LOCAL CONVERGENCE OF A PRIMAL-DUAL METHOD FOR DEGENERATE NONLINEAR PROGRAMMING

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Abstract. In recent work, the local convergence behavior of path-following interior-point methods and sequential quadratic programming methods for nonlinear programming has been investigated for the case in which the assumption of linear independence of the active constraint gradients at the solution is replaced by the weaker Mangasarian-Fromovitz constraint qualification. In this paper, we describe a stabilization of the primal-dual interior-point approach that ensures rapid local convergence under these conditions without enforcing the usual centrality condition associated with path-following methods. The stabilization takes the form of perturbations to the coefficient matrix in the step equations that vanish as the iterates converge to the solution.

Key words. Nonlinear Programming, Degeneracy, Interior-Point Methods.

AMS subject classifications. 90C30, 90C31

1. Introduction. We consider the nonlinear programming (NLP) problem in the following form:

$$\min f(x), \quad \text{subject to } g(x) = 0,\ x \geq 0, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, and $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ are smooth functions. We write the Lagrangian function ℓ for this problem as

$$\ell(x, \lambda, z) \ = \ f(x) + g(x)^T \lambda - x^T z.$$

The Karush-Kuhn-Tucker (KKT) conditions are satisfied at a point x if there exist Lagrange multipliers $\lambda \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$ such that

$$\nabla_x \ell(x, \lambda, z) = \nabla f(x) + \nabla g(x)\lambda - z = 0, \tag{1.2a}$$

$$g(x) = 0, \tag{1.2b}$$

$$XZe = 0, \tag{1.2c}$$

$$x \geq 0, \quad z \geq 0, \tag{1.2d}$$

where

$$X = \text{diag}(x_i)_{i=1}^n, \quad Z = \text{diag}(z_i)_{i=1}^n,$$

and e represents the vector of ones of the appropriate dimension (in this case n).

The standard primal-dual interior-point approach for solving (1.1) is to apply a Newton-like method to the square system of nonlinear equations formed by (1.2a), (1.2b), and (1.2c), modifying the search direction and step length to ensure that the

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inequalities (1.2d) are satisfied *strictly* by each iterate. As noted below, our modification relaxes the latter condition, allowing some components of x and z to become zero or negative when significant progress toward the solution is achieved by doing so. We show that our strategy achieves local superlinear convergence even when the standard assumption of linear independence of the active constraint gradients (known as LICQ) is replaced by the weaker Mangasarian-Fromovitz constraint qualification (abbreviated as MFCQ). We sometimes use the term "degenerate problems" to refer to problems for which MFCQ but not LICQ is satisfied at the solution x^*.

The local convergence theory of interior-point methods for (1.1) is developed in the papers by El-Bakry et al. [4], Martinez, Parada, and Tapia [13], and Yamashita and Yabe [22] for primal-dual methods and in the papers by Coleman and Li [3], Heinkenschloss, Ulbrich, and Ulbrich [10], and Vicente [18] for affine-scaling methods. Byrd, Liu, and Nocedal [2] describe a hybrid primal-dual approach. The algorithms in these papers have the classical properties of Newton and quasi-Newton methods, and all make the LICQ assumption.

In many application problems, however, LICQ is generically not satisfied. An example is the class of optimal control and design problems with inequalities or bounds involving the state variables; see [11]. Consider the problem

$$\min f(y,u), \quad \text{subject to } g(y,u)=0,\ y\geq 0,\ u\geq 0\,,$$

in which $y\in\mathbb{R}^{n_y}$ represents state variables, $u\in\mathbb{R}^{n_u}$ represents control variables, $g(y,u):\mathbb{R}^{n_y}\times\mathbb{R}^{n_u}\to\mathbb{R}^{n_y}$ models the state equation, and the partial Jacobian $\nabla_y g(y,u)$ is nonsingular at all points of interest. The LICQ condition is satisfied at a point (y,u) if the matrix

$$\begin{bmatrix} \nabla_y g(y,u)^T & \nabla_u g(y,u)^T \\ I_{N_y} & 0 \\ 0 & I_{N_u} \end{bmatrix}$$

has full row rank, where N_y and N_u are the active sets $\{i : y_i = 0\}$ and $\{i : u_i = 0\}$, respectively, and I_{N_y} and I_{N_u} are the submatrices of the identity formed by the rows corresponding to indices in N_y and N_u, respectively. A necessary condition for nondegeneracy is therefore that

$$n_y + |N_y| + |N_u| \leq n_y + n_u,$$

that is,

$$|N_y| + |N_u| \leq n_u.$$

In other words, the number of control and state active variables (those at their bounds) must not exceed the number of control variables. In situations where $n_y \gg n_u$, this requirement is unlikely to be satisfied.

Based on previous work by Wright [19, 20] on SQP methods for degenerate problems, we introduce a stabilization for the primal-dual interior-point approach that allows a measure of the error to converge quadratically to zero even when the problem is degenerate. Related results were obtained by Ralph and Wright [15, 16] for monotone variational inequalities obtained as the optimality conditions of convex programming problems. The latter papers establish both global and rapid local convergence properties by imposing a centrality condition on the iterates; they require

monotonicity/convexity, but not the stabilization parameter of this paper. In a related paper, Tseng [17] describes a primal-dual interior-point method for monotone nonlinear complementarity problems in which superlinear convergence occurs even when the "primal" solution is not unique. His work also imposes centrality and guarantees global convergence. Another topic of interest, not considered here, concerns the behavior of interior-point methods when strict complementarity does not hold at the solution. Heinkenschloss, Ulbrich, and Ulbrich [10] describe an affine-scaling interior-point method that converges superlinearly for minimization problems with simple bounds, without the assumption of strict complementarity.

The stabilized primal-dual method that we introduce in this paper results from modifying the theory for the classical primal-dual method in two respects. The first modification appears in the familiar matrix used in the computation of the primal-dual step. The diagonal blocks in this matrix are modified to ensure that those of their diagonal elements that are converging to zero do so at a controlled rate and remain strictly positive. The other modification is that a step length of 1 may be taken even when it leads to new iterates that have nonpositive components, provided that some conditions are satisfied that ensure that the resulting step makes excellent progress toward the solution.

The paper is organized as follows. In Section 2 we detail our assumptions and notation. The stabilized primal-dual method is presented in Section 3, and the main convergence results are presented in Section 4. Finally, we discuss alternative approaches in Section 5.

2. Assumptions and Notation. Except for relaxing the LICQ to MFCQ, our assumptions are ones traditionally made in standard analyses of superlinear local convergence. We make second-order sufficient assumptions (see below) that ensure that x^* is a strict local solution. Because of MFCQ, the multipliers λ^* and z^* that satisfy (1.2) are not necessarily unique. We define the following sets:

$$\mathcal{S}_D = \{(\lambda^*, z^*) : \ (x^*, \lambda^*, z^*) \text{ satisfy (1.2)}\}, \quad \mathcal{S} = \{x^*\} \times \mathcal{S}_D.$$

The analysis of the stabilized primal-dual method presented in Section 3 makes use of the partition of the index set $\{1, 2, \ldots, n\}$ into active and inactive index sets at the point x^*, defined as follows:

$$B \ = \ \{i = 1, 2, \ldots, n \ : \ x_i^* > 0\}, \qquad N \ = \ \{i = 1, 2, \ldots, n \ : \ x_i^* = 0\}.$$

We will prove in Section 4 that the stabilized primal-dual method exhibits a quadratic rate of local convergence to a point x^* that satisfies the following set of assumptions:

A.1 The functions f and g are twice continuously differentiable in a neighborhood of x^*.

A.2 The point x^* satisfies the KKT conditions (1.2).

A.3 Strict complementarity holds; that is, for some $(\lambda^*, z^*) \in \mathcal{S}_D$, we have $z_i^* > 0$ for all $i \in N$.

A.4 The following second-order sufficient condition is satisfied:

$$\text{there exists } \sigma > 0 \text{ such that } \ d^T \nabla^2_{xx} \ell(x^*, \lambda^*, z^*) d \ \geq \ \sigma \|d\|^2$$

for all $(\lambda^*, z^*) \in \mathcal{S}_D$ and for all d such that

$$\begin{bmatrix} \nabla g_B^{*\,T} & \nabla g_N^{*\,T} \\ 0 & -I \end{bmatrix} \begin{bmatrix} d_B \\ d_N \end{bmatrix} = 0.$$

(Here, ∇g_B^* contains the partial derivatives of g with respect to x_i, evaluated at x^*, for $i \in B$; similarly for ∇g_N^*.)

A.5 The Mangasarian-Fromovitz constraint qualification [12] (MFCQ) is satisfied at x^*, that is, $\nabla g(x^*)$ has full column rank, and there exists a vector w such that

$$\nabla g(x^*)^T w = 0, \quad w_i > 0 \text{ for all } i \in N.$$

Given Assumptions A.1–A.5, it is well known that x^* is a strict local minimizer for (1.1). Another important result is the following characterization of the boundedness of $\mathcal{S}_D$ proved by Gauvin [7]:

LEMMA 2.1. *Let Assumptions A.1–A.4 hold. The set $\mathcal{S}_D$ is bounded if and only if the MFCQ A.5 holds.*

Given $x \in \mathbb{R}^n$, we denote by x_- the vector whose elements are $\min(x_i, 0)$ for $i = 1, 2, \ldots, n$. We use $(x, z)_-$ to denote the vector in $\mathbb{R}^{2n}$ in which the first n components contain x_- and the last n components contain z_-.

Given two continuous functions $\Psi_1(\cdot)$ and $\Psi_2(\cdot)$ that map some space to $\mathbb{R}$, we say that $\Psi_1(x) = \mathcal{O}(\Psi_2(x))$ if there exist constants ϵ and $\kappa(\epsilon)$ such that

$$|\Psi_2(x)| \in [0, \epsilon] \quad \Longrightarrow \quad |\Psi_1(x)| \leq \kappa(\epsilon)|\Psi_2(x)|\,.$$

We say that $\Psi_1(x) = \Theta(\Psi_2(x))$ if both $\Psi_1(x) = \mathcal{O}(\Psi_2(x))$ and $\Psi_2(x) = \mathcal{O}(\Psi_1(x))$. We use $\|\cdot\|$ to denote the Euclidean norm of a matrix or a vector.

3. Stabilizing the Primal-Dual Method. A standard primal-dual approach for solving (1.1) computes steps based on the linearization of the following perturbed version of (1.2a), (1.2b), and (1.2c):

$$\nabla_x \ell(x, \lambda, z) = 0, \tag{3.1a}$$

$$g(x) = 0, \tag{3.1b}$$

$$XZe = \mu_c e, \tag{3.1c}$$

for some parameter $\mu_c > 0$. In addition, all iterates are typically required to satisfy the strict positivity conditions $x > 0$, $z > 0$. Acceptable performance of this approach depends on a number of factors, among them the choice of parameter $\mu_c > 0$, which we discuss below. Linearization of (3.1c) yields

$$Z\Delta x + X\Delta z \;=\; -XZe + \mu_c e\,. \tag{3.2}$$

This equation is combined with linearizations of (3.1a) and (3.1b) to yield the linear system that is solved at each iteration of a typical primal-dual interior-point method.

Since our conditions do not assume a unique primal-dual solution point, we are interested in the convergence of the sequence of iterates (x, λ, z) to some point in the primal-dual solution set $\mathcal{S}$. Difficulties may arise when some of the coefficients of Δx and Δz in the linearization (3.2)—that is, the elements of the vectors x and z—are very close to zero. Many path-following methods enforce the centrality conditions $x_i z_i \geq \gamma\mu$, where $\gamma > 0$ is a constant and μ is a measure of the distance of the current iterate (x, λ, z) to the primal-dual solution set $\mathcal{S}$ (about which see more below). This condition ensures that the pairwise products $x_i z_i$ remain roughly in balance as they approach zero and that even the small components of x and z are bounded below by a multiple of μ. In this paper, we consider an alternative to the centrality condition

that achieves the same effect: We modify the system (3.2) to ensure explicitly that the coefficients of Δx and Δz do not approach zero too rapidly. Our modified system is

$$\tilde{Z}\Delta x + \tilde{X}\Delta z \;=\; -XZe + \mu_c e, \tag{3.3}$$

where

$$\tilde{X} = \operatorname{diag}(\tilde{x}_i)_{i=1}^n, \quad \tilde{x}_i = \max(\mu_{\min}, x_i), \quad i = 1, 2, \ldots, n, \tag{3.4a}$$

$$\tilde{Z} = \operatorname{diag}(\tilde{z}_i)_{i=1}^n, \quad \tilde{z}_i = \max(\mu_{\min}, z_i), \quad i = 1, 2, \ldots, n, \tag{3.4b}$$

where $\mu_{\min}$ is a positive value, specified below, that varies like the distance of the current iterate (x, λ, z) to the solution set $\mathcal{S}$. For those x_i and z_i that do not approach zero, the modification (3.4) has no effect, and we recover the same coefficients as in (3.2) for these components.

Linearization of the nonlinear equation (3.1a) is given by

$$\nabla^2_{xx}\ell(x, \lambda, z)\Delta x + \nabla g(x)\Delta\lambda - \Delta z \;=\; -\nabla_x \ell(x, \lambda, z), \tag{3.5}$$

while linearization of the feasibility condition (3.1b) yields

$$\nabla g(x)^T \Delta x = -g(x). \tag{3.6}$$

The conjunction of the linear equations (3.5), (3.6), and (3.3) defines the linear system that provides the step $(\Delta x, \Delta\lambda, \Delta z)$:

$$\begin{bmatrix} \nabla^2_{xx}\ell(x, \lambda, z) & \nabla g(x) & -I \\ \nabla g(x)^T & 0 & 0 \\ \tilde{Z} & 0 & \tilde{X} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta z \end{bmatrix} = - \begin{bmatrix} \nabla_x \ell(x, \lambda, z) \\ g(x) \\ XZe - \mu_c e \end{bmatrix}. \tag{3.7}$$

When the iterates satisfy a centering condition, and when the norms of $g(x)$ and $\nabla_x \ell(x, \lambda, z)$ are bounded by a constant multiple of $x^T z$ at all iterates, it suffices to estimate the distance of (x, λ, z) to $\mathcal{S}$ by $x^T z$. A measure that holds in more general circumstances, whenever conditions A.1–A.5 are satisfied and for all points (x, λ, z) sufficiently close to the solution set, is given by

$$\mu(x, \lambda, z) \stackrel{\text{def}}{=} \left\| (\nabla_x \ell(x, \lambda, z), g(x), \min(x, z)) \right\|. \tag{3.8}$$

Here, $\min(x, z)$ denotes a vector of length n whose ith component is the minimum of x_i and z_i. Several authors (for example, Facchinei, Fisher, and Kanzow [5], Hager and Gowda [9, Lemma 2], and Wright [19, Theorem A.1]) have proved that under Assumptions A.1–A.5 we have that

$$\operatorname{dist}((x, \lambda, z), \mathcal{S}) = \Theta(\mu(x, \lambda, z)). \tag{3.9}$$

We assume that the algorithmic parameter $\mu_{\min}$ is chosen to satisfy

$$\mu_{\min} = \Theta(\mu(x, \lambda, z)). \tag{3.10}$$

For instance, it can be set to $\mu_{\min} = \chi\mu(x, \lambda, z)$ for some positive constant χ.

We now describe our stabilized primal-dual method.

ALGORITHM 3.1 (Stabilized Primal-Dual Method).

Choose parameter $\sigma \in (1,2)$;
Choose initial point (x,λ,z) with $(x,z) > 0$;
repeat
Choose $\mu_{\min} > 0$ satisfying (3.10);
Choose centering parameter $\mu_c \in [0,\mu]$ to satisfy

$$\mu_c = \mathcal{O}(\mu(x,\lambda,z)^2);$$

Compute $(\Delta x, \Delta\lambda, \Delta z)$ by solving (3.7);
if both of the following conditions hold:

$$\|(x+\Delta x, z+\Delta z)_-\| \le \mu(x,\lambda,z)^\sigma, \tag{3.11a}$$

$$\mu(x+\Delta x, \lambda+\Delta\lambda, z+\Delta z) \le \mu(x,\lambda,z)^\sigma; \tag{3.11b}$$

Set $\alpha = 1$;
else
Choose step parameter $\tau \in (0,1)$ to satisfy

$$1-\tau = \mathcal{O}(\mu(x,\lambda,z));$$

Choose α to satisfy the condition

$$\alpha \in \left(0, \tau \frac{-1}{\min(X^{-1}\Delta x, Z^{-1}\Delta z, -1)}\right];$$

end (if)
Define the new iterate by

$$(x,\lambda,z) \leftarrow (x,\lambda,z) + \alpha(\Delta x, \Delta\lambda, \Delta z);$$

until convergence.

We denote the iterates generated by this method by (x^k, λ^k, z^k), $k = 0, 1, 2, \ldots$.

The method above departs from the classical primal-dual interior-point approach in two major respects. First, the computation of the step in (3.7) is obviously different, since X and Z have been replaced by $\tilde{X}$ and $\tilde{Z}$. Second, a step length of 1 may be taken even when it produces new iterates that have nonpositive components, provided that the conditions (3.11a) and (3.11b) are satisfied. These conditions ensure that such an event happens only when the step makes excellent progress toward the solution. The stabilization (3.3),(3.4) ensures that the *next* iteration is still well defined, in the sense that $\tilde{X}$ and $\tilde{Z}$ still have strictly positive diagonal elements after the step is taken.

The "else" branch of the condition statement ensures that the step length α is chosen to move no more than a fraction τ of the distance to the boundary along the calculated direction. In convex nonlinear programs, other conditions may be imposed on μ_c and α, to enforce centrality or to balance the amounts by which the respective KKT conditions (1.2a)–(1.2c) are violated by the current iterate (see, for example, Ralph and Wright [15, 16]). In nonconvex problems, decrease of a merit function may be required, and other modifications of the search direction (related to imposition of a trust-region bound, for instance) may be imposed; see, for example, Byrd, Hribar, and Nocedal [1], Forsgren and Gill [6], Gay, Overton, and Wright [8]. Since our interest is in local convergence behavior, we have not included in Algorithm 3.1 any

of the safeguards that are needed to ensure desirable global convergence behavior. We believe, however, that such safeguards could be incorporated without interfering with the local convergence behavior that we describe in the next section. In particular, if decrease of a merit function is required for global convergence, we expect that a step satisfying (3.11b) will decrease the merit function, since such a step makes good progress toward the solution set $\mathcal{S}$. (It is for this reason that we have included the condition (3.11b) among our acceptance criteria; this condition is not needed for the local convergence theory developed in the next section.)

4. Convergence of the Stabilized Primal-Dual Method. In this section, we establish rapid local convergence of Algorithm 3.1 once it enters a certain neighborhood of the solution set $\mathcal{S}$. The main technical result, Lemma 4.1, establishes that the length of the step is $\mathcal{O}(\mu(x,\lambda,z))$ when calculated at any point in the neighborhood in question, provided that the centering parameter μ_c meets certain requirements. Lemma 4.2 shows that the conditions (3.11a), (3.11b) are eventually satisfied by the step, while Theorem 4.3 essentially summarizes the result.

Because of Assumption A.3, there is a $\gamma > 0$ such that for each $\epsilon > 0$ the following neighborhood is nonempty:

$$\mathcal{N}_\gamma(\epsilon) \;=\; \Big\{(x,\lambda,z) \;:\; \begin{array}{l} \|(x,\lambda,z)-(x^*,\lambda^*,z^*)\| \le \epsilon\,, \\ \text{for some } (\lambda^*,z^*)\in\mathcal{S}_D \text{ with } z^*_N \ge \gamma e \end{array}\Big\}, \tag{4.1}$$

where z^*_N is the subvector of z^* formed by the components in N. Loosely speaking, this set is a strictly complementary neighborhood of the solution set $\mathcal{S}$. Note that by (3.9) and (3.10), there is a constant κ_0 such that for all ϵ sufficiently small, we have

$$\mu(x,\lambda,z) \le \kappa_0\epsilon \;\text{ for all }\; (x,\lambda,z)\in\mathcal{N}_\gamma(\epsilon). \tag{4.2}$$

Our main result on the length of the step obtained by solving the system (3.7) is as follows.

LEMMA 4.1. *Let Assumptions A.1–A.5 hold, and suppose that for some constant C we have*

$$\mu_c \le C\mu(x,\lambda,z)^2. \tag{4.3}$$

Then for any $\gamma > 0$, there exist positive constants ϵ and $\kappa_1(\epsilon,\gamma)$ such that

$$\|(\Delta x,\Delta\lambda,\Delta z)\| \;\le\; \kappa_1(\epsilon,\gamma)\mu(x,\lambda,z), \tag{4.4}$$

for all points $(x,\lambda,z)\in\mathcal{N}_\gamma(\epsilon)$.

Proof. Suppose γ is such that $\mathcal{N}_\gamma(\epsilon)$ is nonempty for all $\epsilon > 0$. We show that the desired result holds for some sufficiently small ϵ and sufficiently large $\kappa_1(\epsilon,\gamma)$. Since the proof is long, we divide it into five sections labeled (A)-(E) and provide a brief descriptive heading for each section. To simplify the notation, we drop the argument (x,λ,z), using in particular μ to denote $\mu(x,\lambda,z)$.

(A) Transforming the system (3.7) via a singular value decomposition (SVD) of the active constraint Jacobian. Using the partition $\{1,\ldots,n\} = B\cup N$, the linear

system (3.7) can be rewritten as follows:

$$(4.5)\quad \begin{bmatrix} \nabla^2_{xx}\ell_{BB} & \nabla^2_{xx}\ell_{BN} & \nabla g_B & -I & 0 \\ \nabla^2_{xx}\ell_{NB} & \nabla^2_{xx}\ell_{NN} & \nabla g_N & 0 & -I \\ \nabla g_B^T & \nabla g_N^T & 0 & 0 & 0 \\ \tilde{Z}_B & 0 & 0 & \tilde{X}_B & 0 \\ 0 & \tilde{Z}_N & 0 & 0 & \tilde{X}_N \end{bmatrix} \begin{bmatrix} \Delta x_B \\ \Delta x_N \\ \Delta\lambda \\ \Delta z_B \\ \Delta z_N \end{bmatrix} = - \begin{bmatrix} \nabla_x \ell_B \\ \nabla_x \ell_N \\ g \\ (XZe)_B - \mu_c e_B \\ (XZe)_N - \mu_c e_N \end{bmatrix},$$

where e_B and e_N denote vectors of the form $(1, 1, \ldots, 1)^T$ with $|B|$ and $|N|$ elements, respectively. We can eliminate Δz_B immediately to obtain

$$(4.6)\quad \Delta z_B \;=\; -\tilde{X}_B^{-1}\tilde{Z}_B \Delta x_B - \tilde{X}_B^{-1} X_B Z_B e_B + \mu_c \tilde{X}_B^{-1} e_B.$$

By substituting in (4.5) and scaling the last block row by $-\tilde{Z}_N^{-1}$, we obtain

$$\begin{aligned} &\begin{bmatrix} \nabla^2_{xx}\ell_{BB} + \tilde{X}_B^{-1}\tilde{Z}_B & \nabla^2_{xx}\ell_{BN} & \nabla g_B & 0 \\ \nabla^2_{xx}\ell_{NB} & \nabla^2_{xx}\ell_{NN} & \nabla g_N & -I \\ \nabla g_B^T & \nabla g_N^T & 0 & 0 \\ 0 & -I & 0 & -\tilde{X}_N \tilde{Z}_N^{-1} \end{bmatrix} \begin{bmatrix} \Delta x_B \\ \Delta x_N \\ \Delta\lambda \\ \Delta z_N \end{bmatrix} \\ (4.7)\qquad &= \; - \begin{bmatrix} \nabla_x \ell_B + \tilde{X}_B^{-1} X_B Z_B e_B - \mu_c \tilde{X}_B^{-1} e_B \\ \nabla_x \ell_N \\ g \\ -\tilde{Z}_N^{-1} X_N Z_N e_N + \mu_c \tilde{Z}_N^{-1} e_N \end{bmatrix}. \end{aligned}$$

At this point we require the SVD of the Jacobian matrix of the active constraints at x^*, which we write as follows:

$$(4.8)\quad \begin{bmatrix} {\nabla g_B^*}^T & {\nabla g_N^*}^T \\ 0 & -I \end{bmatrix} = \begin{bmatrix} U & V \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{U}^T \\ \hat{V}^T \end{bmatrix},$$

where $U \in \mathbb{R}^{(m+|N|)\times p}$, $V \in \mathbb{R}^{(m+|N|)\times(m+|N|-p)}$, $S \in \mathbb{R}^{p\times p}$, $\hat{U}^T \in \mathbb{R}^{p\times n}$, $\hat{V}^T \in \mathbb{R}^{(n-p)\times n}$, and p is the rank of the Jacobian matrix. By partitioning the rows of V in an obvious way, we have from (4.8) that

$$(4.9)\quad 0 = \begin{bmatrix} V_1^T & V_2^T \end{bmatrix} \begin{bmatrix} {\nabla g_B^*}^T & {\nabla g_N^*}^T \\ 0 & -I \end{bmatrix} = \begin{bmatrix} V_1^T {\nabla g_B^*}^T & (V_1^T {\nabla g_N^*}^T - V_2^T) \end{bmatrix}.$$

Let us apply a change of variables to Δx_B, Δx_N, $\Delta\lambda$, and Δz_N using the orthogonal bases given by U, V and $\hat{U}, \hat{V}$:

$$\begin{bmatrix} \Delta x_B \\ \Delta x_N \end{bmatrix} \;=\; \hat{U} c_{\hat{U}} + \hat{V} c_{\hat{V}}, \qquad \begin{bmatrix} \Delta\lambda \\ \Delta z_N \end{bmatrix} \;=\; U c_U + V c_V\,.$$

With these expansions, the system (4.7) is equivalent to

$$(4.10)\quad \begin{bmatrix} \hat{U}^T L \hat{U} & \hat{U}^T L \hat{V} & \hat{U}^T J^T U & \hat{U}^T J^T V \\ \hat{V}^T L \hat{U} & \hat{V}^T L \hat{V} & \hat{V}^T J^T U & \hat{V}^T J^T V \\ U^T J \hat{U} & U^T J \hat{V} & U^T M U & U^T M V \\ V^T J \hat{U} & V^T J \hat{V} & V^T M U & V^T M V \end{bmatrix} \begin{bmatrix} c_{\hat{U}} \\ c_{\hat{V}} \\ c_U \\ c_V \end{bmatrix} = \begin{bmatrix} r_{\hat{U}} \\ r_{\hat{V}} \\ r_U \\ r_V \end{bmatrix},$$

where the matrices L, J, and M are given by

$$L = \begin{bmatrix} \nabla^2_{xx}\ell_{BB} + \tilde{X}_B^{-1}\tilde{Z}_B & \nabla^2_{xx}\ell_{BN} \\ \nabla^2_{xx}\ell_{NB} & \nabla^2_{xx}\ell_{NN} \end{bmatrix}, \tag{4.11a}$$

$$J = \begin{bmatrix} \nabla g_B^T & \nabla g_N^T \\ 0 & -I \end{bmatrix}, \tag{4.11b}$$

$$M = \begin{bmatrix} 0 & 0 \\ 0 & -\tilde{X}_N\tilde{Z}_N^{-1} \end{bmatrix}, \tag{4.11c}$$

and the residuals $r_{\hat{U}}$, $r_{\hat{V}}$, r_U, and r_V are given by

$$r_{\hat{U}} = -\hat{U}^T \begin{bmatrix} \nabla_x\ell_B + \tilde{X}_B^{-1}X_BZ_Be_B - \mu_c\tilde{X}_B^{-1}e_B \\ \nabla_x\ell_N \end{bmatrix}, \tag{4.12a}$$

$$r_{\hat{V}} = -\hat{V}^T \begin{bmatrix} \nabla_x\ell_B + \tilde{X}_B^{-1}X_BZ_Be_B - \mu_c\tilde{X}_B^{-1}e_B \\ \nabla_x\ell_N \end{bmatrix}, \tag{4.12b}$$

$$r_U = -U^T \begin{bmatrix} g \\ -\tilde{Z}_N^{-1}X_NZ_Ne_N + \mu_c\tilde{Z}_N^{-1}e_N \end{bmatrix}, \tag{4.12c}$$

$$r_V = -V^T \begin{bmatrix} g \\ -\tilde{Z}_N^{-1}X_NZ_Ne_N + \mu_c\tilde{Z}_N^{-1}e_N \end{bmatrix}. \tag{4.12d}$$

(B) Examining V^TMV and its inverse. Using (4.11c), we write the blocks of (4.10) involving M as follows:

$$\begin{aligned} \begin{bmatrix} U \\ V \end{bmatrix}^T M \begin{bmatrix} U & V \end{bmatrix} &= \begin{bmatrix} U_1^T & U_2^T \\ V_1^T & V_2^T \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -\tilde{X}_N\tilde{Z}_N^{-1} \end{bmatrix} \begin{bmatrix} U_1 & V_1 \\ U_2 & V_2 \end{bmatrix} \\ &= -\begin{bmatrix} U_2^T\tilde{X}_N\tilde{Z}_N^{-1}U_2 & U_2^T\tilde{X}_N\tilde{Z}_N^{-1}V_2 \\ V_2^T\tilde{X}_N\tilde{Z}_N^{-1}U_2 & V_2^T\tilde{X}_N\tilde{Z}_N^{-1}V_2 \end{bmatrix} \\ &\stackrel{\text{def}}{=} \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{12}^T & \bar{M}_{22} \end{bmatrix}. \end{aligned} \tag{4.13}$$

We are especially interested in the block $\bar{M}_{22}$. For ϵ sufficiently small, the definition (4.1) together with (3.10) and (4.2) implies that

$$\tilde{z}_i = \max(z_i, \mu_{\min}) = z_i \geq \gamma/2 > 0\,, \text{ for all } i \in N\,. \tag{4.14}$$

Moreover, (3.8) assures that

$$|x_i| = |\min(x_i, z_i)| \leq \mu(x, \lambda, z)\,, \text{ for all } i \in N\,.$$

Since $\tilde{x}_i = \max(x_i, \mu_{\min}) \geq \mu_{\min}$ and since $\mathcal{S}_D$ is compact, we conclude from (3.10) that the matrix $\tilde{X}_N\tilde{Z}_N^{-1}$ is positive diagonal with all diagonal elements of magnitude $\Theta(\mu)$. Hence all blocks in this system have size $\Theta(\mu)$. We now verify that the $(2,2)$ block—the matrix $\bar{M}_{22} = V_2^T\tilde{X}_N\tilde{Z}_N^{-1}V_2$—is symmetric positive definite with all eigenvalues of magnitude $\Theta(\mu)$. Since the eigenvalues of $\tilde{X}_N\tilde{Z}_N^{-1}$ are positive and behave like $\Theta(\mu)$, the same will be true for the eigenvalues of $\bar{M}_{22}$ if V_2 has full column rank. Suppose for contradiction that there is a vector $u \neq 0$ such that $V_2u = 0$. By multiplying (4.9) from the left by u^T, we obtain that

$$0 = \begin{bmatrix} \nabla g_B^*V_1u \\ \nabla g_N^*V_1u - V_2u \end{bmatrix} = \nabla g^*V_1u.$$

We must have $V_1 u \neq 0$, since otherwise we would have $Vu = 0$, which would contradict orthonormality of V. Hence it follows that $V_1 u$ is a nonzero vector in the null space of ∇g^*. But this contradicts our assumption that ∇g^* has full column rank. Hence no such u exists, and we conclude that V_2 has full column rank. Hence $\bar{M}_{22}$ has all eigenvalues of magnitude $\Theta(\mu)$, and in particular we have

$$\bar{M}_{22}^{-1} = \mathcal{O}(\mu^{-1}). \tag{4.15}$$

(C) Estimating other blocks in the matrix (4.10) and performing a block elimination with the lower right block $\bar{M}_{22}$. From (4.11b) and the estimate

$$\|x - x^*\| \leq \operatorname{dist}((x, \lambda, z), \mathcal{S}) = \Theta(\mu(x, \lambda, z)),$$

we have that

$$\begin{aligned} \begin{bmatrix} U \\ V \end{bmatrix}^T J \begin{bmatrix} \hat{U} & \hat{V} \end{bmatrix} &= \begin{bmatrix} U \\ V \end{bmatrix}^T (J - J^*) \begin{bmatrix} \hat{U} & \hat{V} \end{bmatrix} + \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{O}(\mu). \end{aligned} \tag{4.16}$$

We have for ϵ sufficiently small that

$$\tilde{x}_i \geq x_i \geq (1/2) x_i^* > 0, \quad \text{for all } (x, \lambda, z) \in \mathcal{N}_\gamma(\epsilon) \text{ and all } i \in B. \tag{4.17}$$

Since we can force z_i, $i \in B$, to be arbitrarily small by appropriate choice of ϵ, we have from (3.8) that

$$|z_i| = |\min(x_i, z_i)| \leq \mu(x, \lambda, z), \quad \text{for all } i \in B,$$

which in turn implies, by (3.10), that $\tilde{z}_i = \max(z_i, \mu_{\min}) = \Theta(\mu)$ for all $i \in B$. Hence, we obtain

$$\tilde{X}_B^{-1} \tilde{Z}_B = \Theta(\mu). \tag{4.18}$$

Moreover, we have

$$\nabla_{xx}^2 \ell(x, \lambda, z) = \nabla_{xx}^2 \ell(x^*, \hat{\lambda}, \hat{z}) + \mathcal{O}(\mu),$$

where $(\hat{\lambda}, \hat{z})$ is the closest element in $\mathcal{S}_D$ to the current iterate (λ, z). It follows from this observation, (4.11a), and (4.18) that

$$\begin{bmatrix} \hat{U}^T L \hat{U} & \hat{U}^T L \hat{V} \\ \hat{V}^T L \hat{U} & \hat{V}^T L \hat{V} \end{bmatrix} = \begin{bmatrix} \hat{U}^T \nabla_{xx}^2 \ell(x^*, \hat{\lambda}, \hat{z}) \hat{U} & \hat{U}^T \nabla_{xx}^2 \ell(x^*, \hat{\lambda}, \hat{z}) \hat{V} \\ \hat{V}^T \nabla_{xx}^2 \ell(x^*, \hat{\lambda}, \hat{z}) \hat{U} & \hat{V}^T \nabla_{xx}^2 \ell(x^*, \hat{\lambda}, \hat{z}) \hat{V} \end{bmatrix} + \mathcal{O}(\mu). \tag{4.19}$$

By substituting (4.13), (4.16), and (4.19) into the matrix of (4.10), we obtain

$$\begin{aligned} &\begin{bmatrix} \hat{U}^T \nabla_{xx}^2 \ell(x^*, \hat{\lambda}, \hat{z}) \hat{U} & \hat{U}^T \nabla_{xx}^2 \ell(x^*, \hat{\lambda}, \hat{z}) \hat{V} & S & 0 \\ \hat{V}^T \nabla_{xx}^2 \ell(x^*, \hat{\lambda}, \hat{z}) \hat{U} & \hat{V}^T \nabla_{xx}^2 \ell(x^*, \hat{\lambda}, \hat{z}) \hat{V} & 0 & 0 \\ S & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{M}_{22} \end{bmatrix} \\ &\quad + \begin{bmatrix} \mathcal{O}(\mu) & \mathcal{O}(\mu) & \mathcal{O}(\mu) & \mathcal{O}(\mu) \\ \mathcal{O}(\mu) & \mathcal{O}(\mu) & \mathcal{O}(\mu) & \mathcal{O}(\mu) \\ \mathcal{O}(\mu) & \mathcal{O}(\mu) & \mathcal{O}(\mu) & \mathcal{O}(\mu) \\ \mathcal{O}(\mu) & \mathcal{O}(\mu) & \mathcal{O}(\mu) & 0 \end{bmatrix}. \end{aligned} \tag{4.20}$$

Taking the last row of the matrix in (4.20), substituting into (4.10), and using the property (4.15), we have that

$$\begin{aligned} c_V &= \bar{M}_{22}^{-1}\left(r_V + \mathcal{O}(\mu)c_{\hat{U}} + \mathcal{O}(\mu)c_{\hat{V}} + \mathcal{O}(\mu)c_U\right) \\ &= \mathcal{O}(\mu^{-1})r_V + \mathcal{O}(1)c_{\hat{U}} + \mathcal{O}(1)c_{\hat{U}} + \mathcal{O}(1)c_U. \end{aligned} \tag{4.21}$$

By substituting this expression for c_V into (4.10), using the form of the matrix exposed in (4.20), we obtain, after some reordering, the following reduced system:

$$\left\{ \begin{bmatrix} S & \hat{U}^T\nabla^2_{xx}\ell(x^*,\hat{\lambda},\hat{z})\hat{V} & \hat{U}^T\nabla^2_{xx}\ell(x^*,\hat{\lambda},\hat{z})\hat{U} \\ 0 & \hat{V}^T\nabla^2_{xx}\ell(x^*,\hat{\lambda},\hat{z})\hat{V} & \hat{V}^T\nabla^2_{xx}\ell(x^*,\hat{\lambda},\hat{z})\hat{U} \\ 0 & 0 & S \end{bmatrix} + \mathcal{O}(\mu) \right\} \begin{bmatrix} c_U \\ c_{\hat{V}} \\ c_{\hat{U}} \end{bmatrix} = \begin{bmatrix} r_{\hat{U}} \\ r_{\hat{V}} \\ r_U \end{bmatrix} + \mathcal{O}(\|r_V\|). \tag{4.22}$$

Because of the second-order condition of Assumption A.4, the matrix

$$\hat{V}^T\nabla^2_{xx}\ell(x^*,\hat{\lambda},\hat{z})\hat{V}$$

is uniformly nonsingular for all $(\hat{\lambda},\hat{z})$ in the compact set $\mathcal{S}_D$. Hence, the matrix in (4.22) is an $\mathcal{O}(\mu)$ perturbation of a nonsingular matrix.

(D) Estimating the size of right-hand side components in (4.10). Because of (4.17), we have that $\tilde{X}_B^{-1} = \mathcal{O}(1)$. We have too from (3.8), (3.10), and the compactness of $\mathcal{S}_D$ that

$$\begin{aligned} |x_i z_i| &= |\min(x_i, z_i)\max(x_i, z_i)| \\ &= \mathcal{O}(|\min(x_i, z_i)|) \\ &= \mathcal{O}(\mu), \quad \text{all } i = 1, 2, \ldots, n. \end{aligned} \tag{4.23}$$

From (3.8) and (3.10), we have immediately that $\nabla_x\ell(x,\lambda,z) = \mathcal{O}(\mu)$, while $\mu_c = \mathcal{O}(\mu^2)$ from (4.3). By substituting all these estimates in (4.12a) and (4.12b), we obtain

$$r_{\hat{U}} = \mathcal{O}(\mu), \quad r_{\hat{V}} = \mathcal{O}(\mu). \tag{4.24}$$

For r_U, we have from (4.14) and the compactness of $\mathcal{S}_D$ that $\tilde{z}_i = z_i = \Theta(1)$ for all $i \in N$ and ϵ sufficiently small. So, from (4.23), we get

$$\tilde{Z}_N^{-1}X_N Z_N e_N = x_N = \mathcal{O}(\mu).$$

From (3.8) and (3.10) we have $g(x) = \mathcal{O}(\mu)$, while from (4.3) we have $\mu_c = \mathcal{O}(\mu^2)$. By combining (4.12c) and these relationships, we obtain

$$r_U = \mathcal{O}(\mu). \tag{4.25}$$

Finally, using (3.8), (3.10), (4.3), (4.9), and (4.14) in (4.12d), we have for r_V that

$$\begin{aligned} r_V &= -V^T \begin{bmatrix} g(x) \\ -\tilde{Z}_N^{-1}X_N Z_N e_N + \mu_c \tilde{Z}_N^{-1}e_N \end{bmatrix} \\ &= -V^T \begin{bmatrix} g(x^*) + \nabla g(x^*)^T(x - x^*) + \mathcal{O}(\|x - x^*\|^2) \\ -(Xe - X^*e)_N + \mu_c \tilde{Z}_N^{-1}e_N \end{bmatrix} \\ &= -V^T \begin{bmatrix} \nabla g_B^{*\,T} & \nabla g_N^{*\,T} \\ 0 & -I \end{bmatrix} \begin{bmatrix} x_B - x_B^* \\ x_N - x_N^* \end{bmatrix} - V^T \begin{bmatrix} \mathcal{O}(\|x - x^*\|^2) \\ \mu_c \tilde{Z}_N^{-1}e_N \end{bmatrix} \\ &= \mathcal{O}(\mu^2). \end{aligned} \tag{4.26}$$

(E) Estimating the size of the components c_U, $c_{\hat{U}}$, $c_{\hat{V}}$, and c_V. By the observed uniform nonsingularity of the matrix in (4.22) for all μ sufficiently small, we have from (4.24) and (4.25) that

$$c_U,\ c_{\hat{U}},\ c_{\hat{V}}\ =\ \mathcal{O}(\mu)\,.$$

Now we appeal to (4.21), (4.26) and this last result to establish

$$c_V = \mathcal{O}(\mu).$$

Since the step $(\Delta x, \Delta\lambda, \Delta z_N)$ differs by orthogonal transformations from the vector $(c_{\hat{U}}, c_{\hat{V}}, c_U, c_V)$, we have shown so far that

$$\|(\Delta x, \Delta\lambda, \Delta z_N)\| = \mathcal{O}(\mu).$$

To complete the proof, we show that $\Delta z_B = \mathcal{O}(\mu)$, where Δz_B is given by the formula (4.6). From (4.18), the first term on the right-hand side of (4.6) is of size $\mathcal{O}(\mu^2)$. Since $\tilde{X}_B^{-1} = \mathcal{O}(1)$ and, from (4.23), $X_B Z_B = \mathcal{O}(\mu)$, the second term is of size $\mathcal{O}(\mu)$. Finally, we have from $\tilde{X}_B^{-1} = \mathcal{O}(1)$ and (4.3) that the last term is of size $\mathcal{O}(\mu^2)$. Therefore $\Delta z_B = \mathcal{O}(\mu)$ and the proof is complete. □

We have just proved that the norm of the solution $(\Delta x, \Delta\lambda, \Delta z)$ of the linear system (3.7) is bounded by a constant times μ, when the current iterate (x, λ, z) lies inside $\mathcal{N}_\gamma(\epsilon)$ for some ϵ sufficiently small. We now examine the effectiveness of this step in approaching the solution set $\mathcal{S}$ by looking at its effect on the measure defined in (3.8).

LEMMA 4.2. *Let Assumptions A.1–A.5 hold. Then given any $\gamma > 0$ and $\zeta \geq 0$, there exist positive constants ϵ and $\kappa_2(\epsilon, \gamma, \zeta)$ such that for all α with $|1 - \alpha| \leq \zeta\mu(x, \lambda, z)$, we have*

$$\|(x + \alpha\Delta x, z + \alpha\Delta z)_-\| \leq \kappa_2(\epsilon, \gamma, \zeta)\mu(x, \lambda, z)^2, \tag{4.27}$$

$$\mu(x + \alpha\Delta x, \lambda + \alpha\Delta\lambda, z + \alpha\Delta z) \leq \kappa_2(\epsilon, \gamma, \zeta)\mu(x, \lambda, z)^2, \tag{4.28}$$

for all points $(x, \lambda, z) \in \mathcal{N}_\gamma(\epsilon)$.

Proof. We again use μ to denote $\mu(x, \lambda, z)$ in the proof. From the last row in (4.5), and using (4.4) and previous estimates, we have that

$$\Delta x_N = -\tilde{Z}_N^{-1}\tilde{X}_N\Delta z_N - \tilde{Z}_N^{-1}X_N Z_N e_N + \mu_c\tilde{Z}_N^{-1}e_N = -x_N + \mathcal{O}(\mu^2),$$

so that

$$(x_N + \alpha\Delta x_N) = (1 - \alpha)x_N + \mathcal{O}(\mu^2) = \mathcal{O}(\mu^2). \tag{4.29}$$

Similarly, we have from the second-last row in (4.5) that

$$\Delta z_B = -z_B + \mathcal{O}(\mu^2),$$

so that

$$(z_B + \alpha\Delta z_B) = (1 - \alpha)z_B + \mathcal{O}(\mu^2) = \mathcal{O}(\mu^2). \tag{4.30}$$

For the remaining components x_B and z_N, we have from (4.4) that

$$x_B + \alpha\Delta x_B = x_B + \mathcal{O}(\mu) \geq (1/2)x_B^* > 0 \tag{4.31}$$

and

$$z_N + \alpha \Delta z_N = z_N + \mathcal{O}(\mu) \geq (\gamma/2) e_N > 0, \tag{4.32}$$

for all ϵ sufficiently small. By combining (4.29), (4.30), (4.31), and (4.32), we obtain (4.27).

To establish the other bound (4.28), we expand $\nabla_x \ell(x + \alpha\Delta x, \lambda + \alpha\Delta\lambda, z + \alpha\Delta z)$ using Taylor series, use the first row in (3.7), and use (3.8), (4.4), the fact that $1 - \alpha = \mathcal{O}(\mu)$, and boundedness of $\mathcal{N}_\gamma(\epsilon)$ to deduce that

$$\begin{aligned}
&\nabla_x \ell(x + \alpha\Delta x, \lambda + \alpha\Delta\lambda, z + \alpha\Delta z) \\
&= \nabla f(x + \alpha\Delta x) + \nabla g(x + \alpha\Delta x)(\lambda + \alpha\Delta\lambda) - (z + \alpha\Delta z) \\
&= \nabla f(x) + \alpha\nabla^2 f(x)\Delta x + \mathcal{O}(\|\Delta x\|^2) + \nabla g(x)(\lambda + \alpha\Delta\lambda) \\
&\quad + \alpha \sum_{i=1}^{m} (\lambda + \alpha\Delta\lambda)_i \nabla^2 g_i(x)\Delta x - (z + \alpha\Delta z) + \mathcal{O}(\|\Delta x\|^2 \|\lambda + \alpha\Delta\lambda\|) \\
&= (1 - \alpha)\nabla_x \ell(x, \lambda, z) + \mathcal{O}(\|\Delta x\|^2) + \mathcal{O}(\|\Delta x\| \|\Delta\lambda\|) \\
&\quad + \mathcal{O}(\|\Delta x\|^2 \|\lambda + \alpha\Delta\lambda\|) \\
&= \mathcal{O}(\mu^2).
\end{aligned}$$

A similar procedure for $g(x + \alpha\Delta x)$ provides

$$g(x + \alpha\Delta x) = g(x) + \alpha\nabla g(x)^T \Delta x + \mathcal{O}(\|\alpha\Delta x\|^2) = (1 - \alpha) g(x) + \mathcal{O}(\mu^2) = \mathcal{O}(\mu^2).$$

Finally, we note from (4.29) and (4.32) that

$$\min(x_i + \alpha\Delta x_i, z_i + \alpha\Delta z_i) = x_i + \alpha\Delta x_i = \mathcal{O}(\mu^2), \quad \text{for all } i \in N,$$

while similarly from (4.30) and (4.31), we have

$$\min(x_i + \alpha\Delta x_i, z_i + \alpha\Delta z_i) = z_i + \alpha\Delta z_i = \mathcal{O}(\mu^2), \quad \text{for all } i \in B.$$

By substituting these estimates into (3.8), we obtain (4.28). □

Using Lemmas 4.1 and 4.2, we can now prove our final result.

THEOREM 4.3. *Let Assumptions A.1–A.5 hold. Let $\gamma > 0$ be given, and consider a value for ϵ for which Lemmas 4.1 and 4.2 are applicable. If the initial point (x^0, λ^0, z^0) belongs to $\mathcal{N}_{2\gamma}(\epsilon/2)$ and*

$$\mu(x^0, \lambda^0, z^0) \leq \min\left\{ \frac{1}{\kappa_2(\epsilon, \gamma, 0)^{1/(2-\sigma)}},\ \frac{1}{2\kappa_2(\epsilon, \gamma, 0)},\ \frac{\epsilon}{4\kappa_1(\epsilon, \gamma)},\ \frac{\gamma}{2\kappa_1(\epsilon, \gamma)} \right\}, \tag{4.33}$$

then the iterates (x, λ, z) generated by Algorithm 3.1 remain inside the neighborhood $\mathcal{N}_\gamma(\epsilon)$ and converge q-quadratically to a point $(x^*, \lambda^*, z^*) \in \mathcal{S}$.

Proof. In the proof we denote

$$\mu_k \stackrel{\text{def}}{=} \mu(x^k, \lambda^k, z^k),$$

where (x^k, λ^k, z^k), $k = 0, 1, 2 \ldots$ are the iterates generated by Algorithm 3.1. We also use $(\Delta x^k, \Delta\lambda^k, \Delta z^k)$ to denote the step calculated from (3.7) at the iterate (x^k, λ^k, z^k).

From (4.33) we have

$$\kappa_2(\epsilon, \gamma, 0)\mu_0^2 \leq \mu_0^\sigma,$$

so according to (4.27), (4.28), the tests (3.11) are satisfied by $\alpha = 1$, so the unit step will be accepted by the algorithm. Because $(x^0, \lambda^0, z^0) \in \mathcal{N}_{2\gamma}(\epsilon/2)$, and because $\mathcal{S}$ is compact, there exists $(x^*, \hat{\lambda}, \hat{z}) \in \mathcal{S}$ such that

$$\|(x^0 - x^*, \lambda^0 - \hat{\lambda}, z^0 - \hat{z})\| \leq \epsilon/2. \tag{4.34}$$

Using (4.4), (4.34), and the choice of μ_0, we have that

$$\|(x^0 + \Delta x^0 - x^*, \lambda^0 + \Delta\lambda^0 - \hat{\lambda}, z^0 + \Delta z^0 - \hat{z})\| \leq \epsilon/2 + \kappa_1(\epsilon, \gamma)\mu_0 \leq 3\epsilon/4, \tag{4.35}$$

Using (4.33) again, we obtain from Lemma 4.1 that

$$\|\Delta z_N^0\| \leq \kappa_1(\epsilon/2, 2\gamma)\mu_0 \leq \kappa_1(\epsilon, \gamma)\mu_0 \leq \gamma/2 \tag{4.36}$$

where we have used the relationship $\kappa_1(\epsilon/2, 2\gamma) \leq \kappa_1(\epsilon, \gamma)$, which is a consequence of $\mathcal{N}_{2\gamma}(\epsilon/2) \subset \mathcal{N}_\gamma(\epsilon)$. From (4.36) we have

$$z_N^0 + \Delta z_N^0 \geq 2\gamma e - (\gamma/2)e = (3\gamma/2)e. \tag{4.37}$$

Since the full step is taken to obtain (x^1, λ^1, z^1), we conclude from (4.35) and (4.37) that $(x^1, \lambda^1, z^1) \in \mathcal{N}_{3\gamma/2}(3\epsilon/4)$. Moreover, because of (4.28) and (4.33), we have also that

$$\mu_1 \leq \kappa_2(\epsilon, \gamma, 0)\mu_0^2 \leq \frac{\mu_0}{2} \leq \min\left\{ \frac{1}{\kappa_2(\epsilon, \gamma, 0)^{1/(2-\sigma)}}, \frac{1}{2\kappa_2(\epsilon, \gamma, 0)}, \frac{\epsilon}{8\kappa_1(\epsilon, \gamma)}, \frac{\gamma}{4\kappa_1(\epsilon, \gamma)} \right\}.$$

Considering the next iteration, we can show in a similar fashion that the step length $\alpha = 1$ is acceptable for $(\Delta x^1, \Delta\lambda^1, \Delta z^1)$, that $(x^2, \lambda^2, z^2) \in \mathcal{N}_{5\gamma/4}(7\epsilon/8)$, and that

$$\mu_2 \leq \frac{\mu_1}{2} \leq \min\left\{ \frac{1}{\kappa_2(\epsilon, \gamma, 0)^{1/(2-\sigma)}}, \frac{1}{2\kappa_2(\epsilon, \gamma, 0)}, \frac{\epsilon}{16\kappa_1(\epsilon, \gamma)}, \frac{\gamma}{8\kappa_1(\epsilon, \gamma)} \right\}.$$

By using induction, we obtain that $(x^k, \lambda^k, z^k) \in \mathcal{N}_\gamma(\epsilon)$ and $\mu_{k+1} \leq \kappa_2(\epsilon, \gamma, 0)\mu_k^2$ for all k. Since $\mu(x^k, \lambda^k, z^k) \to 0$ (so that $\text{dist}\left((x^k, \lambda^k, z^k), \mathcal{S}\right) \to 0$), and since $\mathcal{S}$ is compact, all limit points of the sequence $\left\{(x^k, \lambda^k, z^k)\right\}$ lie in $\mathcal{S}$. However, it is easy to see from (4.4) and the property $\mu_{k+1} \leq \mu_k/2$ for all k that the sequence of iterates is Cauchy. Hence, there is a single limit point $(x^*, \lambda^*, z^*) \in \mathcal{S}$. □

5. Discussion. Our assumption that some iterate (x^k, λ^k, z^k) eventually enters a neighborhood of the form $\mathcal{N}_\gamma(\epsilon)$ (4.1) is a significant one. In general, it will be necessary to add enhancements to Algorithm 3.1 to ensure that it holds. The key requirement is that the iterates should avoid the situation in which for some index i we have both $x_i^k \to_k 0$ and $z_i^k \to_k 0$. The condition that (x, λ, z) remain in a central path neighborhood, which we do *not* assume in our analysis, serves to meet this requirement in the convex case.

An alternative approach would be to include an enhancement that makes explicit guesses of the index sets B and N, and then modifies the iterates in a way that keeps them close to the set $\mathcal{S}$ but moves the components z_i for $i \in N$ away from their lower bound of zero where necessary. Strategies that make explicit guesses have been proposed for linear complementary problems in which the solutions fail to satisfy strict complementarity; see for example Potra and Sheng [14]. It follows that an added advantage of this approach would be to extend our technique to the case in

which Assumption A.3 is violated. A recent report by one of the authors (Wright [21]) proposes to use linear programming subproblems to identify the index sets B and N. Under similar assumptions to the ones used above, this approach is shown to make a correct determination of B and N from any point (x, λ, z) sufficiently close to $\mathcal{S}$. An adjustment strategy for replacing such a point with another point that lies in a neighborhood of the form $\mathcal{N}_\gamma(\epsilon)$ is also described in [21]. This strategy could be incorporated into Algorithm 3.1, allowing us to prove rapid local convergence from a starting point that is close to $\mathcal{S}$ but not necessarily inside a neighborhood of the form of $\mathcal{N}_\gamma(\epsilon)$.

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