BILEVEL PROGRAMMING: INTRODUCTION, HISTORY, AND OVERVIEW, *BP*

The **bilevel programming** (BP) problem is a **hierarchical optimization** problem where a subset of the variables is constrained to be a solution of a given optimization problem parameterized by the remaining variables. The BP problem is a **multilevel programming** problem with two levels. The hierarchical optimization sctructure appears naturally in many applications when lower level actions depend on upper level decisions. The applications of bilevel and multilevel programming include *transportation* (taxation, network design, trip demand estimation), *management* (coordination of multi-divisional firms, network facility location, credit allocation), *planning* (agricultural policies, electric utility), and *optimal design*.

In mathematical terms, the BP problem consists of finding a solution to the upper level problem

$$\begin{aligned} \text{minimize}_{x,y} \quad & F(x,y) \\ \text{subject to} \quad & g(x,y) \leq 0, \end{aligned}$$

where y, for each value of x, is the solution of the lower level problem:

$$\begin{aligned} \text{minimize}_{y} \quad & f(x,y) \\ \text{subject to} \quad & h(x,y) \leq 0, \end{aligned}$$

with $x \in \Re^{nx}$, $y \in \Re^{ny}$, $F,\ f : \Re^{nx+ny} \to \Re$, $g : \Re^{nx+ny} \to \Re^{nu}$, and $h : \Re^{nx+ny} \to \Re^{nl}$. The lower level problem is also referred as the follower's problem or the inner problem. In a similar way, the upper level problem is also called the leader's problem or the outer problem. One could generalize the BP problem in different ways. For instance, if either x or y or both are restricted to take integer values we would obtain an integer BP problem [20]. Or, if we replace the lower level problem by a variational inequality we would get a generalized BP problem [14].

For each value of the upper level variables x, the lower level constraints $h(x,y) \leq 0$ define the constraint set $\Omega(x)$ of the lower level problem:

$$\Omega(x) \ = \ \Big\{ y : \ h(x,y) \leq 0 \Big\}.$$

Then, the set $M(x)$ of solutions for the lower level problem is given by minimizing the lower level function $f(x,y)$ for all values in $\Omega(x)$ of the lower level variables y:

$$M(x) = \Big\{ y : \ y \in argmin\{ f(x,y) : \ y \in \Omega(x) \} \Big\}.$$

Given these definitions the BP problem can be reformulated as:

$$\begin{aligned} \text{minimize}_{x,y} \quad & F(x,y) \\ \text{subject to} \quad & g(x,y) \leq 0, \\ & y \in M(x). \end{aligned}$$

The feasible set

$$\Big\{ (x,y) : \ g(x,y) \leq 0, \ y \in M(x) \Big\}.$$

of the BP problem is called the induced or inducible region. The induced region is usually nonconvex and, in the presence of upper level constraints, can be disconnected or even empty. In fact, consider the following BP problem

$$\begin{aligned} \text{minimize}_{x,y} \quad & x - 2y \\ \text{subject to} \quad & -x + 3y - 4 \leq 0, \end{aligned}$$

where y, for each value of x, is the solution of:

$$\begin{aligned} \text{minimize}_{y} \quad & x + y \\ \text{subject to} \quad & \begin{array}{rcl} x - y & \leq & 0, \\ -x - y & \leq & 0. \end{array} \end{aligned}$$

For this problem we have:

$$\Omega(x) \ = \ \{ y : \ y \geq |x| \},$$

and

$$M(x) \ = \ |x|.$$

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bilevel programming
hierarchical optimization
multilevel programming
transportation
management
planning
optimal design

Thus, the induced region is given by:

$$\{(x,y):\ -x+3y-4\le 0,\ y\in M(x)\} =$$
$$\{(x,y):\ y=-x,\ -1\le x\le 0\}$$
$$\cup$$
$$\{(x,y):\ y=x,\ 0\le x\le 2\},$$

which is nonconvex but connected. If the upper level constraints were changed to

$$\begin{aligned} -x+3y-4 &\le 0, \\ -y+\tfrac{1}{2} &\le 0, \end{aligned}$$

then the induced region would become

$$\{(x,y):\ y=-x,\ -1\le x\le -\tfrac{1}{2}\}$$
$$\cup$$
$$\{(x,y):\ y=x,\ \tfrac{1}{2}\le x\le 2\},$$

which would be a disconnected set. In either case the BP problem has two local minimizers $(-1,1)$ and $(2,2)$ and one global minimizer $(2,2)$.

This simple example illustrates many features of bilevel programming like the nonconvexity and the disconnectedness of the induced region and the existence of different local minimizers. In this example the induced region is compact. In fact, compactness of the induced region is important for the existence of a global minimizer and can be guaranteed under appropriate conditions [8].

The original formulation for bilevel programming appeared in 1973, in a paper authored by J. Bracken and J. McGill [4], although is was W. Candler and R. Norton [6] that first used the designation bilevel and multilevel programming. However, it was not until the early eighties that these problems started receiving the attention they deserve. Motivated by the **game theory** of `H. Stackelberg` [19], several authors studied bilevel programming intensively and contributed to its proliferation in the mathematical programming community.

The theory of bilevel programming focuses on forms of optimality conditions and complexity results. A number of authors ([7], [15], just to cite a few) have established original forms of optimality conditions for bilevel programming by either considering reformulations of the BP problem or by making use of **nondifferentiable optimization** concepts or even by appealing to the geometry of the induced region. The complexity of the problem has been addressed by a number of authors. It has been proved that even the linear BP problem, where all the involved functions are affine, is a strongly NP-Hard problem [9]. It is not hard to construct a linear BP problem where the number of local minima grows exponentially with the number of variables [5]. Other theoretical results of interest have been established connecting bilevel programming to other fields in mathematical programming. For instance, one can show that minimax problems and linear, integer, bilinear and quadratic programming problems are special cases of BP. Other classes of problems different from but related to BP are **multiobjective optimization** problems and static Stackelberg problems. See [21] for references in these topics.

Many researchers have designed algorithms for the solution of the BP problem. One class of techniques consists of extreme point algorithms and has been mostly applied to the linear BP problem because for this problem, if there is a solution, then there is at least one global minimizer that is an extreme point of Ω [16]. Two other classes of algorithms are branch and bound algorithms and complementarity pivot algorithms that have in common the fact that exploit the complementarity part of the necessary optimality conditions of the lower level problem (assumed convex in y so that the necessary optimality conditions, under an appropriate constraint qualification, are also sufficient). These two classes of algorithms have been applied mostly to the case where the upper level is linear and the lower level is linear or convex quadratic (see for instance [9] and [11]) and, as the extreme point algorithms, find a global minimizer of the BP problem. On the other hand, the algorithms designed to solve nonlinear forms

game theory
`H. Stackelberg`
nondifferentiable optimization
multiobjective optimization

of BP appeal to descent directions (see, among others [13] and [17]) and penalty functions (for instance [1]) and are expected to find a local minimizer.

The reader can find more about bilevel programming in the book by Shimizu, Ishizuka, and Bard [18], in the survey papers [2], [3], [10], [12], [22], and in the bibliography review by Vicente and Calamai [21].

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