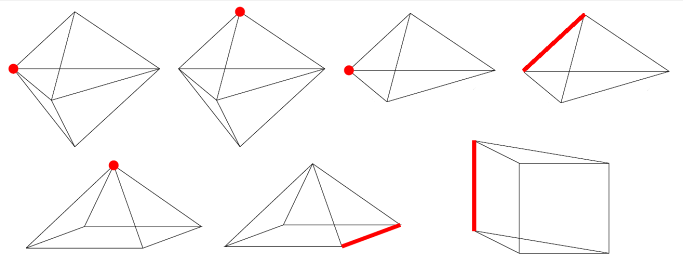

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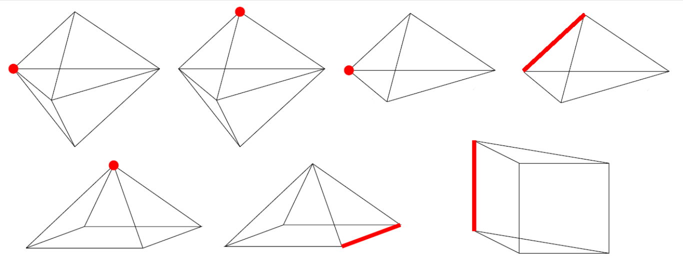

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♯	Construction	Vertices of a psd-minimal embedding	Facet Types	Dual	f-vector
1	Δ_4	$\{-e_{1234}, e_1, e_2, e_3, e_4\}$	5S	Self	$(5,10,10,5)$
2	$(\Delta_1 \times \Delta_1) * \Delta_1$	$\{\pm e_1, \pm e_2, e_3, e_4\}$	4S,2Py	Self	$(6,13,13,6)$
3		$\{0, 2e_1, 2e_2, 2e_3, e_{12} - e_3, e_4, e_{34}\}$	3S,2Py,2B	Self	$(7,17,17,7)$
4	$\Delta_3 \times \Delta_1$	$\{-e_{123}, e_1, e_2, e_3\} + \{\pm e_4\}$	2S,4Pr	5	$(8,16,14,6)$
5	$\Delta_3 \oplus \Delta_1$	$\{-e_{123}, e_1, e_2, e_3, \pm e_4\}$	8S	4	$(6,14,16,8)$
6	$\Delta_2 \times \Delta_2$	$\{-e_{12}, e_1, e_2\} + \{-e_{34}, e_3, e_4\}$	6Pr	7	$(9,18,15,6)$
7	$\Delta_2 \oplus \Delta_2$	$\{-e_{12}, e_1, e_2, -e_{34}, e_3, e_4\}$	9S	6	$(6,15,18,9)$
8	$(\Delta_2 \times \Delta_1) * \Delta_0$	$\{e_4\} \cup (\{-e_{12}, e_1, e_2\} + \{\pm e_3\})$	2S,1Pr,3Py	9	$(7,15,14,6)$
9	$(\Delta_2 \oplus \Delta_1) * \Delta_0$	$\{-e_{12}, e_1, e_2, \pm e_3, e_4\}$	6S,1B	8	$(6,14,15,7)$
10		$\{0, e_1, e_2, e_3, e_{13}, e_{23}, e_4, e_{14}\}$	1S,2Pr,4Py	11	$(8,18,17,7)$
11		$\{e_1, e_2, e_3, e_4, -2e_1 - e_{24}, -e_{13} - 2e_2, -2e_{12}\}$	4S,4Py	10	$(7,17,18,8)$
12		$\{0, e_1, e_2/2, e_3, e_4, e_{14}, e_{12}/2, e_{13}, e_2 + 4e_{34}\}$	3Pr,3Py,2B	13	$(9,22,21,8)$
13		$\{e_1, e_2, 9/4e_3, e_4, e_{124}/2, e_{13}, e_2 + e_3/4, e_{34}\}$	2S,6Py,1B	12	$(8,21,22,9)$
14	$(\Delta_2 \oplus \Delta_1) \times \Delta_1$	$\{0, e_1, e_2, e_3, e_4, e_{12}, e_{23}, e_{24}, 2e_{13} + e_4, 2e_{13} + e_{24}\}$	6Pr,2B	15	$(10,23,21,8)$
15	$(\Delta_2 \times \Delta_1) \oplus \Delta_1$	$\{e_1, 2e_2, e_3, 2e_4, e_2 + 2e_3, e_2 + 4e_4, 2e_1 + e_2, e_{134}\}$	4S,6Py	14	$(8,21,23,10)$
16	$(\Delta_1 \times \Delta_1 \times \Delta_1) * \Delta_0$	$(\{\pm e_1\} + \{\pm e_2\} + \{\pm e_3\}) \cup \{e_4\}$	1C,6Py	17	$(9,20,18,7)$
17	$(\Delta_1 \oplus \Delta_1 \oplus \Delta_1) * \Delta_0$	$\{\pm e_1, \pm e_2, \pm e_3, e_4\}$	1O,8S	16	$(7,18,20,9)$
18		$\{0, e_1, e_2/2e_4, e_{234}, e_{23}, e_{24}/2, e_{134}, e_{13}\}$	2Pr,4Py,2B	19	$(9,22,21,8)$
19		$\{0, e_1, e_3, e_4, e_{14}, e_{23}, e_{24}, e_{234}\}$	1O,4S,4Py	18	$(8,21,22,9)$
20	$((\Delta_1 \times \Delta_1) * \Delta_0) \times \Delta_1$	$\{\pm e_1, \pm e_2, e_3\} + \{\pm e_4\}$	1C,4Pr,2Py	21	$(10,21,18,7)$
21	$((\Delta_1 \times \Delta_1) * \Delta_0) \oplus \Delta_1$	$\{\pm e_1, \pm e_2, e_3, e_3/2 \pm e_4\}$	8S,2Py	20	$(7,18,21,10)$
22		$\{0, 2e_1, 2e_3, 2e_4, e_{12}, e_{123}, e_{1234}, 2e_{24}, 2e_{34}\}$	6Py,3B	23	$(9,24,24,9)$
23		$\{0, e_1, e_3, e_4, e_{12}, e_{123}, e_{23}, e_{24}, e_{234}\}$	2O,3S,1Pr,3Py	22	$(9,24,24,9)$
24		$\{0, 2e_1, 2e_2, 2e_3, 2e_4, e_{123}, e_{124}, e_{134}, e_{1234}, e_{234}\}$	10B	25	$(10,30,30,10)$
25		$\{e_1, e_2, e_3, e_4, e_{12}, e_{13}, e_{14}, e_{23}, e_{24}, e_{34}\}$	5O,5S	24	$(10,30,30,10)$
26	$(\Delta_1 \times \Delta_1 \times \Delta_1) \oplus \Delta_1$	$(\{\pm e_1\} + \{\pm e_2\} + \{\pm e_3\}) \cup \{\pm e_4\}$	12Py	27	$(10,28,30,12)$
27	$(\Delta_1 \oplus \Delta_1 \oplus \Delta_1) \times \Delta_1$	$\{\pm e_1, \pm e_2, \pm e_3\} + \{\pm e_4\}$	2O,8Pr	26	$(12,30,28,10)$
28	$\Delta_1 \times \Delta_1 \times \Delta_2$	$\{\pm e_1\} + \{\pm e_2\} + \{-e_{34}, e_3, e_4\}$	3C,4Pr	29	$(12,24,19,7)$
29	$\Delta_1 \oplus \Delta_1 \oplus \Delta_2$	$\{\pm e_1, \pm e_2, -e_{34}, e_3, e_4\}$	12S	28	$(7,19,24,12)$
30	$\Delta_1 \times \Delta_1 \times \Delta_1 \times \Delta_1$	$\{\pm e_1\} + \{\pm e_2\} + \{\pm e_3\} + \{\pm e_4\}$	8C	31	$(16,32,24,8)$
31	$\Delta_1 \oplus \Delta_1 \oplus \Delta_1 \oplus \Delta_1$	$\{\pm e_1, \pm e_2, \pm e_3 \pm e_4\}$	16S	30	$(8,24,32,16)$

Binomial slack ideals

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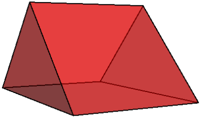

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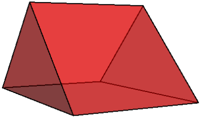

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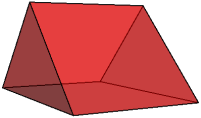

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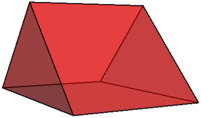

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Proposition

If I_P is binomial then P is psd-minimal.

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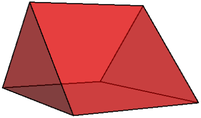

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The first 11 classes of the table and all d-polytopes with $d+2$ vertices have binomial slack ideals.

General technique

How to derive psd-minimality conditions:

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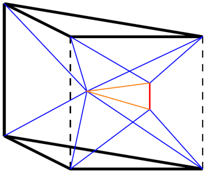
Example (Class 18)



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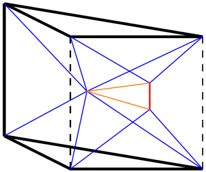

$$\overline{S_P}(x) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & x_3 & 0 & 0 & 1 \\ 0 & 1 & x_1 & 0 & 0 & 1 & x_9 & 0 \\ 0 & 1 & 1 & 0 & 0 & x_6 & 0 & x_{12} \\ 1 & 0 & 0 & x_2 & x_4 & 0 & x_{10} & 0 \\ 1 & 0 & 0 & 1 & x_5 & 0 & 0 & x_{13} \\ 0 & 0 & 0 & 1 & 0 & x_7 & x_{11} & 0 \\ 0 & 0 & 0 & 1 & 0 & x_8 & 0 & x_{14} \end{pmatrix}$$

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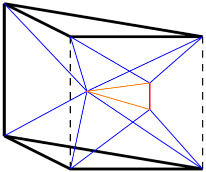

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$$I_P = \langle x_{12} + x_{13} - x_{14} - 1, x_{11} - x_{14}, x_{10} - x_{13}, x_9 + x_{13} - x_{14} - 1, x_8 - 1, x_7 - 1, x_6 - 1, x_5 - 1, x_4 - 1, x_3 - 1, x_2 - 1, x_1 - 1\rangle$$

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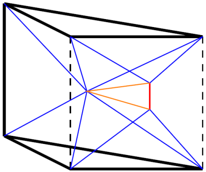

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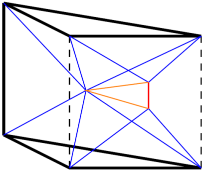

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Now we impose $y_9^2 = x_9$, $y_{10}^2 = x_{10}$ and

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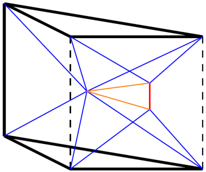

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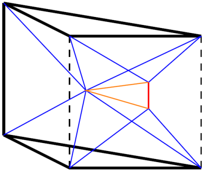

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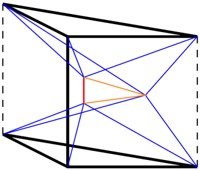
We get a linear space living inside the slack variety

An even more interesting example (class 12)



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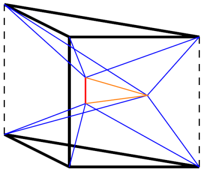

$$\overline{S_P}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ x_1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & x_1 & 0 & 0 & 1 & 1 & 0 & 0 \\ x_2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & x_2 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1-x_1-x_2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1-x_1-x_2 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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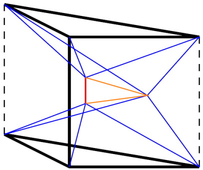

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It is psd-minimal if and only if

$$x_1^4 + 2x_1^3x_2 + 3x_1^2x_2^2 + 2x_1x_2^3 + x_2^4 - 2x_1^3 - 2x_2^3 + x_1^2 - 2x_1x_2 + x_2^2 = 0$$

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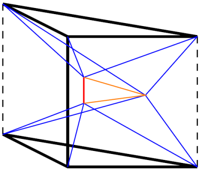

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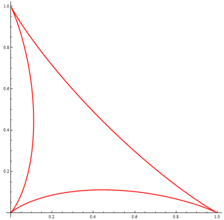

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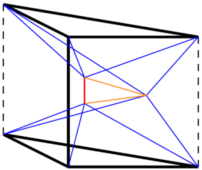

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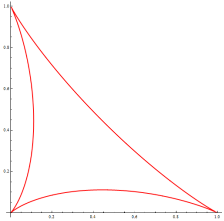

In particular:

- The positive square root does not work.
- The support does not have rank 5.

Open questions

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- Is the slack ideal of a combinatorially psd-minimal polytope always binomial?

Conclusion

G., Pashkovich, Robinson and Thomas.
Four Dimensional Polytopes of Minimum PSD Rank.
arXiv:1506.00187

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G., Pashkovich, Robinson and Thomas.
Four Dimensional Polytopes of Minimum PSD Rank.
arXiv:1506.00187

Thank you