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SEPARATING A SUPERCLASS OF COMB INEQUALITIES IN PLANAR GRAPHS

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Many classes of valid and facet-inducing inequalities are known for the family of polytopes associated with the *Symmetric Travelling Salesman Problem* (STSP), including *subtour elimination*, 2-*matching* and *comb* inequalities. For a given class of inequalities, an *exact separation algorithm* is a procedure which, given an LP relaxation vector x^*, finds one or more inequalities in the class which are violated by x^*, or proves that none exist. Such algorithms are at the core of the highly successful *branch-and-cut* algorithms for the STSP. However, whereas polynomial time exact separation algorithms are known for subtour elimination and 2-matching inequalities, the complexity of comb separation is unknown.

A *partial* answer to the comb problem is provided in this paper. We define a *generalization* of comb inequalities and show that the associated separation problem can be solved efficiently when the subgraph induced by the edges with $x_e^* > 0$ is planar. The separation algorithm runs in $\mathcal{O}(n^3)$ time, where n is the number of vertices in the graph.

1. Introduction. The famous *Symmetric Travelling Salesman Problem* (STSP) is the $\mathcal{NP}$-hard problem of finding a minimum cost Hamiltonian cycle (or *tour*) in a complete undirected graph (see Lawler et al. 1985). The most successful optimization algorithms at present (e.g., Padberg and Rinaldi 1991, Applegate et al. 1995) are based upon a formulation of the STSP as an integer linear program due to Dantzig, Fulkerson and Johnson (1954), which we now describe.

Let G be a complete graph with vertex set V and edge set E. For each edge $e \in E$, let c_e be the cost of traversing e. Given any $S \subset V$, let $E(S)$ denote the set of edges with both end-vertices in S and $\delta(S)$ denote the set of edges with one end-vertex in S and the other in $V \backslash S$. When $S = \{i\}$, we write $\delta(i)$ rather than $\delta(\{i\})$ for brevity. Define the 0-1 variable x_e for each $e \in E$, taking the value 1 if e is traversed, 0 otherwise. Finally, for any $F \subseteq E$, let $x(F)$ denote $\sum_{e \in F} x_e$. The STSP can now be formulated as:

$$\min c^T x$$
subject to

$$x(\delta(i)) = 2 \quad (\forall i \in V), \tag{1}$$

$$x(E(S)) \leq |S| - 1 \quad (\forall S \subset V : 1 \leq |S| \leq |V| - 1), \tag{2}$$

$$-x_e \leq 0 \quad (\forall e \in E), \tag{3}$$

$$x \in Z^{|E|}. \tag{4}$$

Equations (1) are called *degree equations*. The inequalities (2) are known as *subtour elimination constraints* or SECs and inequalities (3) are called *nonnegativity inequalities*. An SEC with $|S| = 2$ is a mere upper bound, since, if $S = e = \{u, v\}$, then (2) reduces to

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$x_e \leq 1$. Note that we allow 'degenerate' SECs in which $|S| = 1$. A degenerate SEC reduces to the trivial inequality $0 \leq 0$.

The convex hull in $\Re^{|E|}$ of vectors satisfying (1)–(4) is called a *Symmetric Travelling Salesman Polytope*. The polytope defined by (1)–(3) is called the *Subtour Elimination Polytope*. These two polytopes are denoted by STSP(n) and SEP(n), respectively, where $n := |V|$. It is known (Grötschel and Padberg 1979a) that SEP(n) = STSP(n) for $n \leq 5$, but that STSP(n) is strictly contained in SEP(n) for $n \geq 6$. Thus, more linear inequalities are needed to describe STSP(n) when $n \geq 6$. In the past twenty years, a great deal of research has been conducted into finding improved linear descriptions of STSP(n) and many classes of valid inequalities have been discovered. Moreover, many of these valid inequalities have been proven to be induce *facets* — proper faces of maximum dimension — and therefore essential in any linear description. Space does not permit a full review of this literature and the reader is referred to Nemhauser and Wolsey (1988) and Jünger et al. (1995, 1997).

The most famous class of facet-inducing inequalities for STSP(n) are the *comb* inequalities of Grötschel and Padberg (1979a,b). Let $p \geq 3$ be an odd integer. Let $H \subset V$ and $T_j \subset V$ for $j = 1, \ldots, p$ be such that $H \cap T_j \neq \emptyset$ and $T_j \backslash H \neq \emptyset$ for $j = 1, \ldots, p$, and also let the T_j be vertex-disjoint. The comb inequality is:

$$x(E(H)) + \sum_{j=1}^{p} x(E(T_j)) \leq |H| + \sum_{j=1}^{p} |T_j| - (3p+1)/2. \tag{5}$$

The set H is called the *handle* of the comb and the T_j are called *teeth*. The validity of comb inequalities in the special case where $|H \cap T_j| = 1$ for all j was proved by Chvátal (1973a). If, in addition, $|T_j \backslash H| = 1$ for all j, the comb inequalities reduce to 2-*matching* inequalities, first discovered by Edmonds (1965).

The highly successful *branch-and-cut* algorithms for the STSP (see Padberg and Rinaldi 1990, 1991; Applegate et al. 1995) use such polyhedral results in the following way. First, linear programming is used to optimize over the polyhedron defined by the degree equations and the bounds $0 \leq x_e \leq 1$ for all $e \in E$. If the optimal solution vector x^* represents a tour, then the STSP instance has been solved. If not, then a search begins for valid inequalities (such as SECs, 2-matching or comb inequalities) which are violated by the current x^*. If any are found, then these are added to the linear program, which is then resolved. This process is repeated iteratively until no more violated inequalities can be found. If the resulting x^* still does not represent a tour, then the entire procedure is embedded into a branch-and-bound framework.

The success of a branch-and-cut approach therefore depends crucially on the identification of violated inequalities. This leads to the idea of an *exact separation algorithm* (see Grötschel et al. 1988, Padberg and Rinaldi 1990). For a given class of inequalities (e.g., comb inequalities), an exact separation algorithm is a procedure which takes a given x^* as input and returns one or more violated inequalities in that class, or a proof that none exist.

A desirable property of an exact separation algorithm is that it runs in polynomial time. Unfortunately, useful polynomial time exact separation algorithms are known only for the SECs and the 2-matching inequalities (see Padberg and Grötschel 1985, Padberg and Rao 1982). Recently, Carr (1996, 1997) showed that the set of all valid inequalities for the STSP can be partitioned into an infinite number of classes such that each class is polynomially separable, but the order of the polynomial for even the simplest classes is large. Moreover, the complexity of comb separation is unknown. (The class of comb inequalities is divided into an infinite number of classes in the scheme of Carr, one for each possible number of teeth.)

Because of this difficulty, researchers have turned to *heuristic* separation algorithms, viz., procedures which *sometimes* detect violated inequalities, but which may occasionally fail (see Jünger et al. 1995, 1997 for surveys).

The present paper is inspired by three important papers due to Applegate et al. (1995), Fleischer and Tardos (1999), and Caprara et al. (2000). In the first paper, an exact algorithm is given for finding *maximally violated* comb inequalities, i.e., comb inequalities which are violated by $\frac{1}{2}$. (It is impossible for a comb inequality to be violated by more than $\frac{1}{2}$ if $x^* \in \mathrm{SEP}(n)$.) The authors claim that this algorithm performs quite well in practice, although it runs in exponential time in the worst case. In the second paper, it is shown how to improve the algorithm in the case where the *support graph* is planar. (The support graph, denoted by G^*, is the subgraph of G induced by the edge set $E^* := \{e \in E : x_e^* > 0\}$.) The Fleischer-Tardos algorithm runs in $\mathcal{O}(n^2 \log n)$ time and always finds a violated comb inequality whenever a maximally violated comb inequality exists. The third paper defines a generalization of comb inequalities, called *mod*-2 *cuts*, and shows how to detect maximally violated mod-2 cuts in $\mathcal{O}(n^2|E^*|)$ time. (In fact, the ideas in this last paper are applicable to any integer programming problem.)

The present paper represents another step forward in this line of research. We define a new class of inequalities for the STSP, called *domino-parity* (DP) inequalities, which are shown to be intermediate in generality between the comb inequalities and the mod-2 cuts. Then we show that the associated separation problem can be solved exactly in $\mathcal{O}(n^3)$ time provided that $x^* \in \mathrm{SEP}(n)$ and G^* is planar. This result is of practical as well as theoretical importance, since vectors $x^* \in \mathrm{SEP}(n)$ with planar support often arise when solving planar, Euclidean STSP instances (Jünger 1999).

The outline of the paper is as follows. In §2, the DP inequalities are defined and shown to be a proper generalization of the comb inequalities. In §3, the separation algorithm is described. In §4, it is shown how to adapt the algorithm to the so-called *Graphical TSP* (Cornuéjols et al. 1985, Naddef and Rinaldi 1992). Concluding comments are made in §5.

2. Domino-parity inequalities.

In this section, we define the domino-parity (DP) inequalities and show how they relate to known valid inequalities for the STSP. We begin with some definitions and notation.

DEFINITION. A *domino* is a pair $\{A, B\}$ with the following properties:
- $\emptyset \neq A, B \subset V$,
- $A \cap B = \emptyset$,
- $A \cup B \neq V$.

The significance of dominoes to comb separation was first stressed by Applegate et al. (1995) and elaborated on by Fleischer and Tardos (1999). However, their definition of *domino* is more restrictive than ours, since they also require the SECs for A, B and $A \cup B$ to be tight (i.e., satisfied at equality) for x^*. Given a domino $\{A, B\}$, we will let $E(A : B)$ denote the set of edges with one end-vertex in A and the other in B.

DEFINITION. Let r be a positive integer and suppose that $E_1, \ldots, E_r$ are edge-sets, i.e., $E_j \subset E$ for $j = 1, \ldots, r$. For each $e \in E$, define

$$\mu_e = |\{j \in \{1, \ldots, r\} : e \in E_j\}|. \tag{6}$$

That is, μ_e denotes the number of edge-sets in which e appears. It may be thought of as the *multiplicity* of e with respect to the chosen edge sets.

DEFINITION. A set of edge-sets $E_1, \ldots, E_r$ is said to *support the cut* $\delta(H)$ if $\delta(H) = \{e \in E : \mu_e \text{ is odd}\}$.

THEOREM 1. *Let p be a positive odd integer, $\{A_j, B_j\}$ for $j = 1, \ldots, p$ be dominoes and let $H \subset V$ satisfy $1 \le |H| \le |V| - 1$. Suppose that $F \subseteq E$ is such that $\{E(A_1 : B_1), \ldots, E(A_p : B_p), F\}$ supports the cut $\delta(H)$ and define μ_e accordingly. Then the* domino parity *inequality*

$$x(E(H)) + \sum_{j=1}^{p} x(E(A_j \cup B_j)) - \sum_{e \in E} \lfloor \mu_e/2 \rfloor x_e \le |H| + \sum_{j=1}^{p} |A_j \cup B_j| - (3p+1)/2 \tag{7}$$

is valid for STSP(n), where $\lfloor a \rfloor$ denotes the greatest integer not bigger than a.

PROOF. Sum the SECs for A_j, B_j and $A_j \cup B_j$ for $j = 1, \ldots, p$ to obtain

$$2\sum_{j=1}^{p} x(E(A_j \cup B_j)) - \sum_{j=1}^{p} x(E(A_j : B_j)) \le 2\sum_{j=1}^{p} |A_j \cup B_j| - 3p. \tag{8}$$

Add the nonnegativity inequalities for the edges $e \in F$ to obtain:

$$2\sum_{j=1}^{p} x(E(A_j \cup B_j)) - \sum_{e \in E} \mu_e x_e \le 2\sum_{j=1}^{p} |A_j \cup B_j| - 3p. \tag{9}$$

Now sum together the degree equations for $i \in H$ to obtain:

$$2x(E(H)) + x(\delta(H)) = 2|H|. \tag{10}$$

Add (9) and (10) to obtain:

$$2\left(x(E(H)) + \sum_{j=1}^{p} x(E(A_j \cup B_j))\right) - \sum_{e \in \delta(H)} (\mu_e - 1)x_e - \sum_{e \in E \setminus \delta(H)} \mu_e x_e \le 2\left(|H| + \sum_{j=1}^{p} |A_j \cup B_j|\right) - 3p. \tag{11}$$

The fact that $\{E(A_1 : B_1), \ldots, E(A_p : B_p), F\}$ supports the cut $\delta(H)$ implies that every term on the left-hand side of (11) is an even integer. Dividing by two and rounding down the right-hand side yields (7).

The DP inequalities are a proper generalization of the comb inequalities, as expressed in the following two propositions.

PROPOSITION. *Comb inequalities are DP inequalities.*

PROOF. For $i = 1, \ldots, p$, set $A_j = H \cap T_j$ and $B_j = T_j \setminus H$. Set $F = \delta(H) \setminus (E(A_1 : B_1) \cup \cdots \cup E(A_p : B_p))$. This gives $\mu_e = 1$ for $e \in \delta(H)$, $\mu_e = 0$ otherwise. The result follows from the fact that $T_j = A_j \cup B_j$.

PROPOSITION. *The class of comb inequalities is strictly contained in the class of facet-inducing DP inequalities.*

PROOF. It is only necessary to produce one facet-inducing DP inequality which is not a comb inequality. Consider the following DP inequality defined for STSP(12), with $H = \{1, \ldots, 6\}$ and $p = 7$. For $j = 1, \ldots, 6$, set $A_j = \{j\}$, $B_j = \{j+6\}$. Set $A_7 = \{3, 4, 9, 10\}$

and $B_7 = \{5, 6, 11, 12\}$ and construct F accordingly. It can be checked that the resulting DP inequality takes the form:

$$x(E(H)) + \sum_{j=1}^{7} x(E(A_j \cup B_j)) - x(C) \leq 15,$$

where C contains the edges $\{3,5\}$, $\{3,6\}$, $\{4,5\}$, $\{4,6\}$, $\{9,11\}$, $\{9,12\}$, $\{10,11\}$ and $\{10,12\}$. This inequality can be shown to be facet-inducing for STSP(12) and distinct from the comb inequalities using techniques described in Naddef and Rinaldi (1992).

The DP inequalities can also be viewed as a special kind of *mod*-2 *cut*; see Caprara et al. (2000). A mod-2 cut for the STSP is an inequality which is obtained by summing together a number of degree equations, SECs and nonnegativity inequalities in such a way that, in the resulting inequality, every left-hand side coefficient is even but the right-hand side is odd, and then dividing this inequality by two and rounding down the right-hand side. This technique is a special case of the *integer rounding* procedure introduced by Chvátal (1973b) (see also Nemhauser and Wolsey 1988, Caprara and Fischetti 1996). Inspection of the proof of Theorem 1 immediately reveals that DP inequalities are mod-2 cuts. However, DP inequalities have a special structure, in that the SECs used in the derivation are partitioned into triplets (one triplet for each domino). It turns out that this is a genuine limitation:

Proposition. *The class of facet-inducing DP inequalities is strictly contained in the class of facet-inducing mod-2 cuts.*

Proof. In Caprara et al. (2000) it was shown that the facet-inducing *extended comb* inequalities (see Jünger et al. 1995) are mod-2 cuts. The simplest extended comb inequality which is not an ordinary comb inequality is defined for STSP(8). It can be shown (by exhaustive enumeration) that this inequality is not a DP inequality.

It should also be pointed out that the class of DP inequalities is not as "clean" as the class of comb inequalities, in the following sense. There are DP inequalities which are not facet-inducing and which are dominated by inequalities which are not DP inequalities (Boyd 1999).

We close this section with a definition and a couple of useful lemmas.

Definition. For a given x^*, the *weight* of a domino $\{A, B\}$, denoted by $w(A, B)$, is the quantity $2|A \cup B| - 3 - 2x^*(E(A \cup B)) + x^*(E(A:B))$.

Note that $w(A, B)$ is the sum of the slacks of the three SECs on A, B and $A \cup B$.

Lemma 1. *If $x^* \in SEP(n)$, the slack of a DP inequality is equal to*

$$\left(x^*(F) - 1 + \sum_{j=1}^{p} w(A_j, B_j) \right) \Big/ 2.$$

Proof. The slack of (8) is obviously equal to $\sum_{j=1}^{p} w(A_j, B_j)$. This implies that the slack of (9) equals $x^*(F) + \sum_{j=1}^{p} w(A_j, B_j)$. Since $x^* \in \mathrm{SEP}(n)$, the equation (10) must be satisfied. Hence, the slack of (9) and (11) are identical. The result follows from the fact that the DP inequality is obtained from (11) by dividing by two and subtracting $\frac{1}{2}$ from the right-hand side.

Obviously, a DP inequality is violated if and only if its slack is negative. Therefore:

LEMMA 2. *If $x^* \in SEP(n)$, every domino used in the derivation of a violated DP inequality must have weight less than* 1.

PROOF. The weight of a domino is the sum of the slacks of three SECs. Therefore, if $x^* \in \mathrm{SEP}(n)$, all dominoes have nonnegative weight and, of course, $x^*_e \geq 0$ for all $e \in E$. The result then follows from Lemma 1.

3. The separation algorithm. In this section we show that DP inequalities can be separated in polynomial time provided that $x^* \in \mathrm{SEP}(n)$ and G^* is planar. The proof has two main strands. First, we build on the results in Fleischer and Tardos (1999) to show that "useful" dominoes have certain desirable properties when G^* is planar. Then, we use planar duality to transform the separation problem into a problem which is already known to be well solved.

Throughout this section, we use the notation $E^* := \{e \in E : x^*_e > 0\}$ and $E^*(A_j : B_j) = E(A_j : B_j) \cap E^*$. We begin with a definition and a lemma.

DEFINITION. A set $\{E_1, \ldots, E_r\}$ with $E_j \subset E^*$ for $j = 1, \ldots, r$ is said to *support a cut in G^** if the edges e with μ_e odd form a cut in G^*.

LEMMA 3. *Suppose $x^* \in SEP(n)$. A DP inequality is violated if and only if there is an odd integer $p \geq 1$, dominoes $\{A_1, B_1\}, \ldots, \{A_p, B_p\}$ and a set $K \subseteq E^*$ of edges such that*:
- *$\{E^*(A_1 : B_1), \ldots, E^*(A_p : B_p), K\}$ supports a cut in G^*,*
- *$\sum_{j=1}^{p} w(A_j, B_j) + x^*(K) < 1$.*

PROOF. The necessity of the first condition follows from the definitions and the necessity of the second condition follows from Lemma 1. So we need only prove sufficiency. If the first condition holds, set H to be one shore of the cut which is supported in G^* by $\{E^*(A_1 : B_1), \ldots, E^*(A_p : B_p), K\}$. There is then a unique $F \subseteq E$ such that $\{E(A_1 : B_1), \ldots, E(A_p : B_p), F\}$ supports the cut $\delta(H)$ in G. Now note that F satisfies $K = F \cap E^*$ and, therefore, $x^*(F \backslash K) = 0$. Thus, if the second condition holds, the resulting DP inequality is violated by Lemma 1.

Note that the $p + 1$ edge sets of Lemma 3 are not required to be disjoint, as they are in the case of a comb inequality.

In light of Lemmas 2 and 3, dominoes which have a weight less than 1 are of particular interest. Next we show that such dominoes have a special structure.

LEMMA 4. *If $x^* \in SEP(n)$ and $\{A, B\}$ is a domino such that $w(A, B) < 1$, then the SECs for A, B and $A \cup B$ must all have slack less than* 1.

PROOF. By definition, $w(A, B)$ is the sum of the slacks of the three SECs mentioned. Since $x^* \in SEP(n)$, all of these three slacks are nonnegative and the result follows.

LEMMA 5. *If $x^* \in SEP(n)$ and $\{A, B\}$ is a domino such that $w(A, B) < 1$, then the subgraphs induced in G^* by A, B and $A \cup B$ are all connected.*

PROOF. We prove it only for A. The proofs for B and $A \cup B$ are similar. By Lemma 4, the slack of the SEC for A must be less than 1. If the subgraph induced by A is *not* connected, then A can be partitioned into A_1, A_2 in such a way that $x^*(E(A_1 : A_2)) = 0$. Also, since $x^* \in SEP(n)$, $x^*(E(A_1)) \leq |A_1| - 1$ and $x^*(E(A_2)) \leq |A_2| - 1$ holds. But this implies that $x^*(E(A)) = x^*(E(A_1)) + x^*(E(A_2)) + x^*(E(A_1 : A_2)) \leq |A| - 2$, contradicting the fact that the slack of the SEC on A is less than 1.

Next, we invoke planar duality. We denote by $\bar{G}^*$ the planar dual of G^*; $\bar{E}^*$, $\bar{e}$, etc., are defined accordingly. It will be assumed that $\bar{G}^*$ is edge-weighted; namely, that each $\bar{e} \in \bar{E}^*$ is given a weight of x^*_e.

Lemma 6. *Suppose that G^* is planar, $x^* \in SEP(n)$ and $\{A,B\}$ is a domino such that $w(A,B) < 1$. Then there are two distinct vertices s and t in $\bar{G}^*$ such that each of the following edge-sets is an (s,t)-path in $\bar{G}^*$:*
- $\bar{E}^*(A:B)$,
- $\bar{E}^*(A:V\backslash(A \cup B))$,
- $\bar{E}^*(B:V\backslash(A \cup B))$.

These three (s,t)-paths are edge-disjoint and also have no vertices in common other than s and t.

Proof. Lemma 5 implies that $\bar{\delta}^*(A \cup B)$ forms a cycle in $\bar{G}^*$. This cycle divides the plane into two regions, one containing $A \cup B$ and one containing $V\backslash(A \cup B)$. The edges in $\bar{E}^*(A:B)$ must form a path dividing the first of these regions into two planar regions: one containing A and the other containing B. Let s be where one end of the path meets $\bar{\delta}^*(A \cup B)$ and let t be where the other end of the path meets $\bar{\delta}^*(A \cup B)$. The result follows from the fact that $\bar{\delta}^*(A \cup B) = \bar{E}^*(A:V\backslash(A \cup B)) \cup \bar{E}^*(B:V\backslash(A \cup B))$.

The following lemma allows us to express the weight of a domino directly in terms of the three paths given by Lemma 6.

Lemma 7. *When $x^* \in SEP(n)$ and G^* is planar, the weight of a domino $\{A,B\}$ is equal to the sum of the weights of the edges in the three paths mentioned in Lemma* 6, *minus* 3.

Proof. The weight was defined in §2 as $w(A,B) := 2|A \cup B| - 3 - 2x^*(E(A \cup B)) + x^*(E(A:B))$. But when $x^* \in \mathrm{SEP}(n)$, the degree equations for $i \in A \cup B$ imply the identity $2|A \cup B| = 2x^*(E(A \cup B)) + x^*(\delta(A \cup B))$. Substituting for $|A \cup B|$ then yields $w(A,B) = x^*(E^*(A:B)) + x^*(\delta^*(A \cup B)) - 3 = x^*(E^*(A:B)) + x^*(E^*(A:V\backslash(A \cup B))) + x^*(E^*(B:V\backslash(A \cup B))) - 3$ and the result follows from the fact that the weight of a dual edge $\bar{e}$ is x^*_e.

We will call the path formed by $\bar{E}^*(A:B)$ a *domino path*.

The next step in the argument is to formulate a dual version of Lemma 3. A graph is *Eulerian* if every vertex has even degree. It is well known (see, e.g., Orlova and Dorfman 1972; Hadlock 1975), that a cut in a planar graph corresponds to an Eulerian subgraph in the dual and vice-versa. (If the removal of the edges of the cut disconnects the graph into more than two components, then the corresponding Eulerian subgraph will also be disconnected. However, the arguments which follow are valid even if this is the case.) This suggests that the following definition will be useful.

Definition. Let r be a positive integer and suppose that $E_1, \ldots, E_r$ are edge-sets satisfying $E_j \subset E^*$ for $j = 1, \ldots, r$. The collection $\{\bar{E}_1, \ldots, \bar{E}_r\}$ is said to *support an Eulerian subgraph in* $\bar{G}^*$ if the edges $\bar{e}$ for which μ_e is odd form an Eulerian subgraph in $\bar{G}^*$.

The definition implies that $\{\bar{E}_1, \ldots, \bar{E}_r\}$ supports an Eulerian subgraph in $\bar{G}^*$ if and only if $\{E_1, \ldots, E_r\}$ supports a cut in G^*. This allows us to formulate the following dual form of Lemma 3.

Lemma 8. *Suppose $x^* \in SEP(n)$ and G^* is planar. A DP inequality is violated if and only if there are an odd number of domino paths and a set $\bar{K} \subseteq \bar{E}^*$ such that*:

- *The domino paths, together with the edges in $\bar{K}$, support an Eulerian subgraph in $\bar{G}^*$.*
- *The weight of the dominoes associated with the domino paths and the weight of the edges in $\bar{K}$ sum to less than* 1.

PROOF. Follows from Lemma 3 and the definitions.

The problem now is that there could be a large number of domino paths and we do not know how to find them efficiently. However, our task is made easier by the following observation.

LEMMA 9. *Suppose that a set of domino paths and a set* $\bar{K} \subseteq \bar{E}^*$ *support an Eulerian subgraph in* $\bar{G}^*$ *as required by Lemma* 8. *Suppose that* $\{A, B\}$ *is a domino corresponding to one of these domino paths. Let* s *and* t *be vertices in* $\bar{G}^*$ *such that the domino path corresponding to* $\{A, B\}$ *is an* (s, t)*-path in* $\bar{G}^*$. *Suppose that* $\{A', B'\}$ *is another domino whose domino path is also an* (s, t)*-path in* $\bar{G}^*$. *Then the set of domino paths obtained by replacing* $\{A, B\}$ *with* $\{A', B'\}$, *together with* $\bar{K}$, *also supports an Eulerian subgraph in* $\bar{G}^*$.

PROOF. Any domino path from s to t meets every vertex apart from s and t an even number of times. Hence, the domino paths for $\{A, B\}$ and $\{A', B'\}$ are interchangeable.

This implies that, for a fixed s and t, we need only consider *one* domino whose domino path is an (s, t)-path; namely, one of minimum weight. Lemma 7 implies that we can find domino by finding three edge-disjoint (s, t)-paths in $\bar{G}^*$ of minimum total weight. This can be done by solving a minimum cost flow problem: set all edge capacities to 1 and send a flow of 3 units from s to t. To solve such a minimum cost flow problem, it suffices (see, e.g., Orlin 1993) to compute three shortest augmenting paths in $\bar{G}^*$. Since $\bar{G}^*$ is planar, each of these shortest augmenting path problems can be solved in $\mathcal{O}(n)$ time using the algorithm of Henziger et al. (1997).

One apparent complication is that the flow algorithm might return a set of 3 (s, t)-paths such that 2 of them share vertices other than s and t, which should not be permitted if we want the paths to represent a domino. However, it is easy to show that, if this occurs, the two paths concerned will contain at least two edge-disjoint cycles, and therefore the weight of the domino will be at least 1. Thus, if a 'bad' set of (s, t)-paths is returned, then there is no domino with weight less than 1 whose domino path connects s and t.

By repeating this procedure $\mathcal{O}(n^2)$ times, once for each pair (s, t), we obtain a set of $\mathcal{O}(n^2)$ dominoes such that a DP inequality with maximum violation, if any, uses only dominoes in that set. We call these the *optimal* dominoes.

Now we define a labelled supergraph of $\bar{G}^*$, denoted by M^*, which will prove to be useful in what follows. First, label each edge of $\bar{E}^*$ *even*. Then, for each vertex pair (s, t), add an edge connecting s and t, labelled *odd*, with weight equal to the weight of the optimal domino whose domino path is an (s, t)-path. We can express Lemma 8 in terms of M^* as follows.

THEOREM 2. *A DP inequality is violated if and only if there is an odd cycle* (*i.e.*, *a cycle containing an odd number of odd edges*) *in* M^*, *of weight less than* 1.

PROOF. First we prove sufficiency. Suppose that such an odd cycle exists. By replacing each odd edge in the cycle by the corresponding domino path, we obtain a set of domino paths and ordinary (even) edges. This set of domino paths and edges supports and Eulerian subgraph in M^* and therefore in $\bar{G}^*$. Therefore a DP inequality is violated by Lemma 8. Next, necessity. Suppose that a DP inequality is violated. Then, by Lemma 8, there is a set of domino paths and a set of edges, of total weight less than 1, which supports an Eulerian subgraph in $\bar{G}^*$. By replacing each domino path with the corresponding (odd) edge, we obtain an Eulerian subgraph in M^*, which may or may not be connected. At least one connected component must contain an odd number of odd edges. A traversal of this connected component is an odd cycle of weight less than 1.

The problem of finding a minimum weight odd cycle in M^* can be solved in $\mathcal{O}(n^3)$ time by solving $\mathcal{O}(n)$ shortest path problems in a suitable graph (see Gerards and Schrijver, 1986). This leads to the main result of this paper:

THEOREM 3. *DP inequalities can be separated in $\mathcal{O}(n^3)$ time when $x^* \in SEP(n)$ and G^* is planar.*

PROOF. Finding the optimal dominoes takes $\mathcal{O}(n^3)$ time. Once this is done, constructing M^* only takes $\mathcal{O}(n^2)$ time. Finally, finding the minimum weight odd cycle takes $\mathcal{O}(n^3)$ time.

4. Application to the graphical TSP. An important variant of the STSP is the so-called *Graphical Travelling Salesman Problem* or GTSP (Cornuéjols et al. 1985, Naddef and Rinaldi 1992). The GTSP is a relaxation of the STSP in that it is permitted to visit vertices more than once and to traverse edges more than once. Also, it is not necessary that $G(V, E)$ be complete. The GTSP can be expressed as:

$$\min c^T x$$

subject to

$$x(\delta(S)) \geq 2 \quad (\forall S \subset V : 1 \leq |S| \leq |V| - 1), \tag{12}$$

$$x(\delta(i)) \quad \text{is even } (\forall i \in V), \tag{13}$$

$$x_e \geq 0 \quad (\forall e \in E), \tag{14}$$

$$x \in Z^{|E|}. \tag{15}$$

We will call (12) *connectivity* inequalities and (13) *evenness conditions*. The convex hull of feasible solutions to (12)–(15) is a polyhedron known as GTSP(G).

Since there is no upper bound on x_e for any $e \in E$, GTSP(G) is unbounded. Therefore, the 2-matching, comb and DP inequalities are not valid for GTSP(G) in the form in which they were presented in §§1 and 2. However, it is possible to use the degree equations (1) to rewrite the comb inequalities as:

$$x(\delta(H)) + \sum_{j=1}^{p} x(\delta(T_j)) \geq 3p + 1. \tag{16}$$

In this form, they *are* valid for GTSP(G), and facet-inducing under certain mild conditions (Cornuéjols et al. 1985).

It turns out that we can apply a similar idea to the DP inequalities. With a little work, it can be shown using the degree equations that the DP inequality (7) is equivalent to:

$$\sum_{j=1}^{p} x(\delta(A_j \cup B_j)) + \sum_{e \in E} \mu_e x_e \geq 3p + 1. \tag{17}$$

In this form, DP inequalities are valid for GTSP(G).

THEOREM 4. *The DP inequality, when written in the form* (17), *is valid for GTSP*(G).

PROOF. Sum the connectivity inequalities for A_j, B_j, and $A_j \cup B_j$ for $j = 1, \ldots, p$ and divide by two to obtain:

$$\sum_{j=1}^{p} x(\delta(A_j \cup B_j)) + \sum_{j=1}^{p} x(E(A_j : B_j)) \geq 3p.$$

Add the nonnegativity inequalities (14) for the edges $e \in F$ to obtain:

$$\sum_{j=1}^{p} x(\delta(A_j \cup B_j)) + \sum_{e \in E} \mu_e x_e \geq 3p.$$

Hence, we have that the left-hand side of (17) is at least $3p$. But, the left-hand side of (17) is equal to:

$$x(\delta(H)) + \sum_{j=1}^{p} x(\delta(A_j \cup B_j)) + \sum_{e \in \delta(H)} (\mu_e - 1)x_e + \sum_{e \in E \setminus \delta(H)} \mu_e x_e. \tag{18}$$

The evenness conditions imply that, in any feasible GTSP solution, $x(\delta(H))$ will be an even integer and so will $x(\delta(A_j \cup B_j))$ for $j = 1, \ldots, p$. Moreover, the set F is chosen so that the remaining terms on the left-hand side of (18) must all be even integers in any feasible solution. Therefore the left-hand side of (17) must be an even integer which is not less than $3p$ and the result follows.

It turns out that the separation algorithm presented in §3 can be adapted to separate the inequalities (17) when solving the GTSP. The only thing which needs to be changed is the assumption that $x^* \in \mathrm{SEP}(n)$. Instead, we need the assumption that x^* satisfies all connectivity and nonnegativity inequalities. Provided that the weight of a domino is defined as in Lemma 7, the argument carries through unchanged. This leads to the following result.

THEOREM 5. *If a GTSP instance is defined on a planar graph G, then it is possible to optimize in polynomial time over the polyhedron defined by all connectivity, nonnegativity and DP inequalities.*

PROOF. The separation problem for nonnegativity inequalities is trivial. The separation problem for connectivity inequalities is a minimum weight cut problem and therefore solvable in polynomial time. Once x^* is known to satisfy the nonnegativity and connectivity inequalities, the separation problem for DP inequalities can be solved in polynomial time as described in this paper. The result follows from the polynomial equivalence of separation and optimization (Grötschel et al. 1988).

5. Discussion. It has been shown that the DP inequalities are a genuine generalization of the comb inequalities and that they can be separated in $\mathcal{O}(n^3)$ time when G^* is planar. It has also been shown how to adapt the results to the GTSP. The results in this paper can also be applied to the *Asymmetric Travelling Salesman Problem* with minor modification: given a DP inequality for STSP(n), one obtains an obvious valid inequality for the corresponding *Asymmetric Travelling Salesman Polytope*, ATSP(n), by giving each directed arc $[i, j]$ the same coefficient as the corresponding undirected edge $\{i, j\}$. The resulting class of valid inequalities is separated in an obvious way.

There are several remaining issues for further research. Of particular interest to the author are the following:

• Is there a polynomial separation algorithm for the *exact* class of comb inequalities, either in the planar case, or in general? (Boyd 1999 has shown that the DP separation algorithm of this paper can return a violated DP inequality even when no violated comb inequality exists, so the DP separation algorithm certainly cannot be used for this purpose.)

• Is the separation problem for *mod*-2 *cuts* for the STSP solvable in polynomial time, either in general, or on special classes of graphs? (Mod-2 separation is known to be $\mathcal{NP}$-hard for general integer programs; see Caprara and Fischetti 1996.)

• If the costs c_e satisfy the triangle inequality, what is the worst-case ratio of the lower bound obtained using SECs, nonnegativity and DP inequalities to the value of the optimal STSP solution? The worst example currently known to the author is $5/6$.

• Chvátal (1973a) calls a graph *weakly Hamiltonian* if the system of degree equations, nonnegativity inequalities, SECs and Chvátal combs has a solution when restricted to the edges in the given graph. In the same paper, various properties are proved for weakly Hamiltonian graphs. What happens if we add the DP inequalities to the system? (The implication is that, for certain planar graphs, one can quickly prove that they are non-Hamiltonian.)

• Under what conditions can a *nonplanar* support graph G^* be *shrunk* (see Padberg and Rinaldi, 1990) without losing any violated DP inequalities?

• Are there more general classes of graphs for which the DP separation problem can be solved in polynomial time?

This paper will conclude with some thoughts on the last of these questions.

The duality between cuts and Eulerian subgraphs, so crucial to the proofs in §3, is also behind a classical proof that the *max-cut problem* is polynomially solvable in planar graphs (Orlova and Dorfman 1972, Hadlock 1975). More recent papers show that the max-cut problem is solvable on graphs which are not contractible to the complete graph K_5 (Barahona 1983) and also on the so-called *weakly bipartite* graphs (Grötschel and Pulleyblank 1981). Planar graphs are not contractible to K_5 by the classical Kuratowski Theorem (see, e.g., Barahona 1983); recently it was shown that graphs not contractible to K_5 are weakly bipartite (Fonlupt et al. 1992). The author therefore believes that a good place to start would be to attempt to find a polynomial DP separation algorithm for graphs in one of these two classes.

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