Local analysis of the feasible primal-dual interior-point method

R. Silva · J. Soares · L.N. Vicente

Abstract In this paper we analyze the rate of local convergence of the Newton primal-dual interior-point method when the iterates are kept strictly feasible with respect to the inequality constraints.

It is shown under the classical conditions that the rate is q-quadratic when the functions associated to the binding inequality constraints are concave. In general, the q-quadratic rate is achieved provided the step in the primal variables does not become asymptotically orthogonal to any of the gradients of the binding inequality constraints.

Some preliminary numerical experience showed that the feasible method can be implemented in a relatively efficient way, requiring a reduced number of function and derivative evaluations. Moreover, the feasible method is competitive with the classical infeasible primal-dual interior-point method in terms of number of iterations and robustness.

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1 Introduction

The local convergence theory of (infeasible) primal-dual interior-point methods for nonlinear programming was developed in the papers by El-Bakry et al. [5] and Yamashita and Yabe [14]. These papers show a q-quadratic rate of local convergence under the classical assumptions (second order sufficient optimality conditions, linear independence of the gradients of functions defining the binding constraints (LICQ), and strict complementarity). The study of q-superlinear convergence for quasi-Newton updates is reported in [10] and [14]. Furthermore, Vicente and Wright [13] proved a q-quadratic rate of convergence for a variant of the primal-dual interior-point method under degeneracy (replacing the LICQ by the Mangasarian–Fromowitz constraint qualification). In these approaches, the corresponding primal-dual interior-point method deals with the multipliers associated to both equality and inequality constraints as independent variables, and the primal-dual step is a Newton step for a perturbation of the first order necessary conditions for optimality. These approaches are infeasible since feasibility, corresponding to equality and, more importantly, to inequality constraints (rather than simple bounds), is only achieved asymptotically. Other rates of convergence for different interior-point methods for nonlinear programming have been established in [2, 3, 9], and [12].

Gould, Orban, Sartenaer, and Toint [7] investigated the rate of convergence of primal-dual logarithmic barrier interior-point methods for linear equality constraints and general inequalities. The log-barrier approach maintains the iterates strictly feasible with respect to the inequality constraints, and the multipliers corresponding to the equalities are treated implicitly as dependent variables. The authors proved q-superlinear convergence, with a rate that may be chosen arbitrarily close to quadratic. Basically, they studied conditions under which a single primal-dual Newton step is strictly feasible and satisfies appropriate log-barrier subproblem termination criteria.

The feasible primal-dual interior-point method of Tits et al. [11] achieves a quadratic rate of local convergence. In this algorithm, the multipliers corresponding to the inequality constraints are updated according to an appropriate formula and do not result directly from the Newton related primal-dual step on the perturbed KKT system of first-order optimality conditions. As we explain below, we are interested in analyzing the local convergence of a feasible primal-dual interior-point method without any special provision or correction formula for these multipliers.

In this paper we analyze the rate of local convergence of the feasible primal-dual interior-point method along the lines of the analyses in [5] and [14]. The aspect considered is that inequality constraints are not converted into equalities using slack variables. The method keeps strict feasibility with respect to the inequality constraints. The other components of the primal-dual interior-point method remain essentially the same: the primal-dual step is a Newton step on the perturbed KKT system and the various parameters are updated appropriately to induce a q-quadratic rate on the sequence of primal-dual iterates.

The material of this paper is organized in the following way. In Sect. 2, we describe the feasible primal-dual interior-point method in detail. The method is analyzed in Sect. 3, where it is shown that the iterates converge locally with a q-quadratic rate in

the case of concave inequalities. The analysis includes the case where the step length is computed inexactly. The nonconcave case is discussed in Sect. 4. The rate remains q-quadratic for nonconcave inequalities as long as the primal component of the step is asymptotically nonorthogonal to the gradients of the (nonconcave) functions defining the binding inequalities. In Sect. 5 we report some numerical results which show that the feasible method is competitive with the infeasible method in terms of number of iterations. We present a scheme to calculate the step length for the feasible method which requires a moderate number of constraint function evaluations. The paper is concluded in Sect. 6 with remarks about the theoretical interest and computational limitation of the analyzed approach.

2 The feasible primal-dual interior-point method

We consider the general nonlinear programming problem written in the form

$$
\begin{aligned}
\min \quad & f(x), \\
\text{s.t.} \quad & h(x) = 0, \\
& g(x) \leq 0,
\end{aligned}
\tag{1}
$$

where $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, $h : \mathbb{R}^n \longrightarrow \mathbb{R}^{m_h}$, and $g : \mathbb{R}^n \longrightarrow \mathbb{R}^{m_g}$. The assumptions on the differentiability of the functions f, g, and h will be stated later. The numbers m_h and m_g are assumed to be positive integers. The material of this paper remains valid in the case where there are no equality constraints ($m_h = 0$).

The Lagrangean function for problem (1) is $\ell : \mathbb{R}^{n+m_h+m_g} \longrightarrow \mathbb{R}$ defined by

$$
\ell(x, y, z) = f(x) + h(x)^\top y + g(x)^\top z,
$$

where x are the primal variables and the pair (y, z) represents the dual variables (or Lagrange multipliers). The gradient and the Hessian of ℓ with respect to the primal variables are given by

$$
\nabla_x \ell(x, y, z) = \nabla f(x) + \nabla h(x) y + \nabla g(x) z,
$$

$$
\nabla_{xx}^2 \ell(x, y, z) = \nabla^2 f(x) + \sum_{j=1}^{m_h} y_j \nabla^2 h_j(x) + \sum_{j=1}^{m_g} z_j \nabla^2 g_j(x),
$$

whenever f, g, and h are twice continuously differentiable at x.

The Karush–Kuhn–Tucker (KKT) first order (necessary optimality) conditions for problem (1) are described by

$$
F_0(x, y, z) \overset{\text{def}}{=} \begin{pmatrix} \nabla_x \ell(x, y, z) \\ h(x) \\ -G(x)z \end{pmatrix} = 0,
\tag{2}
$$

$$
g(x) \leq 0, \quad z \geq 0,
$$

where $G(x) = \text{diag}(g(x))$. As we will see later, the primal-dual interior-point method is based on a perturbation of the conditions (2), given by

$$F_\mu(x, y, z) \overset{\text{def}}{=} \begin{pmatrix} \nabla_x \ell(x, y, z) \\ h(x) \\ -G(x)z - \mu e \end{pmatrix} = 0,$$

$$g(x) < 0, \quad z > 0,$$

where μ is a positive scalar and e is a vector of ones of dimension m_g. Note that, for $\bar{e} = (0, 0, e^\top)^\top \in \mathbb{R}^{n+m_h+m_g}$,

$$F_\mu(x, y, z) = F_0(x, y, z) - \mu\bar{e}. \tag{3}$$

We will also make use of $\|\bar{e}\| = \|e\|$.

The main part of the iterative step of the primal-dual interior-point method consists of the linearization of the perturbed KKT system. One computes a primal-dual step $\Delta w = (\Delta x, \Delta y, \Delta z)$, by solving the linear system of equations

$$F'_\mu(w)\Delta w = -F_\mu(w), \tag{4}$$

for fixed $w = (x, y, z)$ and $\mu > 0$, where $F'_\mu(w)$ is the Jacobian of $F_\mu(w)$. Notice that, from (3), $F'_\mu(w)$ is also the Jacobian of $F_0(w)$. The primal-dual system (4) can be written by blocks in the form

$$\begin{pmatrix} \nabla^2_{xx}\ell(x, y, z) & \nabla h(x) & \nabla g(x) \\ \nabla h(x)^\top & 0 & 0 \\ -Z\nabla g(x)^\top & 0 & -G(x) \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = - \begin{pmatrix} \nabla_x \ell(x, y, z) \\ h(x) \\ -G(x)z - \mu e \end{pmatrix}, \tag{5}$$

where $Z = \text{diag}(z)$.

Most variants of the primal-dual interior-point method keep positive all the variables subject to nonnegativity constraints. In our case, it means keeping the multipliers z positive. The parameter μ is driven to zero asymptotically. Since we are looking at the feasible variant of the primal-dual interior-point method, we must also keep $g(x)$ negative throughout the iterations. The main steps of this feasible variant are described below in Algorithm 2.1. For the purpose of analyzing local convergence, we do not include any stopping criterion.

Algorithm 2.1 (Feasible primal-dual interior-point method)
Choose an initial point $w_0 = (x_0, y_0, z_0)$ with $g(x_0) < 0$ and $z_0 > 0$.
For $k = 0, 1, 2, \ldots$

Step 1. Choose the parameter $\mu_k > 0$.
Step 2. Compute the solution $\Delta w_k = (\Delta x_k, \Delta y_k, \Delta z_k)$ of the system (5), for $x = x_k$, $y = y_k$, and $z = z_k$.
Step 3. Compute a positive step length α_k such that

$$g(x_k + \alpha_k \Delta x_k) < 0 \quad \text{and} \quad z_k + \alpha_k \Delta z_k > 0. \tag{6}$$

Step 4. Define the next iterate $w_{k+1} = (x_{k+1}, y_{k+1}, z_{k+1})$ according to:

$$w_{k+1} = w_k + \alpha_k \Delta w_k. \tag{7}$$

Since the step size α_k must satisfy (6) throughout the iterations, we will impose that

$$\alpha_k = \min\left\{1, \tau_k \min_{i=1,\ldots,m_g}\left\{-\frac{(z_k)_i}{(\Delta z_k)_i} : (\Delta z_k)_i < 0\right\}, \tau_k \min_{i=1,\ldots,m_g} \bar{\alpha}_k^i\right\}, \tag{8}$$

where $\tau_k \in (0, 1)$ and

$$\bar{\alpha}_k^i \leq \min\{\alpha : g_i(x_k + \alpha \Delta x_k) = 0, \alpha > 0\}, \quad i = 1, \ldots, m_g. \tag{9}$$

Whenever the minimum is not achieved, it is assumed by convention that it is set to $+\infty$.

We point out first that when the inequality constraints are of the simple bound type $(-x \leq 0)$, the choice for α_k is of the type given above with the inequalities in (9) satisfied as equalities. In general, when the functions defining the inequality constraints are nonlinear, it might be computationally expensive to actually determine the step lengths $\bar{\alpha}_k^i$ such that

$$\bar{\alpha}_k^i = \min\{\alpha : g_i(x_k + \alpha \Delta x_k) = 0, \alpha > 0\}, \quad i = 1, \ldots, m_g. \tag{10}$$

On the other hand, to get a fast rate of local convergence one cannot compute step lengths $\bar{\alpha}_k^i$ that differ too much from (10). However, it is possible to allow a certain inexactness in this computation. Let us define the residuals

$$r_k^i \stackrel{\text{def}}{=} g_i(x_k + \bar{\alpha}_k^i \Delta x_k), \quad i = 1, \ldots, m_g.$$

We will show that the feasible primal-dual interior-point method will retain local q-quadratic convergence as long as the residuals r_k^i satisfy the condition

$$-r_k^i \leq \min\left\{\sigma(-g_i(x_k)), \frac{-g_i(x_k)c_1\|\Delta w_k\|}{1 + c_1\|\Delta w_k\|}\right\}, \quad i = 1, \ldots, m_g, \tag{11}$$

where $\sigma \in (0, 1)$ and $c_1 > 0$ are chosen independently of the iteration counter k. In Sect. 5.1 we will describe a scheme to compute $\bar{\alpha}_k^i$ that seems to be relatively efficient in practice.

Moreover, to achieve a q-quadratic rate of local convergence, the feasible primal-dual interior-point method must update the parameters $\tau_k \in (0, 1)$ and $\mu_k > 0$ satisfying the classical conditions

$$1 - \tau_k \leq c_2\|F_0(w_k)\|, \tag{12}$$

$$\mu_k \leq c_3\|F_0(w_k)\|^2, \tag{13}$$

where c_2 and c_3 are constants independent of k. Vector and matrix norms in this paper are chosen to be the Euclidean ones.

3 Analysis of local convergence

The local convergence of the feasible primal-dual interior-point method is analyzed at a point x_* satisfying the following assumptions. In what follows, $B(a_*; r)$ denotes the open ball $\{a \in \mathbb{R}^d : \|a - a_*\| < r\}$ of radius r centered at a_*.

(A1) There exists an $\epsilon > 0$ such that the functions f, g, and h are twice continuously differentiable in the ball $B(x_*; \epsilon)$. Moreover, the second order partial derivatives of f, g, and h are Lipschitz continuous in $B(x_*; \epsilon)$.

(A2) The point x_* is feasible and the gradients of the active constraints are linearly independent at x_*.

(A3) There exist Lagrange multipliers y_* and z_* such that $w_* = (x_*, y_*, z_*)$ satisfies the first order KKT conditions and the second order sufficient conditions and such that the pair $(-g(x_*), z_*)$ satisfies the strict complementarity condition $(-g(x_*) + z_* > 0)$.

Assumptions A1–A3 are the classical (nondegenerate) assumptions used to locally analyze interior-point methods. It results from Assumption A3 that the multipliers associated with the inequalities are nonnegative ($z_* \geq 0$) and also that

$$F_0(w_*) = 0. \tag{14}$$

We recall now the basic smoothness results that are required in the proof of the local convergence of the primal-dual interior-point method.

Lemma 3.1 *Let x_* be a point for which Assumptions* A1–A3 *hold and $w_* = (x_*, y_*, z_*)$. Then, there exists a positive constant γ such that*

$$\|F_0(w^1) - F_0(w^2)\| \leq \gamma \|w^1 - w^2\|, \tag{15}$$

$$\|F_0'(w^1) - F_0'(w^2)\| \leq \gamma \|w^1 - w^2\|,$$

$$\|F_0(w^1) - F_0(w^2) - F_0'(w^2)(w^1 - w^2)\| \leq \frac{1}{2}\gamma \|w^1 - w^2\|^2, \tag{16}$$

for all w^1 and w^2 in $B(w_*; \epsilon)$.

The next lemma states that the primal-dual matrix is nonsingular around w_*, in the sense that is of interest to us. For a proof see, for instance, [6].

Lemma 3.2 *Let x_* be a point for which Assumptions* A1–A3 *hold and $w_* = (x_*, y_*, z_*)$. Then the following holds*:

(i) $F_0'(w_*)$ *is nonsingular*;

(ii) $F_0'(w)$ *is nonsingular for w in $B(w_*; \epsilon_{ns})$, for some ϵ_{ns} satisfying $0 < \epsilon_{ns} < \epsilon$.*

From this lemma, it is assured the existence of a constant $\zeta > 0$ such that

$$\|F_0'(w)^{-1}\| = \|F_\mu'(w)^{-1}\| \leq \zeta, \tag{17}$$

for all w in $B(w_*; \epsilon_{ns})$. For such points w, the primal-dual step Δw given by the solution of the system (4) is well-defined and is equal to

$$\Delta w = -F'_\mu(w)^{-1} F_\mu(w). \tag{18}$$

The local asymptotic behavior of the feasible primal-dual interior-point method is studied first for concave binding inequalities.

(A4) The functions g_i, for $i \in \{1, \ldots, m_g\}$ such that $g_i(x_*) = 0$, are concave.

The main part of the analysis is spent proving a lower bound for the length of the step size parameter α_k.

Lemma 3.3 *Let x_* be a point for which Assumptions A1–A4 hold and $w_* = (x_*, y_*, z_*)$. Consider a sequence $\{w_k = (x_k, y_k, z_k)\}$ generated by the feasible primal-dual interior-point method described in Algorithm* 2.1. *If α_k satisfies (8–9) and (11) and $\tau_k \in (0, 1)$ and $\mu_k > 0$ satisfy (12) and (13), then there exist positive constants ε and κ independent of k such that, when*

$$w_k \in B(w_*; \varepsilon), \tag{19}$$

either $\alpha_k = 1$ or the bound

$$1 - \alpha_k \le (1 - \tau_k) + \kappa \zeta(\|F_0(w_k)\| + \mu_k \|e\|), \tag{20}$$

holds for all iterates k.

Proof First we have to set $\varepsilon = \epsilon_{ns}$, where ϵ_{ns} is given in Lemma 3.2.

Using (18), (3), (14), (17), (15), and (19) sequentially, it is easily derived the following bound for the primal-dual step:

$$\begin{aligned}
\|\Delta w_k\| &= \|F'_\mu(w_k)^{-1} F_\mu(w_k)\| \\
&\le \|F'_\mu(w_k)^{-1}\|(\|F_0(w_k)\| + \mu_k \|e\|) \\
&\le \zeta(\gamma \|w_k - w^*\| + \mu_k \|e\|) \\
&\le \zeta(\gamma \varepsilon + \mu_k \|e\|).
\end{aligned}$$

Thus, from the condition (13) on the size of μ_k, and given a constant $\eta > 0$, one can reduce ε if necessary such that

$$\|\Delta w_k\| \le \eta. \tag{21}$$

In particular, it is possible to choose a sufficiently small ε such that

$$\kappa \|\Delta w_k\| \le \tau_k, \tag{22}$$

where κ is defined by

$$\kappa \overset{\text{def}}{=} \max\left\{ \frac{\kappa_2}{1 - \sigma}, \kappa_1 + \kappa_1 c_1 \eta + c_1 \right\}.$$

The constants κ_1 and κ_2 are given by

$$\kappa_1 = 2 \max\left\{\frac{1}{(z_*)_i} : (z_*)_i > 0, i \in \{1, \ldots, m_g\}\right\}$$

and

$$\kappa_2 = 2M_{\nabla g} \max\left\{-\frac{1}{g_i(x_*)} : g_i(x_*) < 0, i \in \{1, \ldots, m_g\}\right\},$$

where $M_{\nabla g}$ is an upper bound on the size of ∇g in $B(x_*; \epsilon)$.

We divide the proof in two separate cases: the case where the step length is defined by a multiplier and the case where the step length is defined by an inequality.

Case where step length is defined by a multiplier. In this first case we assume that there exists an index $i \in \{1, \ldots, m_g\}$ for which $(\Delta z_k)_i < 0$ and

$$\alpha_k = -\tau_k \frac{(z_k)_i}{(\Delta z_k)_i}.$$

If i is such that $(z_*)_i > 0$ then, from the definition of κ and from (22),

$$\alpha_k = \tau_k \frac{(z_k)_i}{-(\Delta z_k)_i} \geq \frac{\tau_k}{\kappa\|\Delta w_k\|} \geq 1.$$

When $(z_*)_i = 0$ (and $g_i(x_*) < 0$), we make use of the primal-dual block equation (see (5))

$$-Z_k \nabla g(x_k)^\top \Delta x_k - G(x_k)\Delta z_k = G(x_k)z_k + \mu_k e,$$

to write

$$-(z_k)_i \nabla g_i(x_k)^\top \Delta x_k - g_i(x_k)(\Delta z_k)_i = g_i(x_k)(z_k)_i + \mu_k,$$

or equivalently,

$$-\frac{(\Delta z_k)_i}{(z_k)_i} = 1 + \frac{\mu_k}{g_i(x_k)(z_k)_i} + p_k^i$$

with

$$p_k^i = \frac{\nabla g_i(x_k)^\top \Delta x_k}{g_i(x_k)} \leq \frac{|\nabla g_i(x_k)^\top \Delta x_k|}{-g_i(x_k)} \leq \kappa\|\Delta w_k\|.$$

Thus, since $\mu_k/(g_i(x_k)(z_k)_i) < 0$,

$$-\frac{(\Delta z_k)_i}{(z_k)_i} \leq 1 + \kappa\|\Delta w_k\|$$

and

$$\alpha_k = \tau_k \frac{(z_k)_i}{-(\Delta z_k)_i} \geq \frac{\tau_k}{1 + \kappa\|\Delta w_k\|} \geq \tau_k(1 - \kappa\|\Delta w_k\|).$$

Case where step length is defined by an inequality. Now we are interested in the case

$$\alpha_k = \tau_k \bar{\alpha}_k^i,$$

for some index $i \in \{1, \ldots, m_g\}$. By applying the mean value theorem, we have

$$r_k^i - g_i(x_k) = g_i(x_k + \bar{\alpha}_k^i \Delta x_k) - g_i(x_k) = \bar{\alpha}_k^i \nabla g_i(x_k + t_k^i \bar{\alpha}_k^i \Delta x_k)^\top \Delta x_k,$$

for some $t_k^i \in (0, 1)$, and the step length $\bar{\alpha}_k^i$ can be written as

$$\bar{\alpha}_k^i = \frac{r_k^i - g_i(x_k)}{\nabla g_i(x_k + t_k^i \bar{\alpha}_k^i \Delta x_k)^\top \Delta x_k}. \tag{23}$$

Since $-r_k^i \le \sigma(-g_i(x_k))$, both the numerator and the denominator in this expression for $\bar{\alpha}_k^i$ are positive.

If i is such that $g_i(x_*) < 0$ then, from the definitions of κ_2 and κ and from (22),

$$
\begin{aligned}
\alpha_k = \tau_k \bar{\alpha}_k^i &\ge \tau_k \frac{(1-\sigma)(-g_i(x_k))}{\nabla g_i(x_k + t_k^i \bar{\alpha}_k^i \Delta x_k)^\top \Delta x_k} \\
&\ge \tau_k \frac{(1-\sigma)(-g_i(x_k))}{\|\nabla g_i(x_k + t_k^i \bar{\alpha}_k^i \Delta x_k)\| \|\Delta x_k\|} \\
&\ge \tau_k \frac{(1-\sigma)}{\kappa_2 \|\Delta w_k\|} \\
&\ge \frac{\tau_k}{\kappa \|\Delta w_k\|} \\
&\ge 1.
\end{aligned}
$$

When $g_i(x_*) = 0$ (and $(z_*)_i > 0$), we must first add and subtract

$$r_k^i \frac{(\Delta z_k)_i}{(z_k)_i} + r_k^i + r_k^i \frac{\mu_k}{g_i(x_k)(z_k)_i}$$

to the right hand side in the primal-dual equation

$$-\nabla g_i(x_k)^\top \Delta x_k = g_i(x_k) \frac{(\Delta z_k)_i}{(z_k)_i} + g_i(x_k) + g_i(x_k) \frac{\mu_k}{g_i(x_k)(z_k)_i}.$$

After division by $g_i(x_k) - r_k^i$, this results in

$$-\frac{\nabla g_i(x_k)^\top \Delta x_k}{g_i(x_k) - r_k^i} = \frac{(\Delta z_k)_i}{(z_k)_i} + 1 + \frac{\mu_k}{g_i(x_k)(z_k)_i}$$

$$+ \frac{r_k^i}{g_i(x_k) - r_k^i} \frac{(\Delta z_k)_i}{(z_k)_i} + \frac{r_k^i}{g_i(x_k) - r_k^i} + \frac{\mu_k r_k^i}{g_i(x_k)(z_k)_i(g_i(x_k) - r_k^i)}.$$

Since the third and the sixth terms in the right hand side of this equality are negative and since, from (11),

$$\frac{r_k^i}{g_i(x_k) - r_k^i} \le c_1 \|\Delta w_k\|,$$

we obtain, from (21),

$$-\frac{\nabla g_i(x_k)^\top \Delta x_k}{g_i(x_k) - r_k^i} \leq 1 + \kappa_1 \|\Delta w_k\| + \kappa_1 c_1 \|\Delta w_k\|^2 + c_1 \|\Delta w_k\|$$

$$\leq 1 + (\kappa_1 + \kappa_1 c_1 \eta + c_1) \|\Delta w_k\|$$

$$\leq 1 + \kappa \|\Delta w_k\|. \tag{24}$$

Now, from the concavity of g_i, we derive

$$-\frac{\nabla g_i(x_k + t_k^i \bar{\alpha}_k^i \Delta x_k)^\top \Delta x_k}{g_i(x_k) - r_k^i}$$

$$= -\frac{\nabla g_i(x_k + t_k^i \bar{\alpha}_i \Delta x_k)^\top \Delta x_k}{g_i(x_k) - r_k^i} + \frac{\nabla g_i(x_k)^\top \Delta x_k}{g_i(x_k) - r_k^i} - \frac{\nabla g_i(x_k)^\top \Delta x_k}{g_i(x_k) - r_k^i}$$

$$= \frac{[\nabla g_i(x_k + t_k^i \bar{\alpha}_k^i \Delta x_k) - \nabla g_i(x_k)]^\top \Delta x_k}{r_k^i - g_i(x_k)} - \frac{\nabla g_i(x_k)^\top \Delta x_k}{g_i(x_k) - r_k^i}$$

$$\leq -\frac{\nabla g_i(x_k)^\top \Delta x_k}{g_i(x_k) - r_k^i}$$

$$\leq 1 + \kappa \|\Delta w_k\| \tag{25}$$

and

$$\alpha_k = \tau_k \bar{\alpha}_k^i = \tau_k \frac{r_k^i - g_i(x_k)}{\nabla g_i(x_k + t_k^i \bar{\alpha}_k^i \Delta x_k)^\top \Delta x_k} \geq \frac{\tau_k}{1 + \kappa \|\Delta w_k\|} \geq \tau_k (1 - \kappa \|\Delta w_k\|).$$

Conclusion. Combining all the four bounds derived for α_k (two in each case considered), one obtains

$$\alpha_k \geq \min\{1, \tau_k(1 - \kappa \|\Delta w_k\|)\} = \tau_k(1 - \kappa \|\Delta w_k\|) \geq \tau_k - \kappa \|\Delta w_k\|.$$

The last inequality above is based on the fact that $\tau_k < 1$ for all k provided ε is chosen small enough. Finally, from this lower bound on α_k, we get

$$0 \leq 1 - \alpha_k \leq (1 - \tau_k) + \kappa \|\Delta w_k\| \leq (1 - \tau_k) + \kappa \zeta(\|F_0(w_k)\| + \mu_k \|e\|),$$

which concludes the proof of the lemma.   □

We can state now the q-quadratic rate of local convergence of Algorithm 2.1. The proof can be found in [14] and we describe it briefly for completeness.

Theorem 3.1 *Let x_* be a point for which Assumptions A1–A4 hold and $w_* = (x_*, y_*, z_*)$. Consider a sequence $\{w_k = (x_k, y_k, z_k)\}$ generated by the feasible primal-dual interior-point method described in Algorithm 2.1. If α_k satisfies (8–9)*

and (11) *and* $\tau_k \in (0, 1)$ *and* $\mu_k > 0$ *satisfy* (12) *and* (13), *then there exists a positive constant* ε *independent of* k *such that, when*

$$w_0 \in B(w_*; \varepsilon),$$

the sequence $\{w_k\}$ *is well defined and converges to* w_*. *Moreover, we have*

$$\|w_{k+1} - w_*\| \le \nu \|w_k - w_*\|^2, \tag{26}$$

for all iterates k, *where* ν *is a positive constant independent of* k.

Proof Let us assume that $\|w_k - w_*\| < \varepsilon$. By applying (7), (18), (3), and (14), we obtain

$$
\begin{aligned}
w_{k+1} - w_* &= (w_k - w_*) + \alpha_k \Delta w_k \\
&= (1 - \alpha_k)(w_k - w_*) \\
&\quad + \alpha_k F'_{\mu_k}(w_k)^{-1}[F_0(w_*) - F_0(w_k) - F'_{\mu_k}(w_k)(w_* - w_k) + \mu_k \bar{e}].
\end{aligned}
$$

Now, using (20), (16), (17), and $\alpha_k \le 1$, we have, for a sufficiently small ε,

$$
\begin{aligned}
\|w_{k+1} - w_*\| &\le (1 - \alpha_k)\|w_k - w_*\| \\
&\quad + \alpha_k \|F'_{\mu_k}(w_k)^{-1}\| \|F_0(w_*) - F_0(w_k) - F'_{\mu_k}(w_k)(w_* - w_k)\| \\
&\quad + \alpha_k \mu_k \|F'_{\mu_k}(w_k)^{-1}\| \|e\| \\
&\le [(1 - \tau_k) + \kappa\zeta\|F_0(w_k)\| + \kappa\zeta\|e\|\mu_k]\|w_k - w_*\| \\
&\quad + \frac{\zeta\gamma}{2}\|w_k - w_*\|^2 + \|e\|\zeta\mu_k.
\end{aligned}
$$

We also known that τ_k and μ_k satisfy (12) and (13). Thus, using the fact that $\|F_0(w_k)\| = \|F_0(w_k) - F_0(w_*)\| \le \gamma\|w_k - w_*\|$, we assure the existence of a constant $\nu > 0$ independent of the iterates such that (26) holds. It is possible to prove by induction that $\{w_k\}$ converges to w_*, if ε is chosen sufficiently small. The inequality (26) shows that the local convergence rate is q-quadratic. $\qquad \square$

4 The nonconcave case

The concavity of the inequality constraint functions was required in (25) when the binding constraint function g_i was responsible for the step size α_k. However, one can see that the method retains a q-quadratic rate in the nonconcave case as long as there exists a positive constant β such that

$$r_k^i - g_i(x_k) = g_i(x_k + \bar{\alpha}_k^i \Delta x_k) - g_i(x_k) \ge \beta \bar{\alpha}_k^i \|\Delta x_k\| \tag{27}$$

for all k and all indices i corresponding to $g_i(x_*) = 0$. In fact, one would get

$$\frac{[\nabla g_i(x_k + t_k^i \bar{\alpha}_k^i \Delta x_k) - \nabla g_i(x_k)]^\top \Delta x_k}{r_k^i - g_i(x_k)} \le \frac{L_{\nabla g_i}}{\beta}\|\Delta x_k\|,$$

where $L_{\nabla g_i}$ is the Lipschitz constant of the function ∇g_i in $B(x_*; \epsilon)$. Then, from (23) and (24),

$$\frac{1}{\bar{\alpha}_k^i} = -\frac{\nabla g_i(x_k + t_k^i \bar{\alpha}_k^i \Delta x_k)^\top \Delta x_k}{g_i(x_k) - r_k^i} \leq 1 + [(\kappa_1 + \kappa_1 c_1 \eta + c_1) + L_{\nabla g_i}/\beta)] \|\Delta w_k\|$$

$$\leq 1 + \kappa \|\Delta w_k\|,$$

after an appropriate redefinition of κ.

The bound (27) is satisfied for k sufficiently large as long as

$$\liminf_{k \longrightarrow +\infty} \nabla g_i(x_k)^\top \frac{\Delta x_k}{\|\Delta x_k\|} = 4\beta > 0. \tag{28}$$

To see why this is true let us expand $g_i(x_k + \bar{\alpha}_k^i \Delta x_k)$ around x_k:

$$g_i(x_k + \bar{\alpha}_k^i \Delta x_k) - g_i(x_k) = \bar{\alpha}_k^i \nabla g_i(x_k)^\top \Delta x_k + \frac{(\bar{\alpha}_k^i)^2}{2} \Delta x_k^\top \nabla^2 g_i(x_k + s_k^i \bar{\alpha}_k^i \Delta x_k) \Delta x_k,$$

for some $s_k^i \in (0, 1)$. Since we are only looking at the cases where $\bar{\alpha}_k^i \leq 1$, one can see that

$$\frac{g_i(x_k + \bar{\alpha}_k^i \Delta x_k) - g_i(x_k)}{\bar{\alpha}_k^i \|\Delta x_k\|} \geq \beta$$

holds for k sufficiently large as long as

$$\|\Delta x_k\| \leq \|\Delta w_k\| \leq \frac{2\beta}{M_{\nabla^2 g_i}},$$

where $M_{\nabla^2 g_i}$ is an upper bound on the size of the Hessian $\nabla^2 g_i$ in $B(x_*; \epsilon)$, requiring again a redefinition of κ.

Condition (28) has no influence if the constraint is concave because, even when this condition is not satisfied, the concavity of the function defining the constraint allows a locally full step ($\alpha_k = 1$) with respect to that constraint.

5 Implementation and numerical results

We have implemented Algorithm 2.1 (the feasible primal-dual interior point method) and tested it on a set of small CUTEr problems [8]. Subsection 5.1 completes the algorithm description, namely, we explain there how to compute, without too much effort, a stepsize that satisfies condition (11). As a baseline for comparison, we have also tested the infeasible primal-dual interior point method (details are explained in Subsect. 5.2). Numerical results are reported in Subsect. 5.3.

5.1 Step size calculation for the feasible method

To achieve a q-quadratic rate of convergence, condition (11) is imposed on the size of the residual $r_k^i = g_i(x_k + \bar{\alpha}_k^i \Delta x_k)$. A simple procedure to calculate $\bar{\alpha}_k^i$ is Newton's method, fully described in Algorithm 5.1 below. Obviously, the scheme is to be applied only to the nonlinear inequality constraints, as for the linear constraints (including bounds) one can easily determine the exact maximum allowed step length.

Algorithm 5.1 (Step size calculation for the feasible method)
Choose an initial step size $\alpha > 0$ (for instance, $\alpha = 0.5$ if $k = 0$ and $\alpha = \bar{\alpha}_{k-1}^i$ for $k \geq 1$). Choose $\sigma \in (0, 1)$ and $c_1 > 0$ (for instance, $\sigma = 10^{-2}$ and $c_1 = 0.5$). Set also $\Delta\alpha = 1$.
While $|\Delta\alpha| > 10^{-3}$ and $-g_i(x_k + \alpha \Delta x_k) > \min\{\sigma(-g_i(x_k)), \frac{-g_i(x_k)c_1\|\Delta w_k\|}{1+c_1\|\Delta w_k\|}\}$ do

$$\text{compute } \Delta\alpha = -\frac{g_i(x_k + \alpha \Delta x_k)}{\nabla g_i(x_k + \alpha \Delta x_k)^\top \Delta x_k} \text{ and replace } \alpha \text{ by } \alpha + \Delta\alpha.$$

After termination set $\bar{\alpha}_k^i = \alpha$.

We incorporated a safeguard for negative values of α. When α became negative we projected it back to the positive axis by setting it to 10^{-2}. This safeguard acted only in two of the problems tested (leading then to a convergent run).

We also experimented an alternative scheme where the calculation of the derivatives $\nabla g_i(x_k + \alpha \Delta x_k)^\top \Delta x_k$ was approximated by $\nabla g_i(x_k)^\top \Delta x_k$. While this alternative seemed not to affect the efficiency of the feasible method (in terms of number of iterations), it provided a less robust method (in the sense that less problems were solved).

5.2 The infeasible primal-dual interior-point method

As a baseline for comparison we also ran the infeasible primal-dual interior-point method reported in Algorithm 5.2. This version of the infeasible method is the same as the feasible method applied to a reformulated problem, where inequality constraints are converted into equalities with the introduction of nonnegative slack variables. The feasible region can thus be written as $\{\hat{x} \in \mathbb{R}^{\hat{n}} : \hat{h}(\hat{x}) = 0, \hat{x} \geq 0\}$ and the Lagrangean function as $\hat{\ell}(\hat{x}, \hat{y}, \hat{z}) = \hat{f}(\hat{x}) + \hat{h}(\hat{x})^\top \hat{y} - \hat{x}^\top \hat{z}$.

Algorithm 5.2 (Infeasible primal-dual interior-point method)
Choose an initial point $w_0 = (\hat{x}_0, \hat{y}_0, \hat{z}_0)$ with $\hat{x}_0 > 0$ and $\hat{z}_0 > 0$.
For $k = 0, 1, 2, \ldots$

Step 1. Choose the parameter $\mu_k > 0$.
Step 2. Compute the solution $\Delta w_k = (\Delta\hat{x}_k, \Delta\hat{y}_k, \Delta\hat{z}_k)$ of the system

$$\begin{pmatrix} \nabla_{\hat{x}\hat{x}}^2 \hat{\ell}(\hat{x}_k, \hat{y}_k, \hat{z}_k) & \nabla\hat{h}(\hat{x}_k) & -I_{\hat{n}\times\hat{n}} \\ \nabla\hat{h}(\hat{x}_k)^\top & 0 & 0 \\ \hat{Z}_k & 0 & \hat{X}_k \end{pmatrix} \begin{pmatrix} \Delta\hat{x}_k \\ \Delta\hat{y}_k \\ \Delta\hat{z}_k \end{pmatrix} = -\begin{pmatrix} \nabla_{\hat{x}}\hat{\ell}(\hat{x}_k, \hat{y}_k, \hat{z}_k) \\ \hat{h}(\hat{x}_k) \\ \hat{X}_k\hat{z}_k - \mu_k e_{\hat{n}} \end{pmatrix},$$

where $\hat{X}_k = \text{diag}(\hat{x}_k)$ and $\hat{Z}_k = \text{diag}(\hat{z}_k)$.

Step 3. Choose $\tau_k \in (0, 1)$. Compute a positive step length α_k such that

$$\alpha_k = \min\left\{1, \tau_k \min_{i=1,\ldots,\hat{n}}\left\{-\frac{(\hat{x}_k)_i}{(\Delta\hat{x}_k)_i} : (\Delta\hat{x}_k)_i < 0\right\},\right.$$
$$\left.\tau_k \min_{i=1,\ldots,\hat{n}}\left\{-\frac{(\hat{z}_k)_i}{(\Delta\hat{z}_k)_i} : (\Delta\hat{z}_k)_i < 0\right\}\right\}.$$

Step 4. Define the next iterate $w_{k+1} = (\hat{x}_{k+1}, \hat{y}_{k+1}, \hat{z}_{k+1})$ according to:

$$w_{k+1} = w_k + \alpha_k \Delta w_k.$$

5.3 Numerical results

We tested both feasible and infeasible primal-dual interior point methods (Algorithms 2.1 and 5.2) on a set of small CUTEr [8] problems. We restricted our attention to problems with nonlinear inequality constraints and for which the initial point x_0, provided by CUTEr, satisfies $g(x_0) < 0$.

The initial primal point given to the infeasible method was the point x_0 provided by CUTEr. When the initial primal values are nonpositive we projected them to the positive axis. In some cases, not all variables are restricted in sign and the infeasible method was adapted to take care of this situation. The remaining components of the initial vector for the infeasible method are defined as follows: $\hat{y}_0 = \text{argmin}_{\hat{y}}\|\nabla\hat{h}(\hat{x}_0)\hat{y} + [-\hat{z}_0^{in} + \nabla\hat{f}(\hat{x}_0)]\|$ and $\hat{z}_0 = \max\{\nabla\hat{h}(\hat{x}_0)\hat{y}_0 + \nabla\hat{f}(\hat{x}_0), \hat{z}_0^{in}\}$, where $\hat{z}_0^{in}$ is a vector of ones of the appropriate size. For the feasible method, x_0 is provided by CUTEr (satisfying $g(x_0) < 0$). Then, we compute $y_0 = \text{argmin}_y\|\nabla h(x_0)y + [\nabla g(x_0)z_0^{in} + \nabla f(x_0)]\|$ and $z_0 = \max\{\text{argmin}_z\|\nabla g(x_0)z + [\nabla h(x_0)y_0 + \nabla f(x_0)]\|, z_0^{in}\}$.

The codes were implemented in Fortran 90 and ran on a Compaq Tru64 (operating system Unix V5.1, 2 GB RAM, Alpha 21264A 667 MHz). For both algorithms, we updated τ_k and μ_k as follows:

$$1 - \tau_k = \min\{10^{-2}, 10^{-2}\|F_0(w_k)\|\} \quad \text{and} \quad \mu_k = \min\{10^{-2}, 10^{-1}\|F_0(w_k)\|^2\}.$$

The stopping criterion was $\|F_0(w_k)\| \leq 10^{-8}$ for the feasible method and $\|(\nabla_{\hat{x}}\hat{\ell}(\hat{x}_k, \hat{y}_k, \hat{z}_k), \hat{h}(\hat{x}_k), \hat{X}_k\hat{z}_k)\| \leq 10^{-8}$ for the infeasible case. The linear algebra was implemented in the dense form using the LAPACK [1] routines DGESV (for the primal-dual systems) and DGELSD (for the least-squares multiplier problems).

The results are reported in Table 1. The legend of the table is as follows: infeasible method (var = number of variables, eq = number of equalities after slacks are incorporated, bd = number of bounds, it = number of iterations, evals = number of function evaluations); feasible method (var = number of variables, eq = number of equalities, linear g = number of linear inequality constraints including bounds, nlinear g = number of nonlinear inequality constraints, it = number of iterations, evals = number of function evaluations). The number of function evaluations counts the associated derivative calls (and accommodates either calls to set the primal-dual data for the systems solves or calls needed to apply Scheme 5.1). When convergence was not

Table 1 Numerical behavior of infeasible and feasible methods on a set of CUTEr problems

Problems	Infeasible method					Feasible method					
	var $\hat{n}$	eq	bd	it	evals	var n	eq	linear g	nlinear g	it	evals
CHACONN1	6	3	3	10	11	3	0	0	3	10	25
ERRINBAR	19	9	15	$\alpha < 10^{-10}$		18	8	14	1	$\alpha < 10^{-10}$	
EXPFITA	27	22	22	$\alpha < 10^{-10}$		5	0	0	22	$\alpha < 10^{-10}$	
EXPFITB	107	102	102	$\alpha < 10^{-10}$		5	0	0	102	$\alpha < 10^{-10}$	
HAIFAS	22	9	9	div		13	0	0	9	div	
HIMMELP2	3	1	5	$\alpha < 10^{-10}$		2	0	4	1	div	
HIMMELP3	4	2	6	$\alpha < 10^{-10}$		2	0	4	2	10	21
HIMMELP4	5	3	7	$\alpha < 10^{-10}$		2	0	4	3	10	21
HS13	3	1	3	$it > 200$		2	0	2	1	$\alpha < 10^{-10}$	
HS16	4	2	5	24	25	2	0	3	2	12	26
HS20	5	3	5	$\alpha < 10^{-10}$		2	0	2	3	12	29
HS24	5	3	5	$\alpha < 10^{-10}$		2	0	2	3	$\alpha < 10^{-10}$	
HS29	4	1	1	12	13	3	0	0	1	15	30
HS30	4	1	7	$\alpha < 10^{-10}$		3	0	6	1	39	78
HS32	4	2	7	14	15	3	1	3	1	$\alpha < 10^{-10}$	
HS33	5	2	6	$it > 200$		3	0	4	2	$\alpha < 10^{-10}$	
HS65	4	1	4	$\alpha < 10^{-10}$		3	0	6	1	$\alpha < 10^{-10}$	
HS67	17	14	20	$\alpha < 10^{-10}$		3	0	6	14	$it > 200$	
HS73	6	3	6	$\alpha < 10^{-10}$		4	1	5	1	$\alpha < 10^{-10}$	
HS84	8	3	16	$\alpha < 10^{-10}$		5	0	10	6	$\alpha < 10^{-10}$	
HS95	10	4	16	$\alpha < 10^{-10}$		6	0	12	4	$\alpha < 10^{-10}$	
HS96	10	4	16	$\alpha < 10^{-10}$		6	0	12	4	$\alpha < 10^{-10}$	
HS109	13	10	20	$\alpha < 10^{-10}$		9	6	18	2	$\alpha < 10^{-10}$	
HS113	18	8	8	$it > 200$		10	0	3	5	$\alpha < 10^{-10}$	
HS117	20	5	20	$it > 200$		15	0	15	5	$\alpha < 10^{-10}$	
HUBFIT	3	1	2	8	9	2	0	1	1	8	16
KSIP	1021	1001	1001	12	13	20	0	0	1001	12	24
LSQFIT	3	1	2	8	9	2	0	1	1	8	16
MATRIX2	8	2	6	15	16	6	0	4	2	$\alpha < 10^{-10}$	
MIFFLIN1	5	2	2	10	11	3	0	0	2	6	20
MINMAXBD	25	20	20	div		5	0	0	20	$\alpha < 10^{-10}$	
NGONE	13	8	14	div		5	0	8	6	$\alpha < 10^{-10}$	
OPTMASS	77	55	11	div		66	44	0	11	7	14
PRODPL0	69	29	69	$\alpha < 10^{-10}$		60	20	60	9	$\alpha < 10^{-10}$	
PRODPL1	69	29	69	div		60	20	65	9	$\alpha < 10^{-10}$	
READING4	4	2	8	$\alpha < 10^{-10}$		2	0	4	4	$\alpha < 10^{-10}$	
ROSENMMX	9	4	4	18	19	5	0	0	4	15	46
SVANBERG	20	10	30	$\alpha < 10^{-10}$		10	0	20	10	19	93
SWOPF	107	102	34	$\alpha < 10^{-10}$		83	78	20	14	$\alpha < 10^{-10}$	
WOMFLET	6	3	3	div		3	0	0	3	$it > 200$	
ZECEVIC2	4	2	6	$\alpha < 10^{-10}$		2	0	4	2	12	29
ZY2	5	2	6	7	8	3	0	4	2	$\alpha < 10^{-10}$	

achieved was either because the maximum number of iterations (200) was achieved, or because the step size became too small ($\alpha < 10^{-10}$), or because the iterates start growing to infinity (div).

Both infeasible and feasible methods converged only for a subset of the problems, given that no globalization strategy was incorporated. In the cases where both converged, the number of iterations was approximately the same. The infeasible method converged for 11 problems whereas the feasible method converged for 15. Actually, this gain in robustness was also observed by looking at the size of the residuals for the problems where both methods did not converge, which showed that the feasible method achieved, in average, smaller residuals.

The procedure to determine the step size for the feasible method (Algorithm 5.1) took a low average number of steps (roughly 2), as we can observe from the difference between the number of function evaluations and the number of iterations taken by this method.

The only problem of the form (1) that satisfies Assumption A4 (concave binding inequalities) is, as far as we could check, problem HS16. The feasible method took much less iterations on this problem than the infeasible method.

6 Concluding remarks

Keeping strict feasibility with respect to the inequality constraints in the way required by the feasible primal-dual interior-point method can be numerically affordable. Although the exact computation of the step sizes requires the solution of a number of nonlinear equations per iteration, it is possible to compute them satisfying inexact requirements in a relatively efficient way.

Our numerical findings show that the proposed method can be particularly efficient when the number of inequality constraints is not very large or when the structure of the functions defining the inequality constraints eases the step size calculation considerably. Strict feasibility can be imposed only partially, with a subset of the problem inequalities being treated by slack variables. The inequalities imposed strictly may be those that restrict the objective function domain, or the domain of some other constraint functions.

The most restrictive aspect of the proposed algorithm seems to be the initial point. However, the issue of finding a primal strictly feasible point ($g(x_0) < 0$) is out of the scope of this paper. See [4], and the references therein, for the determination of feasible points of systems of (nonlinear) equalities and inequalities.

Despite the numerical considerations, looking at the infeasible variant of the primal-dual interior-point method is of interest on itself. It is worth pointing out that the approach presented in this paper covers the infeasible case ([5, 14]) since simple bounds of the type $-x \leq 0$ correspond to concave inequalities. Linear inequality constraints are also concave and can be treated without slack variables for achieving the purpose of fast local convergence. Finally, the observation that the q-quadratic rate is retained in the general nonconcave case provided the angle between the primal step and the gradients of the binding constraints is kept away from ninety degrees, see (28), fits well into the theory of interior-point methods since it corresponds to the notion of centrality.

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