Lifts of Convex Sets

João Gouveia

Universidade de Coimbra

27th July - Optimization 2011

with Pablo Parrilo (MIT) and Rekha Thomas (U.Washington)

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For every even set $A \subseteq \{1, \ldots, n\}$,

$$\sum_{i \in A} x_i - \sum_{i \notin A} x_i \leq |A| - 1$$

is a facet, so we have at least 2^{n-1} facets.

Parity Polytope

There is a much shorter description.

PP_n is the set of $\mathbf{x} \in \mathbb{R}^n$ such that there exists for every odd $1 \leq k \leq n$ a vector $\mathbf{z}_k \in \mathbb{R}^n$ and a real number α_k such that

- $\sum_k \mathbf{z}_k = \mathbf{x}$;
- $\sum_k \alpha_k = 1$;
- $\| \mathbf{z}_k \|_1 = k \, \alpha_k$;
- $0 \leq (\mathbf{z}_k)_i \leq \alpha_k$.

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$O(n^2)$ variables and $O(n^2)$ constraints.

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Canonical LP Lift

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for some affine space L and affine map Φ. We say it is a $\mathbb{R}^k_+$-lift.

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We are interested in the smallest k such that P has a $\mathbb{R}^k_+$-lift, a much better measure of "LP-complexity" .

Two definitions

Let P be a polytope with facets defined by $h_1(\mathbf{x}) \geq 0, \ldots, h_f(\mathbf{x}) \geq 0$, and vertices $p_1, \ldots, p_v$.

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The slack matrix of P is the matrix $S_P \in \mathbb{R}^{v \times f}$ defined by

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Nonnegative Factorization

Given a nonnegative matrix $M \in \mathbb{R}_+^{n \times m}$ we say that it has a k-nonnegative factorization, or a $\mathbb{R}_+^k$-factorization if there exist matrices $A \in \mathbb{R}_+^{n \times k}$ and $B \in \mathbb{R}_+^{k \times m}$ such that

$$M = A \cdot B.$$

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- Does it work for other types of convex sets?
- Can we compare the power of different lifts?
- Does LP solve all polynomial combinatorial problems?

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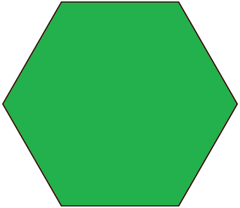

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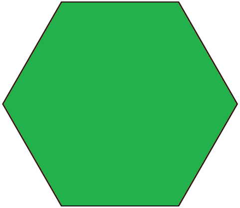

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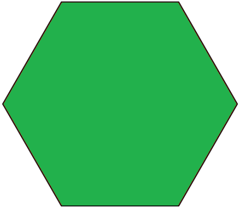

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$$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

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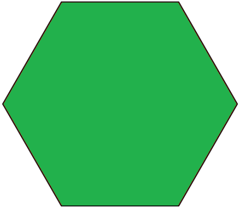

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Hexagon - continued

It is the projection of the slice of $\mathbb{R}^5_+$ cut out by

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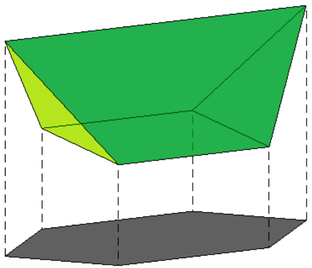

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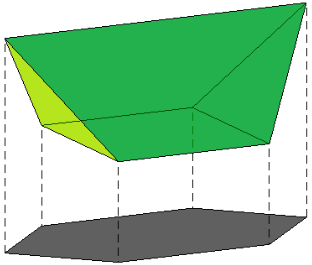

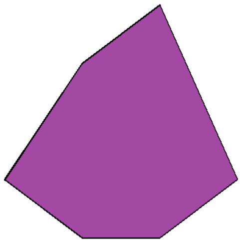

For irregular hexagons a $\mathbb{R}^6_+$-lift is the only we can have.

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Important cases are $\mathbb{R}^n_+$, PSD_n, SOCP_n, CP_n, CoP_n, ...

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Recall that if $K \subseteq \mathbb{R}^l$ is a closed convex cone, $K^* \subseteq \mathbb{R}^l$ is its dual cone, defined by

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Given a nonnegative matrix $M \in \mathbb{R}_+^{n \times m}$ we say that it has a K-factorization if there exist $a_1, \ldots a_n \in K$ and $b_1, \ldots, b_m \in K^*$ such that

$$M_{i,j} = \langle a_i, b_j \rangle .$$

We can now generalize Yannakakis.

Generalized Yannakakis for polytopes

Theorem (G-Parrilo-Thomas)

A polytope P has a K-lift if and only if S_P has a K-factorization.

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Given a convex set $C \subseteq \mathbb{R}^n$, consider its polar set

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$$S_C(x, y) = 1 - \langle x, y \rangle .$$

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Note that this generalizes the slack matrix.

Generalized Yannakakis for convex sets

We can then define a *K-factorization of S_C* as a pair of maps

$$A : \mathrm{ext}(C) \to K \qquad B : \mathrm{ext}(C^\circ) \to K^*$$

such that

$$\langle A(x), B(y) \rangle = S_C(x, y)$$

for all x, y.

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Theorem (G-Parrilo-Thomas)

A convex set C has a K-lift if and only if S_C has a K-factorization.

The Square

The 0/1 square is the projection onto x and y of the slice of PSD_3 given by

$$\begin{bmatrix} 1 & x & y \\ x & x & z \\ y & z & y \end{bmatrix} \succeq 0.$$

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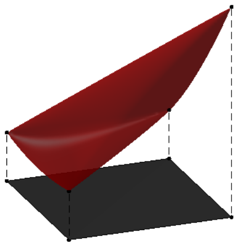

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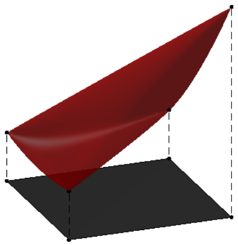

Its slack matrix is given by

$$S_P = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

and should factorize in PSD_3.

Square - continued

$$S_P = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

is factorized by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix},$$

for the rows and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

for the columns.

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- Are there polynomial sized [symmetric] SDP-lifts for the matching polytope? What about LP?
- Are there polynomial sized LP-lifts for the stable set polytope of a perfect graph?
- Which sets are SDP-representable, i.e., which sets have SDP-lifts?

The end

Thank You