Bilevel and Multilevel Programming:

A Bibliography Review

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Abstract.
This paper contains a bibliography of all references central to bilevel and multilevel programming that the authors know of. It should be regarded as a dynamic and permanent contribution since, all the new and appropriate references that are brought to our attention will be periodically added to this bibliography. Readers are invited to suggest such additions, as well as corrections or modifications, and to obtain a copy of the LaTeX and BibTeX files that constitute this manuscript, using the guidelines contained in this paper.

To classify some of the references in this bibliography a short overview of past and current research in bilevel and multilevel programming is included. For those who are interested in but unfamiliar with the references in this area, we hope that this bibliography facilitates and encourages their research.

Keywords. Bilevel (two level), three level and multilevel programming, static Stackelberg problems, hierarchical optimization, minimax problems.

AMS subject classification. 90C26, 90C30, 90C31
1 Introduction and historical notes

Multilevel optimization problems are mathematical programs which have a subset of their variables constrained to be an optimal solution of other programs parameterized by their remaining variables. When these other programs are pure mathematical programs we are dealing with bilevel programming. Three level programming results when these other programs are themselves bilevel programs. By extending this idea it is possible to define multilevel programs with any number of levels.

The (continuous) bilevel programming problem (BPP) is defined as:

\[
\begin{align*}
\min_{x,y} & \quad F(x,y) \\
\text{subject to} & \quad g(x,y) \leq 0,
\end{align*}
\]

where \( y \), for each value of \( x \), is the solution of the so-called lower level problem:

\[
\begin{align*}
\min_y & \quad f(x,y) \\
\text{subject to} & \quad h(x,y) \leq 0,
\end{align*}
\]

with \( x \in \mathbb{R}^{nx}, y \in \mathbb{R}^{ny}, F, f : \mathbb{R}^{nx+ny} \to \mathbb{R}, g : \mathbb{R}^{nx+ny} \to \mathbb{R}^{nu} \) and \( h : \mathbb{R}^{nx+ny} \to \mathbb{R}^{nl} \). Variables \( x \) (\( y \)) are called the upper (respectively lower) level variables, \( g(x,y) \leq 0 \) (\( h(x,y) \leq 0 \)) the upper (lower) level constraints and \( F(x,y) \) (\( f(x,y) \)) the upper (lower) level objective function. Furthermore the relaxed problem associated with BPP can be stated as:

\[
\begin{align*}
\min_{x,y} & \quad F(x,y) \\
\text{subject to} & \quad g(x,y) \leq 0, \quad h(x,y) \leq 0,
\end{align*}
\]

and its optimal value is a lower bound for the optimal value of the BPP. Other important BPP definitions and notations are itemized below.

- the relaxed feasible set (or constraint region),
  \[
  \Omega = \{(x,y) : g(x,y) \leq 0, h(x,y) \leq 0\}.
  \]

- for each \( x \), the lower level feasible set,
  \[
  \Omega(x) = \{y : h(x,y) \leq 0\}.
  \]

- for each \( x \), the lower level reaction set (follower’s feasible region),
  \[
  M(x) = \{y : y \in \arg\min\{f(x,y) : y \in \Omega(x)\}\}.
  \]

- for each \( x \) and any value of \( y \) in \( M(x) \), the lower level optimal value,
  \[
  v(x) = f(x,y)
  \]
The induced region is the feasible set of the BPP. It is usually nonconvex and, in the presence of upper level constraints, can be disconnected or even empty. The reader is referred to T. Edmunds and J. Bard [73] for a short description of the conditions under which the induced region is compact and the BPP has an optimal solution.

The BPP is convex if \( f(x, y) \) and \( h(x, y) \) are convex functions in \( y \) for all values of \( x \) (i.e., if the lower level problem is convex). The convex BPP has received most of the attention in the literature. The advantage of dealing with the convex BPP is that under an appropriate constraint qualification, the lower level problem can be replaced by its Karush-Kuhn-Tucker (KKT) conditions to obtain an equivalent (one-level) mathematical program. However, despite their designation, convex BPPs have nonconvex induced regions that can be disconnected or even empty in the presence of upper level constraints. There are three important classes of convex BPPs, namely:

- the linear BPP, where all functions involved are affine.
- the linear-quadratic BPP, where the lower level objective is a convex quadratic and all remaining functions are affine.
- the quadratic BPP, that differs from the linear-quadratic BPP in that the upper level objective is also a quadratic function.

The original formulation for bilevel programming appeared in 1973, in a paper authored by J. Bracken and J. McGill [41], although it was W. Candler and R. Norton [51] that first used the designation *bilevel* and *multilevel programming*. However, it was not until the early eighties that these problems started receiving the attention they deserve. Motivated by the game theory of H. Stackelberg [150], several authors studied bilevel programming intensively and contributed to its proliferation in the mathematical programming community. At this stage, references such as E. Aiyoshi and K. Shimizu [1], [146], [147], J. Bard and J. Falk [11], [20], W. Bialas, H. Karwan and J. Shaw [34], [35], [37], W. Candler, J. Fortuny-Amat, B. McCarl, R. Norton and R. Townley [50], [51], [52], [53], [77] and U. Wen [163] should be distinguished.

Since 1980 a significant effort has been devoted to understanding the fundamental concepts associated with bilevel programs. At the same time several algorithms have been proposed for solving these problems. Important surveys of these efforts include those by C. Kolstad [99], G. Savard [140] and G. Anandalingam and T. Friesz [8]. Recently, a survey on the linear case has been written by O. Ben-Ayed [25].
2 Properties of bilevel programs

It is our opinion that bilevel programming represents an interesting and rich field of mathematical programming and although some important results have already been obtained it is still a fertile area for research. In this section we list some of the well-known properties of the BPP.

Optimality conditions

Several different optimality conditions have been proposed in the literature.

A first attempt was made by J. Bard [16] using an equivalence with a one-level mathematical program having an infinite and parametric set of constraints. However a counter example to these conditions was discovered by P. Clarke and A. Westerberg [61] and by A. Haurie, G. Savard and D. White [85]. Consequently, two algorithms based on these conditions (proposed in [13], [14] and [158]) are not convergent (see [140]).

Y. Chen and M. Florian [57], S. Dempe [66, 67], Y. Ishizuka [88], J. Outrata [135], and J. Ye and D. Zhu [174] used nonsmooth analysis, whereas Z. Bi and P. Calamai [31] explored the relationship between the BPP and an associated exact penalty function, to derive other necessary and sufficient optimality conditions.

Unlike much of the optimality analysis that has been done for (one-level) mathematical programs these aforementioned contributions have mostly ignored the special geometry of the BPP. To partially address this void G. Savard and J. Gauvin [141] have proposed necessary optimality conditions based on the concept of the steepest descent direction. More directly, L. Vicente and P. Calamai [160] have proposed necessary and sufficient optimality conditions, based on the geometry of the BPP, that are generalizations of the well-known first and second order optimality conditions for mathematical programs.

Complexity

The difficulty and complexity of the BPP is easily confirmed by looking at what might be considered its simplest version, the linear BPP. Examples of linear BPPs with an exponential number of local minima can be generated using a technique proposed by P. Calamai and L. Vicente [48]. R. Jeroslow [92] showed that the linear BPP is NP-Hard. A few years later, J. Bard [19] and O. Ben-Ayed and C. Blair [26], confirmed this result and presented shorter proves. The tightest complexity result is due to P. Hansen, B. Jaumard and G. Savard [81], where it is established that the linear BPP is strongly NP-Hard. Recently, L. Vicente, G. Savard and J. Júdice [162] have shown that checking local optimality in a linear BPP is a NP-Hard problem.
Related problems

The fact that important mathematical programs, such as minimax problems, linear integer, bilinear and quadratic programs, can be stated as special instances of bilevel programs illustrates the importance of these problems.

Although it is a simple matter to see that a minimax problem can be rewritten as a BPP problem, the first authors exploiting the reduction of a bilinear program to a linear BPP were G. Gallo and A. Ülküçü [80]. This result also established that any integer or concave quadratic program could be written as a linear BPP. One might think that any linear BPP can also be reduced to a bilinear program, thereby establishing an equivalence between these problems. However this is not entirely possible since the reciprocal result states that there exists a (penalized) bilinear program whose optimal (global) solutions are also global solutions of the corresponding linear BPP (see [171]). Finally, the reduction of any quadratic program to a quadratic BPP with bilinear objective functions is described in L. Vicente [159].

Although several authors have attempted to establish a link between two objective optimization and bilevel programming (J. Bard [16] and G. Ünlü [158]), none have succeeded thus far in proposing conditions that guarantee that the optimal solution of a given bilevel program is Pareto-optimal or efficient [85] for both upper and lower level objective functions (W. Candler [49], P. Clarke and A. Westerberg [61], A. Haurie, G. Savard and D. White [85], P. Marcotte [115], P. Marcotte and G. Savard [117] and U. Wen and S. Hsu [167]).

The static Stackelberg problem (SSP) can be posed as:

$$\begin{align*}
\min_x & \quad F(x, y) \\
\text{subject to} & \quad g(x, y) \leq 0, \\
& \quad y \in \arg\min \{f(x, y) : h(x, y) \leq 0\},
\end{align*}$$

and differs from the BPP in the way the upper level function is minimized. If the reaction set \( \{y : y \in \arg\min \{f(x, y) : h(x, y) \leq 0\}\} \) is not a singleton for some values of \( x \) with \( g(x, y) \leq 0 \), then a solution of the SSP might not be a solution of the BPP. Comments on this problem and its relationships with game theory can be found in [140].

Other two-level optimization problems might also be confused with bilevel programs. That is the case with the following problem studied by T. Tanino and T. Ogawa [154].

$$\begin{align*}
\min_{x,y} & \quad \mathcal{F}(x) = F(x, v(x)) \\
\text{subject to} & \quad G(x) \leq 0,
\end{align*}$$

where \( y \), for each value of \( x \), is the solution of the second optimization problem:

$$\begin{align*}
\min_y & \quad f(x, y) \\
\text{subject to} & \quad h(x, y) \leq 0,
\end{align*}$$
\( v(x) \) is the optimal value of the second problem parameterized by \( x \), and \( G : \mathbb{R}^{nx} \rightarrow \mathbb{R}^{m} \). Under certain convexity and differentiability assumptions these authors have demonstrated that this problem can be treated as a one-level convex optimization problem and proposed a descent algorithm for its solution.

Authors who have studied generalized bilevel programming problems include T. Friesz et. al. [79], J. Outrata [136] and P. Marcotte and D. Zhu [119] who replaced the BPP lower level problem with a variational inequality problem.

3 Solution of bilevel programs

The algorithms that have been proposed for solving continuous bilevel programming problems may be divided in five different classes. In most cases these algorithms can be tested and compared using the test problem generators proposed by P. Calamai and L. Vicente [46], [48], [47] for generating linear, linear–quadratic and quadratic BPPs.

Extreme point algorithms

Most of these algorithms are applied to the solution of linear BPPs. Every linear BPP with a finite optimal solution shares the important property that at least one optimal (global) solution is attained at an extreme point of the set \( \Omega \). This result was first established by W. Candler and R. Townsley [53] for linear BPPs with no upper level constraints and with unique lower level solutions. Afterwards J. Bard [15] and W. Bialas and M. Karwan [36] proved it under the assumption that \( \Omega \) is bounded. The result for the case where upper level constraints exist has been established by G. Savard [140] under no particular assumptions. We remark that this property is no longer valid for linear–quadratic BPPs.

Based on this property, W. Candler and R. Townsley [53] and W. Bialas and M. Karwan [36] have proposed algorithms that compute global solutions of linear BPPs by enumerating the extreme points of \( \Omega \). Whereas the former algorithm enumerates basis of the lower level problem, the latter, known as the “Kth-best”, enumerates basis of the relaxed problem. Other extreme point approaches for linear BPPs have been proposed by Y. Chen and M. Florian [58], [60], S. Dempe [64], G. Papavassilopoulos [138] and H. Tuy, A. Midgalas and P. Värbrand [156].

L. Vicente, G. Savard and J. Júdice [162] have studied the induced regions of the quadratic BPP and introduced the concepts of extreme induced region points and extreme induced region directions. They proposed extreme point algorithms that compute local star minima and local minima depending on the nature of the upper level objective function.
Branch and bound algorithms

Branch and bound methods are widely applied to convex bilevel programs. Although they are associated with large computational efforts they are also capable of computing global minima. Several approaches exploit the complementarity between the multipliers and the slack variables that arises from the KKT conditions of the lower level problem. That is the case of the algorithms proposed by J. Bard and J. Falk [20] and J. Fortuny-Amat and B. McCarl [77] for the linear case, J. Bard and J. Moore [21] for the linear–quadratic case and F. Al-Khayyal, R. Horst and P. Pardalos [3], J. Bard [18] and T. Edmunds and J. Bard [73] for the quadratic case. Using different branching strategies, P. Hansen, B. Jaumard and G. Savard [81] have proposed a branch and bound algorithm for the solution of the linear BPP that seems particularly efficient for the solution of medium–scale problems.

Although little attention has been given to the case in which some variables are restricted to have integer values J. Bard and J. Moore [22], [124] and U. Wen and Y. Yang [170] have proposed branch and bound procedures for the solution of integer linear instances of the BPP, and T. Edmunds and J. Bard [74] have introduced a branch and bound algorithm for the solution of the integer quadratic BPP.

Complementarity pivot algorithms

The first complementarity pivot algorithm for solving linear BPPs was proposed by W. Bialas, M. Karwan and J. Shaw [37]. This algorithm cannot, as suggested in [36], compute global solutions of linear BPPs (see examples in [26] and [93]).

By combining some of the ideas from the last two classes of algorithms, J. Júdice and A. Faustino proposed the SLCP (sequential linear complementarity problem) algorithm for the computation of ε–global solutions of linear ([93], [94]) and linear–quadratic ([95]) BPPs. This algorithm seems quite efficient for the solution of medium–scale problems.

Another complementarity pivot approach which can be classified as a modified simplex approach was proposed by H. Onal [133].

Descent methods

In this class we include methods incorporating descent directions that are designed to compute stationary points and local minima. A classical example is the steepest descent direction algorithm extended to nonlinear bilevel programming by G. Savard and J. Gauvin [141]. Here the computation of the steepest descent direction for a BPP is done with the help of a linear–quadratic BPP. L. Vicente, G. Savard and J. Júdice [162] studied the application of this algorithm to convex bilevel programming, where the lower level problems are strictly convex quadratic programs, and proposed appropriate stepsize rules to displacements along directions in the induced region.

A second classical algorithm is the one proposed by C. Kolstad and L. Lasdon [100] for the solution of nonlinear BPPs. This algorithm consists of applying
gradient information to the implicit optimization problem:

$$\min_x F(x, y(x))$$

subject to

$$g(x, y(x)) \leq 0$$

where \(\{y(x)\}\) is the lower level reaction set for all values of \(x\). The authors introduced a local estimation of the gradient of \(y\) and applied a BFGS quasi-Newton algorithm to the solution of an unconstrained version of this problem.

Another descent approach can be found in M. Florian and Y. Chen [75].

**Penalty function methods**

Some of the methods in this class can also be classified as descent algorithms. They usually incorporate exact penalty functions and are limited to computing stationary points and local minima. See E. Aiyoshi and K. Shimizu [1], [2], [147], and Z. Bi, P. Calamai and A.R. Conn [32], [33] for the case where the penalty term incorporates the lower level objective function, and Y. Ishizuka and E. Aiyoshi [89] for the case where both objective functions are penalized. The reader is also referred to the work of P. Loridan and J. Morgan on approximation and stability results for bilevel programming that might be of interest for the convergence theory of these and other algorithms, and to Z.-Q. Luo, J.-S. Pang and S. Wu [112] for the derivation of an exact penalty function that only uses the square-root of the complementarity term associated with the lower level quadratic program as the penalty term.

In [171], D. White and G. Anandalingam exploit the penalized bilinear version of a linear BPP and introduce a exact penalty function algorithm that finds a global solution of the linear BPP by solving a sequence of bilinear programs.

## 4 Multilevel programming and applications

As stated before, bilevel programming is a special case of multilevel programming. However, as described by C. Blair [39], the complexity of these problems increases significantly when the number of levels is greater than two. In spite of this, three level and multilevel programming has been studied in the literature by, among others, J. Bard [15], J. Bard and J. Falk [20], H. Benson [29], R. Jan and M. Chen [90] and U. Wen and W. Bialas [166].

The particular structure of bilevel and multilevel programs facilitates the formulation of a number of practical problems that involve an hierarchical decision making process. Among the several applications of bilevel and multilevel programming the following are noteworthy:

- **Transportation** – Network design problem (L. LeBlanc and D. Boyce [102], O. Ben-Ayed, C. Blair, D. Boyce and L. LeBlanc [27], [28], P. Marcotte [114], P. Marcotte and G. Marquis [116] and S. Suh and T. Kim [151]) and trip demand
estimation problem (M. Florian and Y. Chen [75], [76] and R.L. Tobin and T.L. Friesz [155]).

- **Management** – Coordination of multidivisional firms (J. Bard [13]), network facility location with delivered price competition (T. Miller, T. Friesz and R. Tobin [122]) and credit allocation (R. Cassidy and M. Kirby and W. Raike [54]).

- **Planning** – Application of agricultural policies (W. Candler, J. Fortuny-Amat and B. McCarl [50], W. Candler and R. Norton [51], [52] and H. Önal [132]) and electric utility planning (A. Haurie, R. Loulou and G. Savard [83] and B. Hobbs and S. Nelson [86]).

- **Engineering Design** – Optimal design problems (M. Kocvara and J. Outrata [97], [98] and P. Neittaanmäki and A. Stachurski [130]).

We believe that bilevel programming can play an important role in other branches of mathematical programming. For example, bilevel programming can provide a novel approach for analyzing the step selection subproblem in a trust region algorithm for nonlinear equality constrained optimization (see [55]), and has been applied to the discriminant problem [118].

5 **How to contribute and how to get this report**

The subjects covered in this bibliography review are bilevel and multilevel programming and Stackelberg problems when considered as optimization problems – usually called static Stackelberg problems. We have selected contributions in this area that deal with theory issues (properties, existence of solution, optimality conditions and so on), algorithms and numerical results, software and generation of test problems, applications and complexity issues.

References to be cited should be books, articles published in journals or special volumes and technical reports that are available to the broad research community. Conferences and seminar abstracts are not included.

Many of the references listed in our bibliography have been cited in the text however for completeness we have included all qualifying references that we are familiar with.

This bibliography review is updated bianually and is available via email or anonymous ftp. It consists of the Bib/Tex file bilevel-review.bib that contains the bibliographic entries and the LaTeX file bilevel-review.tex that constitutes this manuscript. In order to get these files:

- Most preferably, using anonymous ftp:

  Compressed versions of these two files can be obtained using the procedure
described below. Entries on the left are the prompts (typewriter type style) and example responses (bold typewriter type style) whereas those on the right are comments that describe the corresponding action.

```
% ftp dial.uwaterloo.ca
Name (machine:userid): anonymous
Password: jqpublic@domain
ftp> cd pub/phcalamai/bilevel-review
ftp> binary
ftp> get bilevel-review.tex.Z
ftp> get bilevel-review.bib.Z
ftp> quit
% uncompress bilevel-review.tex.Z
% uncompress bilevel-review.bib.Z
% latex bilevel-review
% bibtex bilevel-review
% latex bilevel-review
% latex bilevel-review
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All contributions, corrections and suggestions are welcome and should be sent to either of the authors’ addresses or (preferably) to the email address listed above.

References


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