$\{0, \frac{1}{2}\}$-Chvátal-Gomory Cuts

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Abstract

Given the integer polyhedron $P_I := conv\{x \in Z^n : Ax \leq b\}$, where $A \in Z^{m \times n}$ and $b \in Z^m$, a *Chvátal-Gomory* (*CG*) *cut* is a valid inequality for P_I of the type $\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$ for some $\lambda \in R^m_+$ such that $\lambda^T A \in Z^n$. In this paper we study $\{0, \frac{1}{2}\}$-*CG cuts*, arising for $\lambda \in \{0, 1/2\}^m$. We show that the associated separation problem, $\{0, \frac{1}{2}\}$-SEP, is equivalent to finding a minimum-weight member of a binary clutter. This implies that $\{0, \frac{1}{2}\}$-SEP is NP-hard in the general case, but polynomially solvable when A is related to the edge-path incidence matrix of a tree. We show that $\{0, \frac{1}{2}\}$-SEP can be solved in polynomial time for a convenient relaxation of the system $Ax \leq b$. This leads to an efficient separation algorithm for a subclass of $\{0, \frac{1}{2}\}$-CG cuts, which often contains wide families of strong inequalities for P_I. Applications to the Clique Partitioning, Asymmetric Traveling Salesman, Plant Location, Acyclic Subgraph and Linear Ordering polytopes are briefly discussed.

1 Introduction

Given an $m \times n$ integer matrix $A = (a_{ij})$ and an m-dimensional integer vector b, let $P := \{x \in R^n : Ax \leq b\}$, $P_I := \text{conv}\{x \in Z^n : Ax \leq b\}$, and assume $P_I \neq P$. We assume, without loss of generality, that each row of (A, b) contains at least one odd coefficient. A *Chvátal-Gomory* (*CG*) *cut* is a valid inequality for P_I of the form $\lambda^T Ax \leq \lfloor \lambda^T b \rfloor$, where $\lambda \in R^m_+$ is such that $\lambda^T A \in Z^n$, and $\lfloor \cdot \rfloor$ denotes lower integer part. Notice that undominated CG cuts only arise for $\lambda \in [0, 1)^m$.

The *rank-1 closure* of P is defined as $P_1 := \{x \in P : \lambda^T Ax \leq \lfloor \lambda^T b \rfloor$, for $\lambda \in [0, 1)^m$ s.t. $\lambda^T A \in Z^n\}$. We define a $\{0, \frac{1}{2}\}$-*Chvátal-Gomory cut* ($\{0, \frac{1}{2}\}$-*cut*, for short) as a CG cut with $\lambda \in \{0, 1/2\}^m$, and define

$$P_{1/2} := \{x \in P : \lambda^T Ax \leq \lfloor \lambda^T b \rfloor, \text{ for } \lambda \in \{0, 1/2\}^m \text{ s.t. } \lambda^T A \in Z^n\},$$

the polyhedron obtained by intersecting P with the half-spaces induced by all $\{0, \frac{1}{2}\}$-cuts. There is an important difference in how P_1 and $P_{1/2}$ depend on A and b. P_1 depends only on the polyhedron P, so not on the actual system $Ax \leq b$ describing it, whereas $P_{1/2}$ is a function of A and b. Notice that neither P_1 nor $P_{1/2}$ are uniquely determined by P_I. Moreover, $P_I \subseteq P_1 \subseteq P_{1/2} \subseteq P$.

Although $P_1 = P$ holds if and only if $P = P_I$, one can have $P_{1/2} = P$ even if $P \neq P_I$; this case occurs, e.g., when $b/2 \in Z^m$. Nevertheless, $\{0, \frac{1}{2}\}$-cuts play an important role in polyhedral theory, in that the following results hold.

It is well known that an $r \times n$ $\{0, \pm 1\}$-matrix Q is totally unimodular if and only if $P = P_I$ for all $d \in Z^r$, where $P := \{x \in R^n : \left[\begin{smallmatrix} Q \\ -I \end{smallmatrix}\right] x \leq \left[\begin{smallmatrix} d \\ 0 \end{smallmatrix}\right]\}$. Similarly, an $r \times n$ $\{0,1\}$-matrix Q is balanced if and only if $P = P_I$ for all $d \in \{1, +\infty\}^r$ (see, e.g., Schrijver, 1986).

Theorem 1 *Let Q be an $r \times n$ $\{0, \pm 1\}$-matrix, and let $P := \{x \in R^n : \left[\begin{smallmatrix} Q \\ -I \end{smallmatrix}\right] x \leq \left[\begin{smallmatrix} d \\ 0 \end{smallmatrix}\right]\}$. Q is totally unimodular if and only if $P = P_{1/2}$ for all $d \in Z^r$.*

Proof. If Q is totally unimodular, then $P = P_{1/2} = P_I$ for all $d \in Z^r$. Assume now Q is not totally unimodular. Then, because of a result of Camion (1965), there exists a square submatrix B of Q with even row and column sums, such that the sum of the entries of B is not divisible by four. Let I_B and J_B index the rows and columns of B, respectively, and define $d \in Z^r$ as follows: $d_i := \sum_{j \in J_B} q_{ij}/2$ if $i \in I_B$; $d_i := M$ otherwise, where $M \geq \max_{i \notin I_B}\{\lceil \sum_{j \in J_B} q_{ij}/2 \rceil\}$. We next construct a point $\tilde{x} \in P$ and a $\{0, \frac{1}{2}\}$-cut (associated say with $\tilde{\lambda} \in \{0, 1/2\}^{r+n}$) which separates $\tilde{x}$ from $P_{1/2}$, thus proving $P \neq P_{1/2}$. For $j = 1, \ldots, n$, let $\tilde{x}_j := 1/2$ if $j \in J_B$; $\tilde{x}_j := 0$ otherwise. For $i = 1, \ldots, r$, let $\tilde{\lambda}_i := 1/2$ if $i \in I_B$; $\tilde{\lambda}_i := 0$ otherwise. Moreover, for $j = 1, \ldots, n$ let $\tilde{\lambda}_{r+j} := 1/2$ if $\sum_{i \in I_B} q_{ij}$ is odd; $\tilde{\lambda}_i := 0$ otherwise. By construction, $\tilde{\lambda}^T \left[\begin{smallmatrix} Q \\ -I \end{smallmatrix}\right] \in Z^n$, and $\tilde{\lambda}^T \left[\begin{smallmatrix} Q \\ -I \end{smallmatrix}\right] \tilde{x} = \frac{1}{4} \sum_{i \in I_B} \sum_{j \in J_B} q_{ij} > \lfloor \frac{1}{4} \sum_{i \in I_B} \sum_{j \in J_B} q_{ij} \rfloor = \lfloor \tilde{\lambda}^T d \rfloor$. □

Theorem 2 *Let Q be an $r \times n$ $\{0, 1\}$-matrix, and let $P := \{x \in R^n : \left[\begin{smallmatrix} Q \\ -I \end{smallmatrix}\right] x \leq \left[\begin{smallmatrix} d \\ 0 \end{smallmatrix}\right]\}$. Q is balanced if and only if $P = P_{1/2}$ for all $d \in \{1, +\infty\}^r$.*

Proof. If Q is balanced then $P = P_{1/2} = P_I$ for all $d \in \{1, +\infty\}^r$. Assume now Q is not balanced. Then, by definition of balancedness, there exists a square submatrix B of Q of odd order, with row and column sums equal to two. By using the same construction as in the proof of Theorem 1, one can define a vector $d \in \{1, +\infty\}^r$ such that $P \neq P_{1/2}$, as claimed. □

In some relevant cases one has $P_{1/2} = P_1 \neq P_I$, as, e.g., when P is defined by the edge inequalities and the nonnegativity constraints of the stable set problem. Moreover, sometimes $P_{1/2} = P_1 = P_I$ as, for example, when P is the solution set of the nonnegativity constraints and the degree constraints for the matching problem; see Edmonds (1965), Edmonds and Johnson (1970). Even in case $P_1 \neq P_{1/2}$, the family of $\{0, \frac{1}{2}\}$-cuts often contains several classes of (facet-inducing) valid inequalities for P_I, which are of valuable use within cutting-plane algorithms for optimization over P_I. This gives us motivation for studying $P_{1/2}$. In particular, we address the following $\{0, \frac{1}{2}\}$-*cut separation* problem ($\{0, \frac{1}{2}\}$-*SEP*), in its recognition version: *Given $x^* \in P$, find $\lambda \in \{0, 1/2\}^m$ such that $\lambda^T A \in Z^n$ and $\lambda^T A x^* > \lfloor \lambda^T b \rfloor$, or prove that no such λ exists.* Because of the well-known equivalence between optimization and separation, the availability of a polynomial-time algorithm for $\{0, \frac{1}{2}\}$-SEP would allow one to optimize, in polynomial time, a linear objective function over $P_{1/2}$; see Grötschel, Lovász and Schrijver (1981).

The paper is organized as follows. In Section 2 we describe the notation and basic definitions used in the sequel. In Section 3 we show that $\{0, \frac{1}{2}\}$-SEP is equivalent to finding a minimum-weight member of a binary clutter. The latter problem is known to be NP-hard, hence so is $\{0, \frac{1}{2}\}$-SEP. We describe some simple reduction procedures for $\{0, \frac{1}{2}\}$-SEP, and discuss two relevant polynomially-solvable special cases, arising when A is related to the edge-path incidence matrix of a tree. In Section 4 we show that $\{0, \frac{1}{2}\}$-SEP can be solved in polynomial time for a convenient relaxation of the system $Ax \le b$. This leads to an efficient separation algorithm for a subclass of $\{0, \frac{1}{2}\}$-cuts that often contains large families of strong inequalities for P_I. Applications to the Clique Partitioning, Asymmetric Traveling Salesman, Plant Location, Acyclic Subgraph and Linear Ordering polytopes are briefly discussed in Section 5, leading to new efficient separation algorithms. Section 6 draws some conclusions.

2 Notation and basic definitions

For any given $z \in Z$ and $q \in Z_+$, let $z \bmod q := z - \lfloor z/q \rfloor q$. As customary, notation $a \equiv b \pmod{q}$ stands for $a \bmod q = b \bmod q$.

Given an integer matrix $Q = (q_{ij})$, let $\overline{Q} = (\overline{q}_{ij}) := Q \bmod 2$ be the *binary support* of Q, i.e., $\overline{q}_{ij} = 1$ if q_{ij} is odd; $\overline{q}_{ij} = 0$ otherwise.

Given an undirected (not necessarily simple) graph $G = (V, E)$ and a node set S, we define $\delta(S) := \{ij \in E : i \in S, j \notin S\}$ and $E(S) := \{ij \in E : i \in S, j \in S\}$. To simplify notation, for $i \in V$ we write $\delta(i)$ instead of $\delta(\{i\})$.

A *cycle* of G is a subset C of E such that $|C \cap \delta(v)|$ is even for all $v \in V$. Let $T \subseteq E$

induce a maximal forest of G. Every edge $e \in E \setminus T$ is then contained in a *fundamental cycle*, say C_e, of the subgraph induced by $T \cup \{e\}$. We denote by $M_{cycle}(G, T)$ the $\{0,1\}$-matrix whose rows are the characteristic vectors of the fundamental cycles of G with respect to T.

A *cut* of G is a subset F of E of the form $F = \delta(S)$ for some $S \subseteq V$. Let $T \subseteq V \times V$ be any tree spanning V (possibly $T \not\subseteq E$). For each $t \in T$, let $S_t \subset V$ be any of the two components of the graph with node set V and edge set $T \setminus \{t\}$. The cuts $\delta(S_t)$, $t \in T$, are called the *fundamental cuts* of G (with respect to T). Let $M_{cut}(G, T)$ denote the $\{0,1\}$-matrix whose rows are the characteristic vectors of the fundamental cuts of G with respect to T. Notice that cuts and cycles intersect in an even number of edges.

Let a *parity label* $f_e \in \{0,1\}$ be assigned to each $e \in E$. A given $F \subseteq E$ is called *odd* if $\sum_{e \in F} f_e \equiv 1 \pmod 2$, *even* otherwise.

A $p \times q$ $\{0,1\}$-matrix M is the *edge-path incidence matrix of a tree* (*EPT matrix*, for short) if there is a tree T on $p+1$ nodes such that each column of M is the characteristic vector of the edges of a path in T. Every EPT matrix M can be *represented* by a graph G and a tree T such that $M = M_{cut}(G, T)$. EPT matrices play an important role in the theory of network matrices, and can be recognized in polynomial time; see, e.g., Schrijver (1986) and Nemhauser and Wolsey (1988). Examples of EPT matrices include the $\{0,1\}$-matrices having no more than two 1's per column, and those in which the 1's in a column occur consecutively. It is well known that EPT matrices are closed under row and column permutations, deletions and duplications. Moreover, if M is an EPT matrix (represented, say, by $G = (V, E)$ and T), then $M' := \begin{bmatrix} M \\ e_i^T \end{bmatrix}$ also is, where e_i^T denotes the i-th row of the identity matrix. Indeed, let uv be the edge of G associated with the i-th column of M. Then M' can be represented by $G' = (V', E')$ and T', where $V' := V \cup \{w\}$, $E' := (E \setminus \{uv\}) \cup \{uw\}$, and $T' := T \cup \{vw\}$.

Let Q be an $r \times t$ $\{0,1\}$-matrix, and $d \in \{0,1\}^r$, $d \neq 0$. The *binary clutter* associated with (Q, d) is defined as

$$\mathcal{C}(Q, d) := \{z \in \{0,1\}^t : Qz \equiv d \pmod 2\}.$$

Associated with every binary clutter is the following *minimum-weight binary clutter problem (MW-BCP)*:

MW-BCP: *Given* $w \in R^t_+$, *solve* $\min\{w^T z : z \in \mathcal{C}(Q, d)\}$.

Well known binary clutters are those associated with odd cycles and odd cuts in a parity labeled graph, and with complements of cuts in a graph. Indeed, the set of the characteristic vectors of the odd cycles of G is the binary clutter $\mathcal{C}(Q,d)$ defined by

$$Q := \begin{bmatrix} f^T \\ M_{cut}(G,T) \end{bmatrix}, d := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{1}$$

where $T \subseteq V \times V$ is an arbitrarily chosen tree spanning V (e.g., $T := \{1j : j \in V \setminus \{1\}\}$). In this case MW-BCP can be solved in polynomial time, as it amounts to finding a minimum-weight odd cycle of G; see, e.g., Grötschel and Pulleyblank (1981) and Gerards and Schrijver (1986) for efficient algorithms.

Analogously, the set of the characteristic vectors of the odd cuts of G is the binary clutter $\mathcal{C}(Q,d)$ defined by

$$Q := \begin{bmatrix} f^T \\ M_{cycle}(G,T) \end{bmatrix}, d := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \tag{2}$$

where T is any maximal forest of G. MW-BCP can be solved efficiently also in this case. Indeed, we first move the parity information from the edges to the nodes of G by defining, for each $v \in V$, the $\{0,1\}$-label $p_v := (\sum_{e\in\delta(v)} f_e) \bmod 2$. With this definition, $\sum_{e\in\delta(S)} f_e = \sum_{v\in S}\sum_{e\in\delta(v)} f_e - 2\sum_{e\in E(S)} f_e$, hence $\delta(S)$ is odd if and only if S contains an odd number of nodes with $p_v = 1$. MW-BCP then amounts to finding a minimum-weight such cut, and can be solved efficiently through the algorithm of Padberg and Rao (1982).

More generally, the decomposition theorem of Seymour (1980) for regular matroids implies that MW-BCP can be solved in polynomial time when $Q = \overline{U}$ for some totally unimodular matrix U.

Finally, the complements of the cuts of G define a binary clutter, arising when $Q = M_{cycle}(G,T)$ for any maximal forest T of G, and $d_i := 1$ if the fundamental cycle C_e associated with the i-th row of Q has odd cardinality; $d_i := 0$ otherwise. Unlike in the previous examples, MW-BCP is known to be NP-hard in this case, as it calls for a minimum-weight complement of a cut, i.e., for a maximum-weight cut in a graph – the well-known MAX-CUT problem. Moreover, the recognition version of MAX-CUT is NP-complete, see Garey and Johnson (1979), hence so is the recognition version of MW-BCP.

3 $\{0,\frac{1}{2}\}$-SEP and binary clutters

CG cuts can be thought of as being obtained in the following way. Let $\mu \in Z_+^m$ and $q \in Z_+$ be such that $\mu^T A \equiv 0 \pmod{q}$ and $\mu^T b = kq + r$ with $k \in Z$ and $r \in \{1,\ldots,q-1\}$. Then $\mu^T Ax \leq kq$ is a valid inequality for P_I. This inequality can equivalently be written as $\mu^T(b - Ax) \geq r$, hence a given $x^* \in P$ violates $\mu^T Ax \leq kq$ if and only if $\mu^T(b - Ax^*) < r$.

Observe that, for every q, it is enough to consider multipliers $\mu_i \in \{0, \ldots, q-1\}$: a larger μ_i leaves the modulo q arithmetic unchanged, but decreases the violation. Furthermore, given the *slack vector* $s^* := b - Ax^*$ the violation only depends on $(A, b) \bmod q$. $\{0, \frac{1}{2}\}$-cuts are produced by the above procedure when $q = 2$. Therefore, $\{0, \frac{1}{2}\}$-SEP (in its optimization version) can be re-phrased as follows.

$\{0, \frac{1}{2}\}$-SEP: *Given $x^* \in P$, solve* $\min\{s^{*T}\mu : \mu \in \mathcal{F}(\overline{A}, \overline{b})\}$, *where*

$$s^* := b - Ax^* \geq 0, \text{ and}$$

$$\mathcal{F}(\overline{A}, \overline{b}) := \{\mu \in \{0,1\}^m : \overline{b}^T \mu \equiv 1 \pmod 2, \overline{A}^T \mu \equiv 0 \pmod 2\}.$$

By construction, there exists a $\{0, \frac{1}{2}\}$-cut violated by the given point x^* if and only if $\min\{s^{*T}\mu : \mu \in \mathcal{F}(\overline{A}, \overline{b})\} < 1$.

It is then clear that $\{0, \frac{1}{2}\}$-SEP and MW-BCP are closely related to each other. Indeed, we have the following result.

Theorem 3 *Problems $\{0, \frac{1}{2}\}$-SEP and MW-BCP are equivalent.*

Proof. The transformation of any instance of $\{0, \frac{1}{2}\}$-SEP to an equivalent instance of MW-BCP is trivial: just define $w := s^*$, $d := [1|0, \ldots, 0]^T$, and $Q := \left[\begin{array}{c}\overline{b}^T\\ \hline \overline{A}^T\end{array}\right]$.

Consider now any instance of MW-BCP. We define $n := r + t + 1$, $m := t + 1$, $b := [2, \ldots, 2|1]^T$, and $A := \left[\begin{array}{c|c} \begin{matrix} Q^T \\ d^T \end{matrix} & 2I \end{array}\right]$. We then construct the following point x^* with $Ax^* \leq b$: let $x_j^* := 0$ for $j = 1, \ldots, r$; $x_{r+i}^* := 1 - w_i/2$ for $i = 1, \ldots, t$; and $x_{r+t+1}^* := 1/2$. By construction, $s^* := b - Ax^*$ equals $[w_1, \ldots, w_t|0]^T$ and, for every $\mu \in \{0,1\}^{t+1}$, $\overline{b}^T \mu \equiv 1 \pmod 2$ if and only if $\mu_{t+1} = 1$. Therefore $\{0, \frac{1}{2}\}$-SEP calls for $z \in \{0,1\}^t$ such that $Qz \equiv d \pmod 2$ and $w^T z$ is a minimum, i.e., it coincides with MW-BCP. $\square$

Corollary 1 *The recognition version of $\{0, \frac{1}{2}\}$-SEP is NP-complete.*

3.1 Reductions

The size of an instance of $\{0, \frac{1}{2}\}$-SEP can in some cases be reduced by applying reduction criteria, which are well known in the context of binary clutters. Some of these criteria are listed below.

(a) Every row i of $Ax \leq b$ with $s_i^* \geq 1$ can be removed.

(b) If $(\overline{A}, \overline{b})$ contains identical rows, only the one with the smallest s_i^* need to be considered.

(c) Let the *row intersection graph* $G(\overline{A})$ be defined as the graph having a node v_i for each row i of $\overline{A}$, and an edge $[v_i, v_k]$ if and only if $\overline{A}$ has a column j with $\overline{a}_{ij} = \overline{a}_{kj} = 1$. Let $C_1, \ldots, C_t$ be the connected components of $G(\overline{A})$. If $t \geq 2$, $\overline{A}$ can be brought to the form of a block diagonal matrix, hence $\{0, \frac{1}{2}\}$-SEP decomposes into t independent subproblems.

(d) Suppose there exists a row h of $\overline{A}$ such that, for some $j \in \{1, \ldots, n\}$, $\overline{a}_{hj} = 1$ and $\overline{a}_{hk} = 0$ for all $k \in \{1, \ldots, n\}$, $k \neq j$. This situation arises, e.g., when the system $Ax \leq b$ contains a lower/upper bound constraint of the form $\pm x_j \leq b_h$. Moreover, suppose this constraint is tight for the given point x^*, i.e., $s_h^* = 0$. Assuming w.l.o.g. $h = m$ and $j = n$, the input $(\overline{A}, \overline{b}, s^*)$ for $\{0, \frac{1}{2}\}$-SEP has the form:

$$\overline{A} = \left[\begin{array}{c|c} M & d \\ 0 \cdots 0 & 1 \end{array} \right], \overline{b} = \left[\begin{array}{c} \beta \\ \overline{b}_m \end{array} \right], \text{ and } s^* = \left[\begin{array}{c} \sigma^* \\ 0 \end{array} \right].$$

Observe that any feasible solution $\mu \in \{0,1\}^m$ of $\{0, \frac{1}{2}\}$-SEP has $\mu_m \equiv \sum_{i=1}^{m-1} \mu_i d_i \pmod 2$. We then define a reduced instance of $\{0, \frac{1}{2}\}$-SEP, whose input is given by (M, f, σ^*), where $f := \beta$ if $\overline{b}_m = 0$, $f := (\beta + d) \bmod 2$ otherwise. One can easily see that there is a one-to-one correpondence between the feasible solutions $\mu \in \{0,1\}^m$ and $\nu \in \{0,1\}^{m-1}$ to the original and reduced $\{0, \frac{1}{2}\}$-SEP, respectively, where $\mu_k = \nu_k$ for $k = 1, \ldots, m-1$, and $\mu_m \equiv d^T \nu \pmod 2$.

3.2 Polynomially-solvable special cases of $\{0, \frac{1}{2}\}$-SEP

The first polynomially-solvable special case we consider, arises when $\overline{A}^T$ is an EPT matrix.

Theorem 4 *$\{0, \frac{1}{2}\}$-SEP can be solved in polynomial time if $\overline{A}^T$ is an EPT matrix.*

Proof. Let $G = (V, E)$ and T represent $\overline{A}^T$, where $|V| = n + 1$, $|E| = m$, and $\overline{A}^T = M_{cut}(G, T)$. By construction, $\mathcal{F}(\overline{A}, \overline{b}) = \mathcal{C}(Q, d)$, where Q and d are defined as in (1), with $f := \overline{b}$. Hence $\{0, \frac{1}{2}\}$-SEP can be solved in polynomial time, as it calls for a minimum-weight odd cycle of G in which s_i^* and $\overline{b}_i$ play the role of the weight and the parity label for the edge associated with the i-th row of $\overline{A}$, respectively. □

Gerards and Schrijver (1986) gave a polynomial-time algorithm for $\{0, \frac{1}{2}\}$-SEP when A is an integer matrix satisfying $\sum_j |a_{ij}| \leq 2$ for each row index i. More generally, Theorem 4 implies that $\{0,\frac{1}{2}\}$-SEP can be solved efficiently when A has, at most, two odd coefficients in each row. Indeed, $\overline{A}^T$ is in this case the EPT matrix associated with the graph $G=(V,E)$ and the star T, where $V:=\{1,\ldots,n+1\}$, $T:=\{[n+1,j]: j=1,\ldots,n\}$, and E has an edge jk for each row i of $\overline{A}$ with $\overline{a}_{ij}=\overline{a}_{ik}=1$, and an edge $[n+1,j]$ for each row i having a single nonzero entry $\overline{a}_{ij}$.

We next consider the situation arising when $\overline{A}=\left[\begin{smallmatrix}M\\I\end{smallmatrix}\right]$, as in the case in which $x\geq 0$ is part of the system $Ax\leq b$.

Theorem 5 *$\{0,\frac{1}{2}\}$-SEP can be solved in polynomial time if $\overline{A}=\left[\begin{smallmatrix}M\\I\end{smallmatrix}\right]$, and M is an EPT matrix.*

Proof. Let $G=(V,E)$ and T represent M, i.e., $M=M_{cut}(G,T)$. The rows of M are indexed by the edges of T, whereas the other rows of $\overline{A}$ can be thought of as being indexed by E. The columns of $\overline{A}$ can then be viewed as the characteristic vectors of the fundamental cycles (with respect to T) of the graph $\tilde{G}:=(V,E\cup T)$, i.e., $\overline{A}^T=M_{cycle}(\tilde{G},T)$. It follows that $\mathcal{F}(\overline{A},\overline{b})=\mathcal{C}(Q,d)$, where Q and d are defined as in (2), with $f:=\overline{b}$. Therefore $\{0,\frac{1}{2}\}$-SEP can be solved in polynomial time, as it calls for a minimum-weight odd cut of $\tilde{G}$, in which s_i^* and $\overline{b}_i$ play the role of the weight and the parity label for the edge of $\tilde{G}$ associated with the i-th row of $\overline{A}$, respectively. □

Padberg and Rao (1982) gave a polynomial-time algorithm for $\{0,\frac{1}{2}\}$-SEP when $P:=\{x\in R^n: Dx\leq d, 0\leq x\leq g\}$, and D is the node-edge incidence matrix of a graph, i.e., when P_I is the capacitated b-matching polytope. More generally, Theorem 5 implies that $\{0,\frac{1}{2}\}$-SEP can be solved efficiently when $P:=\{x\in R^n: d^1\leq Dx\leq d^2, g^1\leq x\leq g^2\}$, and $\overline{D}$ is an EPT matrix. Indeed, in this case $\overline{A}=\left[\begin{smallmatrix}M\\I\end{smallmatrix}\right]$, where $M=\left[\begin{smallmatrix}\overline{D}\\\overline{D}\\I\end{smallmatrix}\right]$ is an EPT matrix (this follows from the fact that M is obtained from $\overline{D}$ by duplicating rows, and by adding rows of the identity matrix).

4 Optimizing over a relaxation of $P_{1/2}$

In view of Corollary 1, it is unlikely that a polynomial-time algorithm for optimizing a linear objective function over $P_{1/2}$ exists. Now let $P':=\{x\in R^n: A'x\leq b'\}\supseteq P$ be a relaxation of P obtained by "weakening" the system $Ax\leq b$ into $A'x\leq b'$, in such a way that the $\{0,\frac{1}{2}\}$-SEP associated with (A',b') can be solved in time polynomial in the size of (A,b). Then clearly one can optimize in polynomial time over the polyhedron $P\cap P'_{1/2}$.

There are several possible relaxations that meet the requirements above. Among them, we study the one obtained by making a systematical use of lower and upper bounds on the variables so as to produce a weakened system $A'x \le b'$ in which A' has, at most, two odd coefficients per row. To be specific, let us assume that the bound constraints $0 \le x \le d$ are part of the system $Ax \le b$ (possibly $d_j = +\infty$ for some j). For each row index i, let $O_i := \{j : a_{ij} \text{ is odd}\}$.

L-weakening

The simplest weakening arises when the lower bound constraints $-x_j \le 0$ for $j = 1, \dots, n$ are systematically added to the inequalities of $Ax \le b$ so as to reduce to, at most, two the number of odd coefficients in each row. This amounts to replacing each inequality $\sum_j a_{ij}x_j \le b_i$ with $|O_i| \ge 3$, by the $\binom{|O_i|}{2}$ *L-weakenings*

$$a_{ih}x_h + a_{ik}x_k + \sum_{j \notin O_i} a_{ij}x_j + \sum_{j \in O_i \setminus \{h,k\}} (a_{ij} - 1)x_j \le b_i$$

for all $h, k \in O_i$, $h < k$. In this way, the weakened system $A'x \le b'$ has $O(mn^2)$ rows. However, in view of Reduction (b) of Section 3, for any given x^* only $O(n^2)$ such inequalities need to be considered explicitly.

U-weakening

Analogously, by making use of the upper bound constraints $x_j \le d_j$ one can weaken $Ax \le b$ by replacing each inequality $\sum_j a_{ij}x_j \le b_i$ with $|O_i| \ge 3$, by the $\binom{|O_i|}{2}$ *U-weakenings*

$$a_{ih}x_h + a_{ik}x_k + \sum_{j \notin O_i} a_{ij}x_j + \sum_{j \in O_i \setminus \{h,k\}} (a_{ij} + 1)x_j \le b_i + \sum_{j \in O_i \setminus \{h,k\}} d_j$$

for all $h, k \in O_i$, $h < k$.

LU-weakening

More generally, one can use both lower and upper bounds on the variables to produce $A'x \le b'$. This amounts to replacing each inequality $\sum_j a_{ij}x_j \le b_i$ with $|O_i| \ge 3$, by the *LU-weakenings*

$$a_{ih}x_h + a_{ik}x_k + \sum_{j \notin O_i} a_{ij}x_j + \sum_{j \in L}(a_{ij} - 1)x_j + \sum_{j \in U}(a_{ij} + 1)x_j \le b_i + \sum_{j \in U} d_j$$

for all $h, k \in O_i$, $h < k$, and for all partitions (L, U) of $O_i \setminus \{h, k\}$.

Although $A'x \le b'$ has, in general, an exponential number of rows, still $\{0, \frac{1}{2}\}$-SEP can be solved in polynomial time. Indeed, for each triple (i, h, k) only two LU-weakenings are worth considering for the given point x^*, namely those with even and odd right-hand side having minimum slack. These two weakenings can be computed, in $O(n)$ time, through a simple dynamic programming scheme that considers, for each $j \in O_i \setminus \{h, k\}$, the two possibilities $j \in L$ or $j \in U$.

As a consequence of the above discussion, one has the following result.

Theorem 6 *One can optimize in polynomial time over the relaxation of $P_{1/2}$ given by $P \cap P'_{1/2}$, where $P' := \{x \in R^n : A'x \le b'\}$ and $A'x \le b'$ is obtained from $Ax \le b$ through LU-weakening.*

5 Applications

Let $\mathcal{H}$ be the family of the $\{0, \frac{1}{2}\}$-cuts that can be derived from the weakened system $A'x \le b'$ obtained from $Ax \le b$ through LU-weakening. For several widely-studied polyhedra, $\mathcal{H}$ contains large classes of inequalities, some of which are known to be facet-inducing for the integer polyhedron P_I. Hence $P \cap P'_{1/2}$ hopefully gives a tight approximation of P_I. Some relevant cases are next briefly discussed.

5.1 The Clique Partitioning Polytope

The clique partitioning problem arises in optimal clustering. We are given a complete undirected graph $G = (V, E)$. An edge set A is called a *clique partitioning* of G if V can be partitioned into disjoint sets $W_1, \ldots, W_k$ such that $A = \bigcup_{i=1}^{k} E(W_i)$. Let

$$P_I := \text{conv}\{x \in \{0,1\}^E : x_{ij} + x_{jk} - x_{ik} \le 1 \text{ for all } i, j, k \in V, |\{i, j, k\}| = 3\}$$

denote the clique partitioning polytope. The constraints $x_{ij} + x_{jk} - x_{ik} \le 1$ are called *triangle inequalities.* Several classes of facet-inducing inequalities for P_I have been studied by Grötschel and Wakabayashi (1990). These include the following *2-chorded odd cycle inequalities.* Let $C = \{e_1, \ldots, e_k\}$, $k \ge 5$ and odd, be a cycle of G, with $e_i = v_i v_{i+1}$ $(i = 1, \ldots, k-1)$ and $e_k = v_k v_1$. To simplify notation, let $v_{k+1} := v_1$ and $v_{k+2} := v_2$. The set $\overline{C} := \{v_i v_{i+2} : i = 1, \ldots, k\}$ is called the set of the *2-chords* of C. The 2-chorded odd cycle inequality associated with C is then defined as

$$\sum_{ij \in C} x_{ij} - \sum_{ij \in \overline{C}} x_{ij} \le \frac{k-1}{2}.$$

To our knowledge, no separation algorithm for these constraints has been proposed in the literature. Recently, Müller, 1993, proposed an odd cycle separation algorithm for a related class of inequalities for the so-called *transitive acyclic subdigraph* polytope.

2-chorded odd cycle inequalities are $\{0, \frac{1}{2}\}$-cuts obtained by combining the following constraints:

$$x_{v_iv_{i+1}} + x_{v_{i+1}v_{i+2}} - 2x_{v_iv_{i+2}} \leq 1 \text{ for } i = 1, \ldots, k,$$

each of which is an L-weakening of a triangle inequality. We observe here that these are not the only $\{0, \frac{1}{2}\}$-cuts one can obtain from weakened triangle inequalities of the form $x_{ij} + x_{jk} - 2x_{ik} \leq 1$. For instance, let $C = \{e_1, \ldots, e_k\}$, $k \geq 3$ and odd, be a cycle of G with $e_i = v_iv_{i+1}$ for $i = 1, \ldots, k$. Given $z \in V \setminus \{v_1, \ldots, v_k\}$, one can add

$$x_{v_iz} + x_{zv_{i+1}} - 2x_{v_iv_{i+1}} \leq 1 \text{ for } i = 1, \ldots, k,$$

weighted by $1/2$, and obtain through rounding the *odd wheel inequality*

$$\sum_{i=1}^{k} x_{v_iz} - \sum_{ij \in C} x_{ij} \leq \frac{k-1}{2}.$$

These inequalities are facet-inducing for P_I (Chopra and Rao, 1993), and can be separated in polynomial time (Deza, Grötschel and Laurent, 1992).

Since the weakened triangle inequalities belong to family $\mathcal{H}$, one can optimize in polynomial time over a relaxation of $P_{1/2}$ whose inequality set contains all 2-chorded odd cycle and odd wheel inequalities.

5.2 The Asymmetric Traveling Salesman Polytope

Let $G = (V, A)$ be a complete and loop-free directed graph. The *Asymmetric Traveling Salesman (ATS)* polytope, P_I, is the convex hull of the incidence vectors of the Hamiltonian circuits (*tours*) of G, i.e.,

$$P_I := \text{conv}\{x \in \{0,1\}^A :$$

$$\sum_{j \in V} x_{ij} = 1, \quad i \in V \tag{3}$$

$$\sum_{i \in V} x_{ij} = 1, \quad j \in V \tag{4}$$

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1, \quad S \subset V, \; |S| \geq 2 \;\}. \tag{5}$$

Inequalities (5) are called *Subtour Elimination Constraints (SEC's)*. Although there are exponentially many SEC's, one can optimize in polynomial time over $P := \{x \in R^A_+ :$ x satisfies (3)-(5)$\}$ since these constraints can be handled efficiently through max-flow separation algorithms.

Let two arcs (i,j) and (h,k) be called *incompatible* if $i=h$, or $j=k$, or $(i,j)=(k,h)$. It is easy to see that the L-weakening of (3)-(5) consists of the inequalities $x_{ij}+x_{hk} \leq 1$ for all pairs (i,j) and (h,k) of incompatible arcs. Therefore, the family $\mathcal{H}$ contains the following valid inequalities, introduced by Balas (1989). A *Closed Alternating Trail (CAT)* is an arc sequence $T := \{a_1, \ldots, a_s\}$ such that each a_i is incompatible with a_{i-1} and a_{i+1}, and compatible with all the other arcs of T (here, $a_0 := a_s$ and $a_{s+1} := a_1$). The CAT is *odd* if the cardinality s of T is odd. By adding one half (and rounding) the constraints $x_{a_i} + x_{a_{i+1}} \leq 1$ for $i = 1, \ldots, s$, one obtains the following *weak odd CAT inequality*:

$$\sum_{(i,j)\in T} x_{ij} \leq \frac{|T|-1}{2}.$$

The computational experience reported in Fischetti and Toth (1994) has shown that these inequalities are useful to speed-up the convergence of a branch-and-cut algorithm for solving hard ATS real-world instances.

Weak odd CAT inequalities can be lifted to become facet-defining for P_I (except in few pathological cases arising for small values of $|V|$). The resulting constraints are the $\{0,\frac{1}{2}\}$-cuts obtained by replacing, in the Chvátal-Gomory derivation, each $x_{ij}+x_{hk} \leq 1$ having $i=h$ or $j=k$, by the corresponding equation (3) or (4), respectively. These inequalities generalize comb inequalities. The complexity of the separation problem for lifted odd CAT inequalities (as well as that for comb inequalities) is open.

5.3 The Uncapacitated Plant Location Polytope

The uncapacitated (or simple) plant location problem has several applications in location and has been extensively studied; see, e.g., Cornuéjols, Nemhauser and Wolsey (1990). Let $G = (V_1 \cup V_2, E)$ be a complete bipartite graph. A feasible solution of the plant location problem is a subset E' of E with $|E' \cap \delta(i)| = 1$ for all $i \in V_1$.

The *uncapacitated plant location* polytope is then defined as

$$P_I := \text{conv}\{\ (x,y) \in \{0,1\}^{E \cup V_2} :$$

$$x_{ij} - y_j \leq 0, \quad \text{for all } i \in V_1, j \in V_2 \tag{6}$$

$$\sum_{ij\in\delta(i)} x_{ij} = 1, \quad \text{for all } i \in V_1 \quad \}. \tag{7}$$

Here, $x_{ij} = 1$ iff the edge ij is chosen in E', and $y_j = 1$ iff $|E' \cap \delta(j)| \neq 0$.

$\{0,\frac{1}{2}\}$-cuts include the following *odd cycle inequalities*

$$\sum_{ij\in C} x_{ij} - \sum_{j\in V_2(C)} y_j \leq \frac{k-1}{2},$$

where C is a cycle of G of length $2k$, with $k \geq 3$ and odd, and $V_i(C)$ contains the k nodes of V_i visited by C $(i = 1, 2)$. Clearly, $|V_1(C)| = |V_2(C)| = k$ as G is bipartite. These inequalities are indeed obtained by adding one half (and rounding) the following constraints

$$x_{ij} - y_j \leq 0, \quad \text{for } ij \in C$$

$$\sum_{ij \in \delta(i) \cap C} x_{ij} \leq 1, \quad \text{for } i \in V_1(C), \tag{8}$$

Notice that (8) is an L-weakening of (7), hence the odd-cycle inequalities belong to the family $\mathcal{H}$.

5.4 The Acyclic Subgraph and Linear Ordering Polytopes

Let $G = (V, A)$ be a complete and loop-free directed graph, and P_{AC} be the convex hull of the incidence vectors of the acyclic subgraphs of G, i.e.,

$$P_{AC} := \text{conv}\{\ x \in \{0,1\}^A :$$

$$\sum_{(i,j) \in C} x_{ij} \leq |C| - 1, \quad \text{for all directed cycles } C \subseteq A \quad \}. \tag{9}$$

P_{AC} is called the *acyclic subgraph* polytope, and has been studied by Grötschel, Jünger and Reinelt (1984, 1985a, 1985b). Let $C_1, \ldots, C_k$ be distinct directed cycles of G. For each $(i, j) \in A$, let

$$\mu_{ij} := |\{h : (i, j) \in C_h\}|,$$

and

$$M := \bigcup_{h=1}^{k} C_h,$$

$$M^* := \{(i, j) \in M : \mu_{ij} \text{ is odd}\}.$$

Moreover, let (M_1^*, M_2^*) be a partition of M^*, with M_1^* or M_2^* possibly empty, and assume $\sum_{h=1}^{k} |C_h| + |M_1^*| - k$ to be odd. By adding one half and rounding the constraints

$$\sum_{(i,j) \in C_h} x_{ij} \leq |C_h| - 1 \quad \text{for } h = 1, \ldots, k$$

$$x_{ij} \leq 1 \quad \text{for } (i, j) \in M_1^*$$

$$-x_{ij} \leq 0 \quad \text{for } (i, j) \in M_2^*$$

one obtains the cut

$$\sum_{(i,j)\in M\setminus M^*} \frac{\mu_{ij}}{2} x_{ij} + \sum_{(i,j)\in M_1^*} \frac{\mu_{ij}+1}{2} x_{ij} + \sum_{(i,j)\in M_2^*} \frac{\mu_{ij}-1}{2} x_{ij} \leq$$

$$\leq \frac{\sum_{h=1}^{k} |C_h| + |M_1^*| - k - 1}{2}. \tag{10}$$

To our knowledge, this class of inequalities is new.

Notice that the left-hand side of (10) may have coefficients greater than 1. If, however, the additional restriction

$$\mu_{ij} \leq 2 \ \text{ for all } (i,j) \in M$$

is imposed (i.e., no more than two cycles overlap in the same arc), when choosing $M_1^* = M^*$ and $M_2^* = \emptyset$ the inequality (10) becomes

$$\sum_{(i,j)\in M} x_{ij} \leq |M| - \frac{k+1}{2}, \tag{11}$$

with k odd since $\sum_{h=1}^{k} |C_h| + |M^*| - k = 2|M| - k$ is required to be odd. If the chosen $C_1, \ldots, C_k$ satisfy some additional technical requirements, see conditions (2.15) to (2.17) in Grötschel, Jünger and Reinelt (1985a), constraint (11) is a so-called *Möbius ladder inequality.* The class of Möbius ladder inequalities contains however members not covered by (10), arising when $\mu_{ij} \geq 3$ for some $(i,j) \in M$. As for the separation problem for (11), we observe that these constraints can equivalently be derived from the following weakening of (9):

$$\sum_{(i,j)\in C_h} x_{ij} + \sum_{(i,j)\in C_h \cap M^*} x_{ij} \leq |C_h| + |C_h \cap M^*| - 1 \ \text{ for } h = 1, \ldots, k. \tag{12}$$

In the special case in which $|C_h \setminus M^*| \leq 2$ holds for all h, these latter inequalities have, at most, 2 odd left-hand side coefficients each. Hence the $\{0, \frac{1}{2}\}$-SEP associated with the system (12) can be solved efficiently, provided we heuristically restrict ourselves to considering a polynomial number of inequalities (12), e.g., those derived from the inequalities (9) with $|C_h| \leq t$ for some fixed t, e.g., $t = 4$.

We next address the so-called *linear ordering* polytope, defined as

$$P_{LO} := \text{conv}\{\ x \in \{0,1\}^A : \ \ (9) \ \text{ and } \\ x_{ij} + x_{ji} = 1, \ \text{ for } 1 \leq i < j \leq |V| \ \ \}. \tag{13}$$

It is well known that, in the definition of P_{LO}, (9) can be replaced by the *triangle inequalities*

$$x_{ij} + x_{jk} + x_{ki} \leq 2 \ \text{ for } i,j,k \in V, i < j, i < k, j \neq k. \tag{14}$$

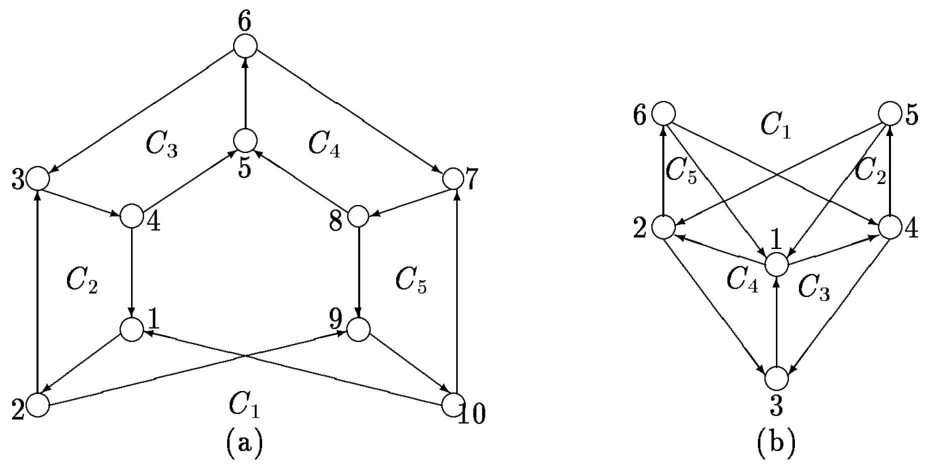

Figure 1: Two Möbius ladders.

Indeed, every cycle-breaking inequality (9) associated with a cycle C with $|C| \geq 4$, say $C = \{(i_1, i_2), (i_2, i_3), \ldots, (i_{|C|}, i_1)\}$, can be obtained by adding $\sum_{(i,j)\in C'} x_{ij} \leq |C'|-1$, $x_{i_1 i_2} + x_{i_2 i_3} + x_{i_3 i_1} \leq 2$, and $-x_{i_1 i_3} - x_{i_3 i_1} = -1$, where $C' := \{(i_1, i_3), (i_3, i_4), \ldots, (i_{|C|}, i_1)\}$ (so, $|C'| = |C| - 1$).

We observe that (13) have 2 odd left-hand side coefficients each, whereas (14) admit the U-weakenings:

$$x_{ij} + 2x_{jk} + x_{ki} \leq 3 \quad \text{for } i, j, k \in V, |\{i, j, k\}| = 3. \tag{15}$$

Therefore one can separate in polynomial time over the family of the $\{0, \frac{1}{2}\}$-cuts obtained by combining (13) and (15). This family contains, among others, the Möbius ladder inequalities covered by Theorem 3.11 in Grötschel, Jünger and Reinelt (1985b). For instance, the Möbius ladder inequality whose support graph is depicted in Figure 1.a is obtained by combining

$$x_{12} + 2x_{23} + x_{31} \leq 3, \quad x_{34} + 2x_{41} + x_{13} \leq 3, \quad -x_{13} - x_{31} = -1$$

$$x_{34} + 2x_{45} + x_{53} \leq 3, \quad x_{56} + 2x_{63} + x_{35} \leq 3, \quad -x_{35} - x_{53} = -1$$

$$x_{56} + 2x_{67} + x_{75} \leq 3, \quad x_{78} + 2x_{85} + x_{57} \leq 3, \quad -x_{57} - x_{75} = -1$$

$$x_{78} + 2x_{89} + x_{97} \leq 3, \quad x_{9,10} + 2x_{10,7} + x_{79} \leq 3, \quad -x_{79} - x_{97} = -1$$

$$x_{9,10} + 2x_{10,1} + x_{19} \leq 3, \quad x_{12} + 2x_{29} + x_{91} \leq 3, \quad -x_{19} - x_{91} = -1,$$

whereas that associated with the graph of Figure 1.b is derived from

$$x_{12} + 2x_{23} + x_{31} \leq 3, \quad x_{14} + 2x_{43} + x_{31} \leq 3$$

$$x_{14} + x_{45} + 2x_{51} \leq 3, \quad x_{12} + x_{26} + 2x_{61} \leq 3$$

$$x_{26} + x_{65} + 2x_{52} \leq 3, \quad x_{45} + x_{56} + 2x_{64} \leq 3, \quad -x_{65} - x_{56} = -1.$$

6 Conclusions

We have considered the family of $\{0, \frac{1}{2}\}$-cuts, and have studied the associated separation problem, $\{0, \frac{1}{2}\}$-SEP. We have shown that $\{0, \frac{1}{2}\}$-SEP does not depend on the actual values of the coefficients of the inequalities used to derive the cut, but only on their parity. This provides a unifying framework for studying some classes of inequalities for different problems such as, e.g., Linear Ordering and Clique Partitioning. We have shown that $\{0, \frac{1}{2}\}$-SEP is equivalent to the problem of finding a minimum-weight member of a binary clutter. This implies that $\{0, \frac{1}{2}\}$-SEP is NP-hard in the general case, but polynomially solvable in two relevant cases that generalize those considered by Gerards and Schrijver (1986) and by Padberg and Rao (1982). We also proved that $\{0, \frac{1}{2}\}$-SEP can be solved in polynomial time for a convenient relaxation of the original polyhedron. Applications to several important problems have been discussed. In some cases, we have discovered exact polynomial separation schemes for large classes of (sometimes new) inequalities. An outcome of the research is that separation sometimes becomes easier if one does not insist in detecting violated cuts belonging to a restricted (and sometimes complicated) class of inequalities, but concentrates on the way these cuts can be derived from the original formulation.

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