(Minimal)SDP-lifts of Polytopes

João Gouveia

University of Coimbra

21st March 2013 - 4th SDP days - CWI

1. Linear Representations and Yannakakis Theorem

Polytopes

The usual way to describe a polytope P is by listing the vertices or giving an inequality description

$$P = \{x \in \mathbb{R}^n : a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq b\},$$

where b and a_i are real vectors and the inequalities are taken entry-wise.

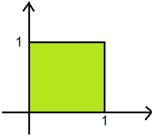
Polytopes

The usual way to describe a polytope P is by listing the vertices or giving an inequality description

$$P = \{x \in \mathbb{R}^n : a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq b\},$$

where b and a_i are real vectors and the inequalities are taken entry-wise.

Example:



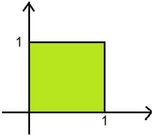
Polytopes

The usual way to describe a polytope P is by listing the vertices or giving an inequality description

$$P = \{x \in \mathbb{R}^n : a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \geq b\},$$

where b and a_i are real vectors and the inequalities are taken entry-wise.

Example:



$$C = \left\{ (x, y) : \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} y \geq \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

Linear Representations of Polytopes

If a polytope has many facets and vertices we would like a better description.

Linear Representations of Polytopes

If a polytope has many facets and vertices we would like a better description.

A linear representation of a polytope P is a description

$$P = \{x : \exists y, a_1 x_1 + \cdots + a_n x_n + a_{n+1} y_1 + \cdots a_{n+m} y_m \geq b\},$$

i.e., a description of P as a projection of a higher dimensional polytope.

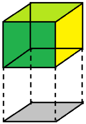
Linear Representations of Polytopes

If a polytope has many facets and vertices we would like a better description.

A linear representation of a polytope P is a description

$$P = \{x : \exists y, a_1 x_1 + \cdots + a_n x_n + a_{n+1} y_1 + \cdots a_{n+m} y_m \geq b\},$$

i.e., a description of P as a projection of a higher dimensional polytope.



Linear representation of a square.

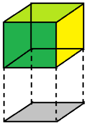
Linear Representations of Polytopes

If a polytope has many facets and vertices we would like a better description.

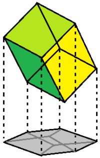
A linear representation of a polytope P is a description

$$P = \{x : \exists y, a_1 x_1 + \cdots + a_n x_n + a_{n+1} y_1 + \cdots a_{n+m} y_m \geq b\},$$

i.e., a description of P as a projection of a higher dimensional polytope.



Linear representation of a square.



Linear representation of an hexagon.

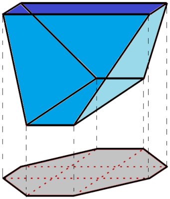
Why?

The projection of a polytope can have many more facets than the original:

Why?

The projection of a polytope can have many more facets than the original:

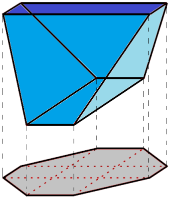
(Ben-Tal + Nemirovski, 2001): A regular n-gon can be written as the projection of a polytope with $2\lceil \log_2(n) \rceil$ sides.



Why?

The projection of a polytope can have many more facets than the original:

(Ben-Tal + Nemirovski, 2001): A regular n-gon can be written as the projection of a polytope with $2\lceil \log_2(n) \rceil$ sides.

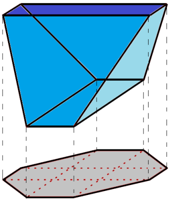


To do linear optimization on the projection we can optimize on the “upper” polytope.

Why?

The projection of a polytope can have many more facets than the original:

(Ben-Tal + Nemirovski, 2001): A regular n-gon can be written as the projection of a polytope with $2\lceil \log_2(n) \rceil$ sides.



To do linear optimization on the projection we can optimize on the “upper” polytope.

Given a polytope P we are interested in finding how small can its linear representation be. This tells us how hard it is to optimize over P using LP.

Slack Matrix

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

Slack Matrix

Let P be a polytope with facets given by $h_1(x) \geq 0, \dots, h_f(x) \geq 0$, and vertices $p_1, \dots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i, j) = h_i(p_j).$$

Slack Matrix

Let P be a polytope with facets given by
$h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Example: For the unit cube.

Slack Matrix

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Example: For the unit cube.

	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	$(1,1,0)$	$(0,1,1)$	$(1,0,1)$	$(1,1,1)$
$x \geq 0$								
$y \geq 0$								
$z \geq 0$								
$1 - x \geq 0$								
$1 - y \geq 0$								
$1 - z \geq 0$								

Slack Matrix

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Example: For the unit cube.

	0 0 0	1 0 0	0 1 0	0 0 1	1 1 0	0 1 1	1 0 1	1 1 1
$x \geq 0$	0	1	0	0	1	0	1	1
$y \geq 0$								
$z \geq 0$								
$1 - x \geq 0$								
$1 - y \geq 0$								
$1 - z \geq 0$								

Slack Matrix

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Example: For the unit cube.

	$(0,0,0)^T$	$(1,0,0)^T$	$(0,1,0)^T$	$(0,0,1)^T$	$(1,1,0)^T$	$(0,1,1)^T$	$(1,0,1)^T$	$(1,1,1)^T$
$x \geq 0$	0	1	0	0	1	0	1	1
$y \geq 0$	0	0	1	0	1	1	0	1
$z \geq 0$								
$1 - x \geq 0$								
$1 - y \geq 0$								
$1 - z \geq 0$								

Slack Matrix

Let P be a polytope with facets given by $h_1(x) \geq 0, \ldots, h_f(x) \geq 0$, and vertices $p_1, \ldots, p_v$.

The slack matrix of P is the matrix $S_P \in \mathbb{R}^{f \times v}$ given by

$$S_P(i,j) = h_i(p_j).$$

Example: For the unit cube.

	$(0,0,0)$	$(1,0,0)$	$(0,1,0)$	$(0,0,1)$	$(1,1,0)$	$(0,1,1)$	$(1,0,1)$	$(1,1,1)$
$x \geq 0$	0	1	0	0	1	0	1	1
$y \geq 0$	0	0	1	0	1	1	0	1
$z \geq 0$	0	0	0	1	0	1	1	1
$1 - x \geq 0$	1	0	1	1	0	1	0	0
$1 - y \geq 0$	1	1	0	1	0	0	1	0
$1 - z \geq 0$	1	1	1	0	1	0	0	0

Nonnegative Factorizations

Let M be an m by n nonnegative matrix.

Nonnegative Factorizations

Let M be an m by n nonnegative matrix.

A nonnegative factorization of M of size k is a pair of nonnegative matrices A, m by k, and B, k by n, such that

$$M = A \times B.$$

Nonnegative Factorizations

Let M be an m by n nonnegative matrix.

A nonnegative factorization of M of size k is a pair of nonnegative matrices A, m by k, and B, k by n, such that

$$M = A \times B.$$

Equivalently, it is a collection of nonnegative vectors $a_1, \cdots, a_m$ and $b_1, \cdots b_n$ in $\mathbb{R}^k$ such that $M_{i,j} = \langle a_i, b_j \rangle$.

Nonnegative Factorizations

Let M be an m by n nonnegative matrix.

A nonnegative factorization of M of size k is a pair of nonnegative matrices A, m by k, and B, k by n, such that

$$M = A \times B.$$

Equivalently, it is a collection of nonnegative vectors $a_1, \cdots, a_m$ and $b_1, \cdots b_n$ in $\mathbb{R}^k$ such that $M_{i,j} = \langle a_i, b_j \rangle$.

Example:

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

Nonnegative Factorizations

Let M be an m by n nonnegative matrix.

A nonnegative factorization of M of size k is a pair of nonnegative matrices A, m by k, and B, k by n, such that

$$M = A \times B.$$

Equivalently, it is a collection of nonnegative vectors $a_1, \cdots, a_m$ and $b_1, \cdots b_n$ in $\mathbb{R}^k$ such that $M_{i,j} = \langle a_i, b_j \rangle$.

Example:

$$M = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 3 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$$

Yannakakis Theorem

Theorem (Yannakakis 1991)

Let P be any polytope and S its slack matrix. Then the following are equal.

Yannakakis Theorem

Theorem (Yannakakis 1991)

Let P be any polytope and S its slack matrix. Then the following are equal.

- The least number of facets of a polytope Q whose projection is P.

Yannakakis Theorem

Theorem (Yannakakis 1991)

Let P be any polytope and S its slack matrix. Then the following are equal.

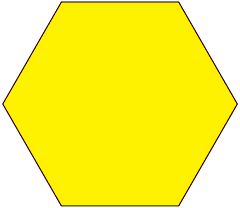
- The least number of facets of a polytope Q whose projection is P.
- The least k such that S has a nonnegative factorization of size k. [$\mathrm{rank}_+(S)$]

Hexagon

Consider the regular hexagon.

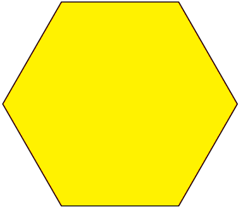
Hexagon

Consider the regular hexagon.



Hexagon

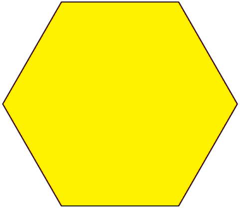
Consider the regular hexagon.



It has a 6×6 slack matrix.

Hexagon

Consider the regular hexagon.

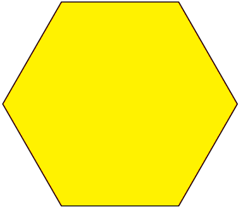


It has a 6×6 slack matrix.

$$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix}$$

Hexagon

Consider the regular hexagon.

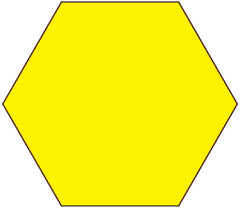


It has a 6×6 slack matrix.

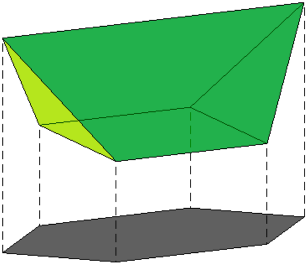
$$\begin{bmatrix} 0 & 0 & 1 & 2 & 2 & 1 \\ 1 & 0 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 0 & 1 & 2 \\ 2 & 2 & 1 & 0 & 0 & 1 \\ 1 & 2 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 1 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Hexagon

Consider the regular hexagon.



It has a 6×6 slack matrix.



2. Semidefinite Representations: General Yannakakis Theorem

Semidefinite Representations

A semidefinite representation of size k of a polytope P is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where A_i and B_i are $k \times k$ real symmetric matrices.

Semidefinite Representations

A semidefinite representation of size k of a polytope P is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where A_i and B_i are $k \times k$ real symmetric matrices.

Given a polytope P we are interested in finding how small can such a description be.

Semidefinite Representations

A semidefinite representation of size k of a polytope P is a description

$$P = \left\{ x \in \mathbb{R}^n \;\middle|\; \exists y \text{ s.t. } A_0 + \sum A_i x_i + \sum B_i y_i \succeq 0 \right\}$$

where A_i and B_i are $k \times k$ real symmetric matrices.

Given a polytope P we are interested in finding how small can such a description be.

This tells us how hard it is to optimize over P using semidefinite programming.

The Square

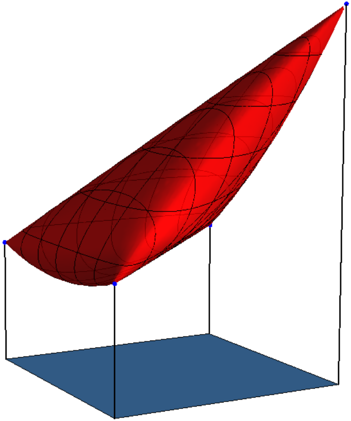
The $0/1$ square is the projection onto x_1 and x_2 of

$$\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1 & y \\ x_2 & y & x_2 \end{bmatrix} \succeq 0.$$

The Square

The $0/1$ square is the projection onto x_1 and x_2 of

$$\begin{bmatrix} 1 & x_1 & x_2 \\ x_1 & x_1 & y \\ x_2 & y & x_2 \end{bmatrix} \succeq 0.$$



Semidefinite Factorizations

Let M be a m by n nonnegative matrix.

Semidefinite Factorizations

Let M be a m by n nonnegative matrix. A PSD_k-factorization of M is a set of $k \times k$ positive semidefinite matrices $A_1, \cdots, A_m$ and $B_1, \cdots B_n$ such that $M_{i,j} = \langle A_i, B_j \rangle$.

Semidefinite Factorizations

Let M be a m by n nonnegative matrix. A PSD_k-factorization of M is a set of $k \times k$ positive semidefinite matrices $A_1, \cdots, A_m$ and $B_1, \cdots B_n$ such that $M_{i,j} = \langle A_i, B_j \rangle$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Semidefinite Factorizations

Let M be a m by n nonnegative matrix. A PSD_k-factorization of M is a set of $k \times k$ positive semidefinite matrices $A_1, \cdots, A_m$ and $B_1, \cdots B_n$ such that $M_{i,j} = \langle A_i, B_j \rangle$.

$$\begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} \quad \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{matrix} \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \end{matrix} \quad \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Semidefinite Yannakakis Theorem

Theorem (G.-Parrilo-Thomas 2011)

A polytope P has a semidefinite representation of size k if and only if its slack matrix has a PSD_k-factorization.

Semidefinite Yannakakis Theorem

Theorem (G.-Parrilo-Thomas 2011)

A polytope P has a semidefinite representation of size k if and only if its slack matrix has a PSD_k*-factorization.*

The psd rank of M, $\mathrm{rank}_{\mathrm{psd}}(M)$ is the smallest k for which M has a PSD_k-factorization.

Semidefinite Yannakakis Theorem

Theorem (G.-Parrilo-Thomas 2011)

A polytope P has a semidefinite representation of size k if and only if its slack matrix has a PSD_k-factorization.

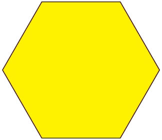
The psd rank of M, $\mathrm{rank}_{\mathrm{psd}}(M)$ is the smallest k for which M has a PSD_k-factorization.

The psd rank of a polytope P is defined as

$$\mathrm{rank}_{psd}(P) := \mathrm{rank}_{psd}(S_P).$$

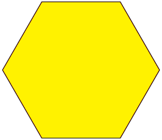
The Hexagon

Consider again the regular hexagon.



The Hexagon

Consider again the regular hexagon.

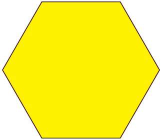


Its 6×6 slack matrix.

$$\begin{bmatrix} 0 & 0 & 2 & 4 & 4 & 2 \\ 2 & 0 & 0 & 2 & 4 & 4 \\ 4 & 2 & 0 & 0 & 2 & 4 \\ 4 & 4 & 2 & 0 & 0 & 2 \\ 2 & 4 & 4 & 2 & 0 & 0 \\ 0 & 2 & 4 & 4 & 2 & 0 \end{bmatrix}$$

The Hexagon

Consider again the regular hexagon.



Its 6×6 slack matrix.

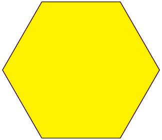
$$\begin{bmatrix} 0 & 0 & 2 & 4 & 4 & 2 \\ 2 & 0 & 0 & 2 & 4 & 4 \\ 4 & 2 & 0 & 0 & 2 & 4 \\ 4 & 4 & 2 & 0 & 0 & 2 \\ 2 & 4 & 4 & 2 & 0 & 0 \\ 0 & 2 & 4 & 4 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

The Hexagon

Consider again the regular hexagon.



Its 6×6 slack matrix.

$$\begin{bmatrix} 0 & 0 & 2 & 4 & 4 & 2 \\ 2 & 0 & 0 & 2 & 4 & 4 \\ 4 & 2 & 0 & 0 & 2 & 4 \\ 4 & 4 & 2 & 0 & 0 & 2 \\ 2 & 4 & 4 & 2 & 0 & 0 \\ 0 & 2 & 4 & 4 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 0 & -1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

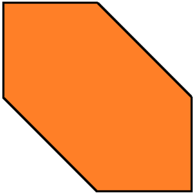
The Hexagon - continued

The regular hexagon must have a size 4 representation.

The Hexagon - continued

The regular hexagon must have a size 4 representation.

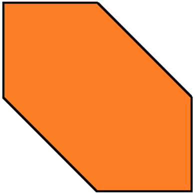
Consider the affinely equivalent hexagon H with vertices $(\pm 1, 0), (0, \pm 1), (1, -1)$ and $(-1, 1)$.



The Hexagon - continued

The regular hexagon must have a size 4 representation.

Consider the affinely equivalent hexagon H with vertices $(\pm 1, 0), (0, \pm 1), (1, -1)$ and $(-1, 1)$.



$$H = \left\{ (x_1, x_2) : \begin{bmatrix} 1 & x_1 & x_2 & x_1 + x_2 \\ x_1 & 1 & y_1 & y_2 \\ x_2 & y_1 & 1 & y_3 \\ x_1 + x_2 & y_2 & y_3 & 1 \end{bmatrix} \succeq 0 \right\}$$

3. Nonnegative and Semidefinite ranks

Basic Facts

Let M be a p by q nonnegative matrix. Then:

Basic Facts

Let M be a p by q nonnegative matrix. Then:

- $\text{rank}(M) \leq \text{rank}_+(M) \leq \min\{p, q\}$.

Basic Facts

Let M be a p by q nonnegative matrix. Then:

- $\text{rank}(M) \leq \text{rank}_+(M) \leq \min\{p, q\}$.
- $\text{rank}(M) \leq \binom{\text{rank}_{\text{psd}}(M)+1}{2}$.

Basic Facts

Let M be a p by q nonnegative matrix. Then:

- $\text{rank}(M) \leq \text{rank}_+(M) \leq \min\{p, q\}$.
- $\text{rank}(M) \leq \binom{\text{rank}_{\text{psd}}(M)+1}{2}$.
- $\text{rank}_{\text{psd}}(M) \leq \text{rank}_+(M)$.

Basic Facts

Let M be a p by q nonnegative matrix. Then:

- $\text{rank}(M) \leq \text{rank}_+(M) \leq \min\{p, q\}$.
- $\text{rank}(M) \leq \binom{\text{rank}_{\text{psd}}(M)+1}{2}$.
- $\text{rank}_{\text{psd}}(M) \leq \text{rank}_+(M)$.

Computing these ranks is hard. In fact checking if $\text{rank}(M) = \text{rank}_+(M)$ is NP-Hard (Vavasis '07).

Basic Facts

Let M be a p by q nonnegative matrix. Then:

- $\text{rank}(M) \leq \text{rank}_+(M) \leq \min\{p, q\}$.
- $\text{rank}(M) \leq \binom{\text{rank}_{\text{psd}}(M)+1}{2}$.
- $\text{rank}_{\text{psd}}(M) \leq \text{rank}_+(M)$.

Computing these ranks is hard. In fact checking if $\text{rank}(M) = \text{rank}_+(M)$ is NP-Hard (Vavasis '07).

Many other complexity questions are open.

Rectangle covering bound

The nonnegative rank of a matrix M is larger than the size of its smallest rectangle cover.

Rectangle covering bound

The nonnegative rank of a matrix M is larger than the size of its smallest rectangle cover.

Example:

$$M = \begin{bmatrix} 0 & 3 & 1 & 4 \\ 7 & 0 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

Rectangle covering bound

The nonnegative rank of a matrix M is larger than the size of its smallest rectangle cover.

Example:

$$M = \begin{bmatrix} 0 & 3 & 1 & 4 \\ 7 & 0 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

Rectangle covering bound

The nonnegative rank of a matrix M is larger than the size of its smallest rectangle cover.

Example:

$$M = \begin{bmatrix} 0 & 3 & 1 & 4 \\ 7 & 0 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

Rectangle covering bound

The nonnegative rank of a matrix M is larger than the size of its smallest rectangle cover.

Example:

$$M = \begin{bmatrix} 0 & 3 & 1 & 4 \\ 7 & 0 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

Rectangle covering bound

The nonnegative rank of a matrix M is larger than the size of its smallest rectangle cover.

Example:

$$M = \begin{bmatrix} 0 & 3 & 1 & 4 \\ 7 & 0 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

Rectangle covering bound

The nonnegative rank of a matrix M is larger than the size of its smallest rectangle cover.

Example:

$$M = \begin{bmatrix} 0 & 3 & 1 & 4 \\ 7 & 0 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

In this case $\text{rank}_+(M) \geq 4$.

Rectangle covering bound

The nonnegative rank of a matrix M is larger than the size of its smallest rectangle cover.

Example:

$$M = \begin{bmatrix} 0 & 3 & 1 & 4 \\ 7 & 0 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

In this case $\text{rank}_+(M) \geq 4$.

The rectangle bound corresponds to the boolean rank and also relates to the minimum communication complexity of a 2-party protocol to compute the support of M.

Hadamard Square Roots

A Hadamard Square Root of a nonnegative matrix M, denoted $\sqrt[H]{M}$, is a matrix whose entries are square roots (positive or negative) of the corresponding entries of M.

Hadamard Square Roots

A Hadamard Square Root of a nonnegative matrix M, denoted $\sqrt[H]{M}$, is a matrix whose entries are square roots (positive or negative) of the corresponding entries of M.

Example:

$M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$;

Hadamard Square Roots

A Hadamard Square Root of a nonnegative matrix M, denoted $\sqrt[H]{M}$, is a matrix whose entries are square roots (positive or negative) of the corresponding entries of M.

Example:

$$M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix};$$

$$\sqrt[H]{M} = \begin{bmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{bmatrix}$$

Hadamard Square Roots

A Hadamard Square Root of a nonnegative matrix M, denoted $\sqrt[H]{M}$, is a matrix whose entries are square roots (positive or negative) of the corresponding entries of M.

Example:

$$M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix};$$

$$\sqrt[H]{M} = \begin{bmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{bmatrix} \text{ or } \sqrt[H]{M} = \begin{bmatrix} -1 & 0 \\ \sqrt{2} & 1 \end{bmatrix}$$

Hadamard Square Roots

A Hadamard Square Root of a nonnegative matrix M, denoted $\sqrt[H]{M}$, is a matrix whose entries are square roots (positive or negative) of the corresponding entries of M.

Example:

$$M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix};$$

$$\sqrt[H]{M} = \begin{bmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{bmatrix} \text{ or } \sqrt[H]{M} = \begin{bmatrix} -1 & 0 \\ \sqrt{2} & 1 \end{bmatrix} \text{ or } \sqrt[H]{M} = \begin{bmatrix} -1 & 0 \\ -\sqrt{2} & 1 \end{bmatrix} \text{ or} \cdots$$

Hadamard Square Roots

A Hadamard Square Root of a nonnegative matrix M, denoted $\sqrt[H]{M}$, is a matrix whose entries are square roots (positive or negative) of the corresponding entries of M.

Example:

$$M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix};$$

$$\sqrt[H]{M} = \begin{bmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{bmatrix} \text{ or } \sqrt[H]{M} = \begin{bmatrix} -1 & 0 \\ \sqrt{2} & 1 \end{bmatrix} \text{ or } \sqrt[H]{M} = \begin{bmatrix} -1 & 0 \\ -\sqrt{2} & 1 \end{bmatrix} \text{ or} \cdots$$

We define $\text{rank}_H(M) = \min\{\text{rank}(\sqrt[H]{M})\}$.

Hadamard Square Roots

A Hadamard Square Root of a nonnegative matrix M, denoted $\sqrt[H]{M}$, is a matrix whose entries are square roots (positive or negative) of the corresponding entries of M.

Example:

$M = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$;

$$\sqrt[H]{M} = \begin{bmatrix} 1 & 0 \\ \sqrt{2} & 1 \end{bmatrix} \text{ or } \sqrt[H]{M} = \begin{bmatrix} -1 & 0 \\ \sqrt{2} & 1 \end{bmatrix} \text{ or } \sqrt[H]{M} = \begin{bmatrix} -1 & 0 \\ -\sqrt{2} & 1 \end{bmatrix} \text{ or} \cdots$$

We define $\text{rank}_H(M) = \min\{\text{rank}(\sqrt[H]{M})\}$.

Hadamard Rank and Semidefinite Rank

Proposition (G.-Robinson-Thomas 2012)

$rank_H(M)$ is the smallest k for which we have a semidefinite factorization of M of size k using only rank one matrices.

Hadamard Rank and Semidefinite Rank

Proposition (G.-Robinson-Thomas 2012)

$rank_H(M)$ is the smallest k for which we have a semidefinite factorization of M of size k using only rank one matrices.
In particular $\mathrm{rank}_{\mathrm{psd}}(M) \leq \mathrm{rank}_H(M)$.

Hadamard Rank and Semidefinite Rank

Proposition (G.-Robinson-Thomas 2012)

$rank_H(M)$ is the smallest k for which we have a semidefinite factorization of M of size k using only rank one matrices.
In particular $\text{rank}_{\text{psd}}(M) \leq \text{rank}_H(M)$.

Corollary

For $0/1$ matrices

$$\text{rank}_{\text{psd}}(M) \leq \text{rank}_H(M) \leq \text{rank}(M).$$

Examples

For $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ we have:

Examples

For $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ we have:

$$\text{rank}_{\text{psd}}(M) = 2, \ \ \text{rank}_H(M) = 3, \ \ \text{rank}(M) = 3.$$

Examples

For $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ we have:

$$\text{rank}_{\text{psd}}(M) = 2, \;\; \text{rank}_H(M) = 3, \;\; \text{rank}(M) = 3.$$

For $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ we have:

Examples

For $M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ we have:

$$\text{rank}_{\text{psd}}(M) = 2, \quad \text{rank}_H(M) = 3, \quad \text{rank}(M) = 3.$$

For $M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ we have:

$$\text{rank}_{\text{psd}}(M) = 2, \quad \text{rank}_H(M) = 2, \quad \text{rank}(M) = 3.$$

Example

Consider the matrix $A \in \mathbb{R}^{n \times n}$ defined by $a_{i,j} = (i - j)^2$.

$$A = \begin{bmatrix} 0 & 1 & 4 & 9 & 16 & \cdots \\ 1 & 0 & 1 & 4 & 9 & \cdots \\ 4 & 1 & 0 & 1 & 4 & \cdots \\ 9 & 4 & 1 & 0 & 1 & \cdots \\ 16 & 9 & 4 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- $\text{rank}(A) = 3$;

Example

Consider the matrix $A \in \mathbb{R}^{n \times n}$ defined by $a_{i,j} = (i - j)^2$.

$$A = \begin{bmatrix} 0 & 1 & 4 & 9 & 16 & \cdots \\ 1 & 0 & 1 & 4 & 9 & \cdots \\ 4 & 1 & 0 & 1 & 4 & \cdots \\ 9 & 4 & 1 & 0 & 1 & \cdots \\ 16 & 9 & 4 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- $\text{rank}(A) = 3$;
- $\text{rank}_{\text{psd}}(A) = 2$;

Example

Consider the matrix $A \in \mathbb{R}^{n \times n}$ defined by $a_{i,j} = (i - j)^2$.

$$A = \begin{bmatrix} 0 & 1 & 4 & 9 & 16 & \cdots \\ 1 & 0 & 1 & 4 & 9 & \cdots \\ 4 & 1 & 0 & 1 & 4 & \cdots \\ 9 & 4 & 1 & 0 & 1 & \cdots \\ 16 & 9 & 4 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- $\text{rank}(A) = 3$;
- $\text{rank}_{\text{psd}}(A) = 2$;
- $\text{rank}_{+}(A) \geq \log_2(n)$ grows with n.

Example

Consider the matrix $A \in \mathbb{R}^{n\times n}$ defined by $a_{i,j} = (i-j)^2$.

$$A = \begin{bmatrix} 0 & 1 & 4 & 9 & 16 & \cdots \\ 1 & 0 & 1 & 4 & 9 & \cdots \\ 4 & 1 & 0 & 1 & 4 & \cdots \\ 9 & 4 & 1 & 0 & 1 & \cdots \\ 16 & 9 & 4 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- $\text{rank}(A) = 3$;
- $\text{rank}_{\text{psd}}(A) = 2$;
- $\text{rank}_+(A) \geq \log_2(n)$ grows with n.

rank_+ can be arbitrarily larger than rank and rank_{psd}.

Lower Bounds for polytopes

Theorem (Goemans)

If a polytope P in $\mathbb{R}^n$ has m vertices, then it has nonnegative rank at least $\log(m)$.

Lower Bounds for polytopes

Theorem (Goemans)

If a polytope P in $\mathbb{R}^n$ has m vertices, then it has nonnegative rank at least $\log(m)$.

Theorem (G.-Parrilo-Thomas 2011)

If a polytope P in $\mathbb{R}^n$ has m vertices, then it has psd rank at least $O\left(\sqrt{\frac{\log(m)}{n\log(\log(m))}}\right)$.

Lower Bounds for polytopes

Theorem (Goemans)

If a polytope P in $\mathbb{R}^n$ has m vertices, then it has nonnegative rank at least $\log(m)$.

Theorem (G.-Parrilo-Thomas 2011)

If a polytope P in $\mathbb{R}^n$ has m vertices, then it has psd rank at least $O\left(\sqrt{\frac{\log(m)}{n\log(\log(m))}}\right)$.

For $P_n = n$-gon, $\text{rank}_+(P_n)$ and $\text{rank}_{\text{psd}}(P_n)$ grow to infinity as n grows, despite $\text{rank}(S_{P_n}) = 3$.

Lower Bounds for polytopes

Theorem (Goemans)

If a polytope P in $\mathbb{R}^n$ has m vertices, then it has nonnegative rank at least $\log(m)$.

Theorem (G.-Parrilo-Thomas 2011)

If a polytope P in $\mathbb{R}^n$ has m vertices, then it has psd rank at least $O\left(\sqrt{\frac{\log(m)}{n\log(\log(m))}}\right)$.

For $P_n = n$-gon, $\text{rank}_+(P_n)$ and $\text{rank}_{\text{psd}}(P_n)$ grow to infinity as n grows, despite $\text{rank}(S_{P_n}) = 3$.

Open questions:

- Separation between rank_{psd} and rank_+ for polytopes?

Lower Bounds for polytopes

Theorem (Goemans)

If a polytope P in $\mathbb{R}^n$ has m vertices, then it has nonnegative rank at least $\log(m)$.

Theorem (G.-Parrilo-Thomas 2011)

If a polytope P in $\mathbb{R}^n$ has m vertices, then it has psd rank at least $O\left(\sqrt{\frac{\log(m)}{n\log(\log(m))}}\right)$.

For $P_n = n$-gon, $\text{rank}_+(P_n)$ and $\text{rank}_{\text{psd}}(P_n)$ grow to infinity as n grows, despite $\text{rank}(S_{P_n}) = 3$.

Open questions:

- Separation between rank_{psd} and rank_+ for polytopes?
- True PSD bound $\sqrt{\log(m)}$?

Upper Bounds for polytopes

Theorem (Fiorini-Rothvoss-Tiwary 2011)

Let P be a generic polytope with m vertices, then $\text{rank}_+(P) \geq \sqrt{2m}$

Upper Bounds for polytopes

Theorem (G.-Robinson-Thomas 2012*)

Let P be a generic polytope with m vertices, then $\text{rank}_{\text{psd}}(P) \geq \sqrt[4]{m}$

Upper Bounds for polytopes

Theorem (G.-Robinson-Thomas 2012*)

Let P be a generic polytope with m vertices, then $\text{rank}_{\text{psd}}(P) \geq \sqrt[4]{m}$

But how bad does it really get? Not much is known.

Upper Bounds for polytopes

Theorem (G.-Robinson-Thomas 2012*)

Let P be a generic polytope with m vertices, then $\text{rank}_{\text{psd}}(P) \geq \sqrt[4]{m}$

But how bad does it really get? Not much is known.

Proposition (Shitov 2013)

All heptagons have nonnegative rank at most 6, hence any m-gon has rank at most $6\lceil \frac{m}{7} \rceil$.

Upper Bounds for polytopes

Theorem (G.-Robinson-Thomas 2012*)

Let P be a generic polytope with m vertices, then $\text{rank}_{\text{psd}}(P) \geq \sqrt[4]{m}$

But how bad does it really get? Not much is known.

Proposition (G.-Robinson-Thomas 2012*)

All hexagons have psd rank 4, hence any m-gon has rank at most $4\lceil \frac{m}{6} \rceil$.

Upper Bounds for polytopes

Theorem (G.-Robinson-Thomas 2012*)

Let P be a generic polytope with m vertices, then
$\text{rank}_{\text{psd}}(P) \geq \sqrt[4]{m}$

But how bad does it really get? Not much is known.

Proposition (G.-Robinson-Thomas 2012*)

All hexagons have psd rank 4, hence any m-gon has rank at most $4\lceil \frac{m}{6} \rceil$.

However, not even the heptagons are totally understood.

Polytopes with minimal representations

Lemma

A polytope of dimension d does not have a semidefinite representation of size smaller than $d + 1$.

Polytopes with minimal representations

Lemma

A polytope of dimension d does not have a semidefinite representation of size smaller than $d+1$.

Using the Hadamard rank we recover an older result.

Theorem (G.-Parrilo-Thomas 2009)

Let P be a polytope with dimension d whose slack matrix S_P is $0/1$. Then P has a semidefinite representation of size $d+1$.

Polytopes with minimal representations

Lemma

A polytope of dimension d does not have a semidefinite representation of size smaller than $d+1$.

Using the Hadamard rank we recover an older result.

Theorem (G.-Parrilo-Thomas 2009)

Let P be a polytope with dimension d whose slack matrix S_P is $0/1$. Then P has a semidefinite representation of size $d+1$.

But we can say much more.

Theorem (G.-Robinson-Thomas 2012)

Let P have dimension d. Then

$$\text{rank}_{\text{psd}}(P) = d+1 \Leftrightarrow \text{rank}_H(S_P) = d+1.$$

SDP-minimal Polytopes

We will say a dimension d polytope P is SDP-minimal if it has a semidefinite representation of size $d + 1$.

SDP-minimal Polytopes

We will say a dimension d polytope P is SDP-minimal if it has a semidefinite representation of size $d + 1$.

On the plane the characterization is easy.

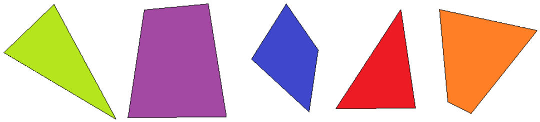
SDP-minimal Polytopes

We will say a dimension d polytope P is SDP-minimal if it has a semidefinite representation of size $d + 1$.

On the plane the characterization is easy.

Proposition

A convex polygon is SDP-minimal if and only if it is a triangle or a quadrilateral.



Octahedra

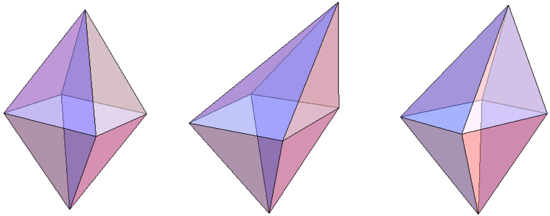
Proposition

If P is combinatorially equivalent to an octahedron then it is SDP-minimal if and only if there are two distinct sets of four coplanar vertices of P.

Octahedra

Proposition

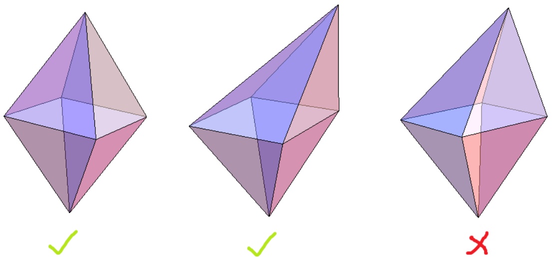
If P is combinatorially equivalent to an octahedron then it is SDP-minimal if and only if there are two distinct sets of four coplanar vertices of P.



Octahedra

Proposition

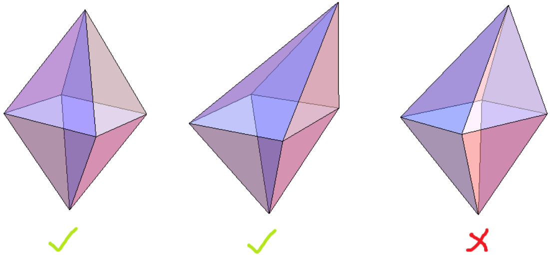
If P is combinatorially equivalent to an octahedron then it is SDP-minimal if and only if there are two distinct sets of four coplanar vertices of P.



Octahedra

Proposition

If P is combinatorially equivalent to an octahedron then it is SDP-minimal if and only if there are two distinct sets of four coplanar vertices of P.



This translates to a dual result on cuboids.

SDP-minimal Polyhedra

Proposition (G.-Robinson-Thomas 2012*)

A polyhedron is SDP-minimal if and only if it is one of the following:

- a simplex;
- a triangular bi-pyramid;
- a triangular prism;
- an octahedron or a cuboid as in the previous proposition.

SDP-minimal Polyhedra

Proposition (G.-Robinson-Thomas 2012*)

A polyhedron is SDP-minimal if and only if it is one of the following:

- a simplex;
- a triangular bi-pyramid;
- a triangular prism;
- an octahedron or a cuboid as in the previous proposition.

Higher dimensions are completely open.

Conclusion

PSD Factorization/rank is an exciting area of research with many recent breakthroughs and many open questions.

Conclusion

PSD Factorization/rank is an exciting area of research with many recent breakthroughs and many open questions.

To read more on this:

Polytopes of Minimum Positive Semidefinite Rank - Gouveia, Robinson and Thomas - arXiv:1205.5306

Lifts of convex sets and cone factorizations - Gouveia, Parrilo and Thomas - arXiv:1111.3164

Thank you