Semigroup invariants of symbolic dynamical systems

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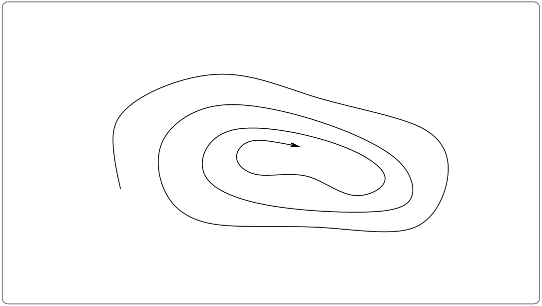
Discretization



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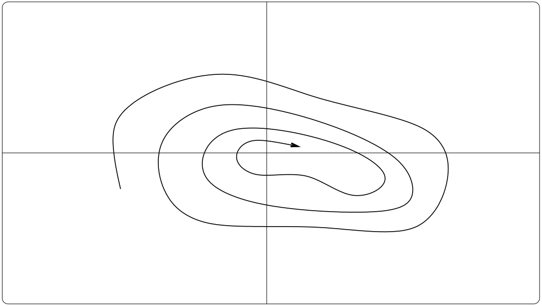

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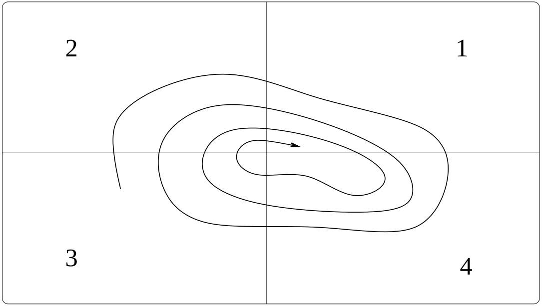

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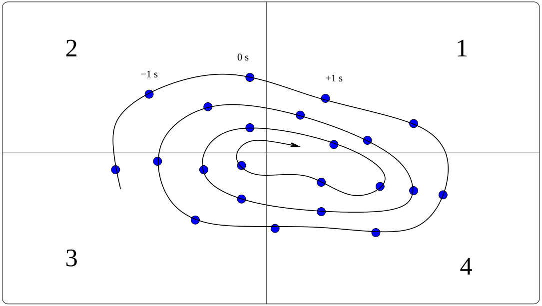

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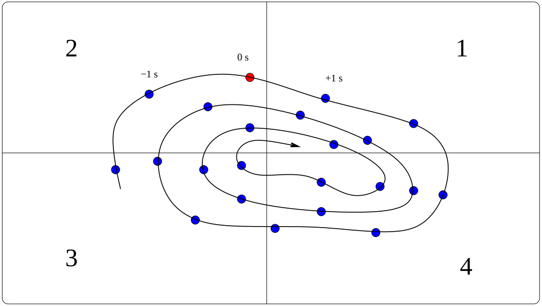

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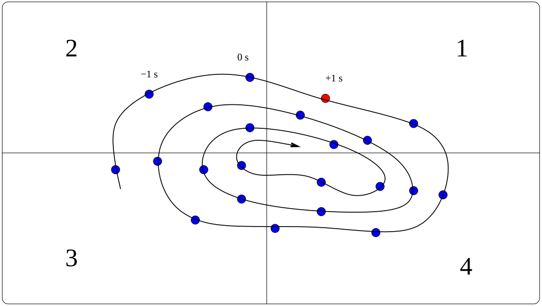

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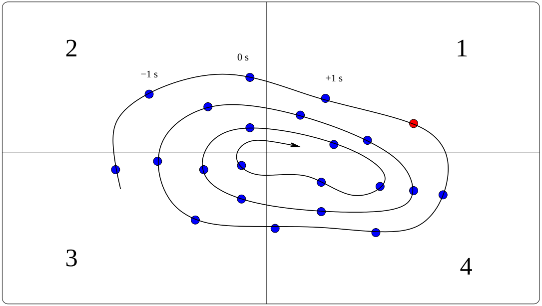
Discretization



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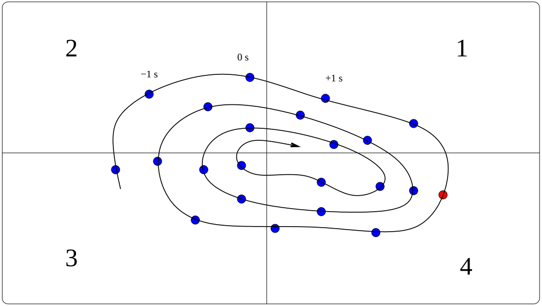
Discretization



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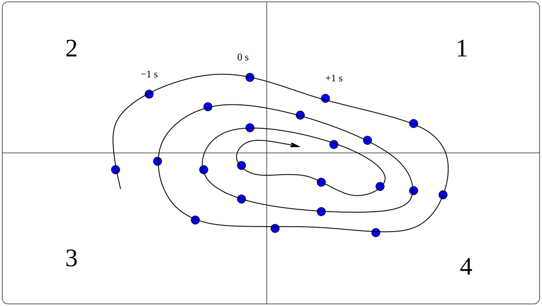
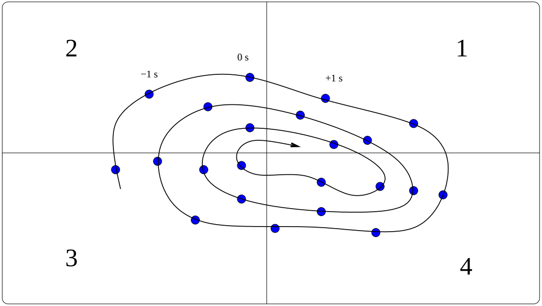
Discretization



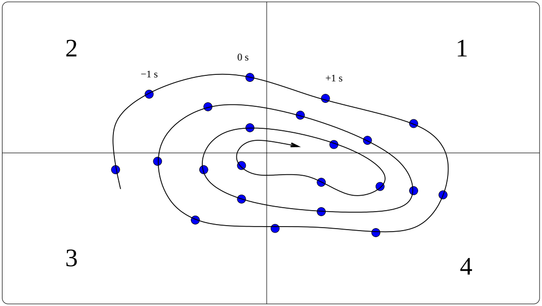
211444333211443321443 . . .

Discretization



$\ldots 32.2114443332114433214 43 \ldots$

Discretization



$$\ldots 32.2114443332114433214 43\ldots$$

This bi-infinite sequence is an element of $\{1,2,3,4\}^{\mathbb{Z}}$, i.e., a mapping from $\mathbb{Z}$ to {1,2,3,4}.

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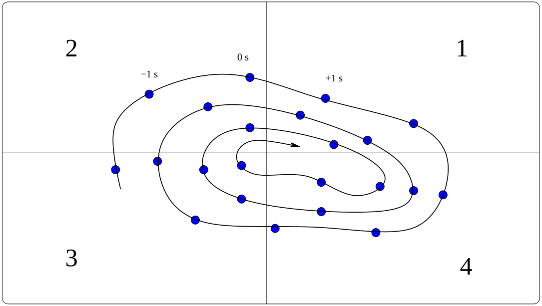

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$$\ldots * * * 32.2114443332114433321443 * \ldots$$

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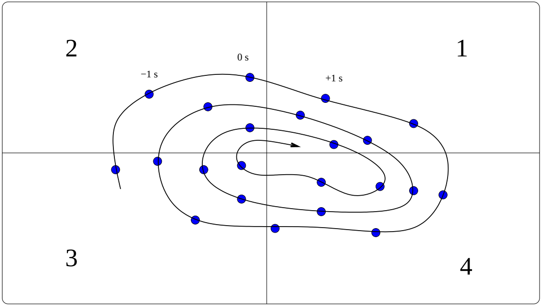

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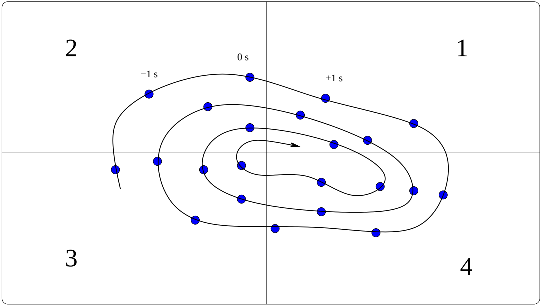
$$\ldots * *322.11444333211443321443 * * \ldots$$

Discretization



$$\ldots 32.2114443332114433214 43\ldots$$

This bi-infinite sequence is an element of $\{1,2,3,4\}^{\mathbb{Z}}$, i.e., a mapping from $\mathbb{Z}$ to {1,2,3,4}.

$$\ldots * 3221.1444333211443321443 * * * \ldots$$

Subshifts

Let A be a finite alphabet.

A symbolic dynamical system of $A^{\mathbb{Z}}$, also called subshift or just shift, is a nonempty subset $\mathcal{X}$ of $A^{\mathbb{Z}}$ such that

- $\mathcal{X}$ is topologically closed,
- $\sigma(\mathcal{X}) = \mathcal{X}$ $\qquad \sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}, \quad x_i \in A.$

The language of a subshift

The language of a subshift

$L(\mathcal{X}) = \{u \in A^+ : u = x_i x_{i+1} \dots x_{i+n}$ for some $x \in \mathcal{X},\ i \in \mathbb{Z},\ n \geq 0\}$

The elements of $L(\mathcal{X})$ are the blocks of $\mathcal{X}$.

Let $\mathcal{X}$ be the least subshift containing

$x = \cdots 32.2114443332114433214443 \cdots$

$L(\mathcal{X}) = L(\mathcal{Y})$ if and only if $\mathcal{X} = \mathcal{Y}$.

Irreducible subshifts:

$$u, v \in L(\mathcal{X}) \Rightarrow \exists w : uwv \in L(\mathcal{X})$$

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$x = \cdots 32.21144433321144332 1443 \cdots$

Then $L(\mathcal{X}) = \{\dots, 3221, 32, 22, 1, 2, 3, 2114433, 4, 21, 11, \dots\}$

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Morphisms between subshifts

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\psi} & \mathcal{Y} \\ \sigma \downarrow & & \downarrow \sigma \\ \mathcal{X} & \xrightarrow{\psi} & \mathcal{Y} \end{array}$$

Isomorphic subshifts are called *conjugate*.
An isomorphism is called *conjugacy*.

Sliding block codes

Let $x \in A^{\mathbb{Z}}$. Given a map $g : A^m \to B$, we can code x through g:

- we choose integers $k, l \geq 0$ such that $m = k + l + 1$;
- we make $y_i = g(x_{[i-k,i+l]})$.

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$$\begin{array}{c} \dots x_{i-4}\, \boxed{x_{i-3}x_{i-2}\color{blue}{x_{i-1}}x_i}\, x_{i+l}x_{i+2}x_{i+3} \dots \\ \Big\downarrow g \\ \dots y_{i-2}\, \boxed{\color{blue}{y_{i-1}}}\, y_i y_{i+1} y_{i+2} \dots \end{array}$$

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$$\ldots x_{i-4}x_{i-3}\boxed{x_{i-2}x_{i-1}x_ix_{i+1}}x_{i+2}x_{i+3}\ldots$$

$$\downarrow g$$

$$\ldots y_{i-2}y_{i-1}\boxed{y_i}y_{i+1}y_{i+2}\ldots$$

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$$\downarrow g$$
$$\cdots y_{i-2}y_{i-1}y_i\boxed{y_{i+1}}y_{i+2}\cdots$$

If $\mathcal{X}$ is a subshift of $A^{\mathbb{Z}}$ then the map $G : \mathcal{X} \to B^{\mathbb{Z}}$ defined by g is continuous and commutes with the shift operation; its image $\mathcal{Y}$ is a subshift of $B^{\mathbb{Z}}$. We say that $G : \mathcal{X} \to \mathcal{Y}$ is a *sliding block code* with block map g, memory k and anticipation l, and we write $G = g^{[-k,l]}$.

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Theorem (Curtis-Hedlund-Lyndon, 1969)

The morphisms between subshifts are precisely the sliding block codes.

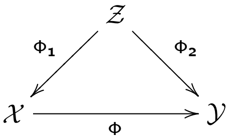
Decomposition Theorem

A sliding block code (respectively, a conjugacy) with memory and anticipation zero is called an 1-*code* (respectively, an 1-*conjugacy*).

Theorem (Williams, 1973)

Every code is the composition of an 1*-code with the inverse of an* 1*-conjugacy.*

$\Phi = \Phi_2 \circ \Phi_1^{-1}$



Some conjugacy invariants

- The number $p_n(\mathcal{X})$ of points with period n.
- The *zeta function*

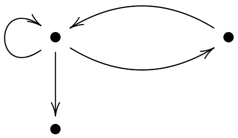
$$\zeta_{\mathcal{X}}(z) = \exp\Big(\sum_{n=1}^{+\infty} \frac{p_n(\mathcal{X})}{n} z^n\Big).$$

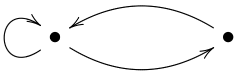
- The *entropy*

$$h(\mathcal{X}) = \lim \frac{1}{n} \log_2 |L(\mathcal{X}) \cap A^n|.$$

Edge shifts

- A subshift that is the set of bi-infinite paths on a graph is called an *edge subshift*.
- Vertices that are not in a bi-infinite path do not intervene in the definition of an edge subshift. Graphs without such vertices are called *essential graphs*.

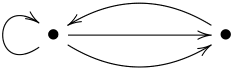




- An edge subshift is irreducible if and only if the corresponding essential graph is strongly connected.

Matrices of non-negative integers

An edge subshift is determined by its adjacency matrix.



$$\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$

Essential graphs correspond to matrices without null rows and null columns. Consider this kind of matrices only.

Let A be a square matrix of nonnegative integers and let Λ be the list of its non-zero eigenvalues, with corresponding multiplicities.

- If $\lambda_A = \max\{|\lambda| : \lambda \in \Lambda\}$ then $\lambda_A \in \Lambda$ and $h(\mathcal{X}_A) = log(\lambda_A)$;
- $\zeta_{\mathcal{X}_A}(z) = [\det(I - zA)]^{-1}$;
- $\zeta_{\mathcal{X}_A}$ and Λ determine each other.

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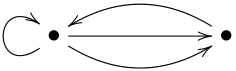

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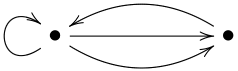

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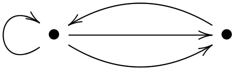

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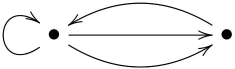

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- $\zeta_{\mathcal{X}_A}$ and Λ determine each other.

The conjugacy problem

Two square matrices of nonnegative integers are *elementary strong shift equivalent* if

$$A = RS \quad \text{and} \quad B = SR$$

for some matrices R, S of nonnegative integers.
The transitive closure of this relation is called *strong shift equivalence*.

Theorem (Williams, 1973)

$\mathcal{X}_A$ and $\mathcal{X}_B$ are conjugate if and only if A and B are strong shift equivalent.

Two square matrices of nonnegative integers are *shift equivalent* if

$$\begin{aligned} A^l &= RS & B^l &= SR \\ AR &= RB & SA &= BS \end{aligned}$$

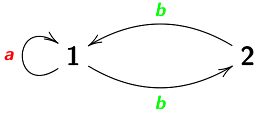
for some matrices R, S of nonnegative integers.

(Kim & Roush, 1990) Shift equivalence is decidable.

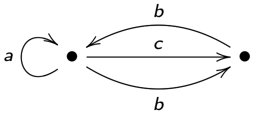
(Kim & Roush, 1999) Strong shift equivalence implies shift equivalence, but the converse is false.

Automata

- An automaton over an alphabet A is a semigroup action over a set Q of states.
- The graphical representation of an automaton is that of a labeled graph.
- The automaton is finite if A and Q are finite.
- A subshift $\mathcal{X}$ is *sofic* if and only if it is *recognized* by an *essential* automaton.
- A sofic subshift is irreducible if and only if it is presented by a *strongly connected* essential automaton.



Symbolic matrices

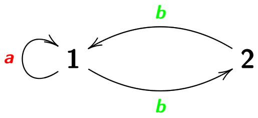


$$\begin{bmatrix} a & b+c \\ b & 0 \end{bmatrix}$$

Fischer cover

Theorem

Every irreducible sofic subshift has a unique minimal strongly connected, deterministic, reduced presentation.



Back to symbolic matrices

Two symbolic adjacency matrices A and B are *strong shift equivalent within Fischer covers* if there is a sequence of symbolic adjacency matrices of Fischer covers

$$A = A_0, A_1, \ldots, A_{l-1}, A_l = B$$

such that for $1 \leq i \leq l$ the matrices A_{i-1} and A_i are elementary strong shift equivalent.

Theorem (Nasu, 1986)

Let $\mathcal{X}$ and $\mathcal{Y}$ be irreducible sofic subshifts and let A and B be the symbolic adjacency matrices of the Fischer covers of $\mathcal{X}$ and $\mathcal{Y}$, respectively.
Then $\mathcal{X}$ and $\mathcal{Y}$ are conjugate if and only if A and B are strong shift equivalent within Fischer covers.

Syntactic congruence

Let L be a language of A^+. The *context of u in L* is the set

$$C_L(u) = \{(x, y) \in A^* \mid xuy \in L\}$$

Define

$$u \equiv_L v \text{ if and only if } C_L(u) = C_L(v).$$

Then $\equiv_L$ is a congruence, the *syntactic congruence of L*.
The quotient

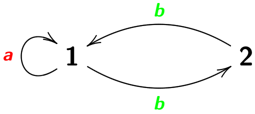
$$S(L) = A^+/{\equiv_L}$$

is the syntactic semigroup of L. The semigroup $S(L)$ is finite if and only if it is recognized by a finite automata.

We denote by $S(\mathcal{X})$ the syntactic semigroup of $L(\mathcal{X})$. The semigroup $S(\mathcal{X})$ is finite if and only if it $\mathcal{X}$ is sofic.

Transition semigroup

Each automaton has a transition semigroup defined by the action of the alphabet.
The transition semigroup of the Fischer cover of $\mathcal{X}$ is $S(\mathcal{X})$.



Some elements of $S(\mathcal{X})$:

$$a = [1, \_] = a^2, \quad b = [2, 1], \quad b^2 = [1, 2],$$

$$ab = [2, \_], \quad aba = [\_, \_] = ab^3a.$$

Green's relations

- Two elements s and t in a semigroup R are $\mathcal{J}$-equivalent if they generate the same principal ideal: $R^1sR^1 = R^1tR^1$.
- Two elements s and t in a semigroup R are $\mathcal{R}$-equivalent if they generate the same principal right ideal: $sR^1 = tR^1$.
- Two elements s and t in a semigroup R are $\mathcal{L}$-equivalent if they generate the same principal left ideal: $R^1s = R^1t$.
- $\mathcal{H} = \mathcal{R} \cap \mathcal{L}$
- $\mathcal{D} = \mathcal{R} \vee \mathcal{L}$. If S is finite then $\mathcal{J} = \mathcal{D}$.

Let H be an $\mathcal{H}$-class. The subsemigroup $T(H)$ of R^1 such that $H \cdot T(H) \subseteq H$ acts on H. If we identify elements of $T(H)$ with the same action, we get a group $\Gamma(H)$, called the Schützenberger group of H.

- If H is a group then $\Gamma(H) \simeq H$
- If H_1 and H_2 are contained in the same $\mathcal{D}$-class then $\Gamma(H_1) \simeq \Gamma(H_2)$.

Egg-Box Diagram of a $\mathcal{D}$-class

	*						*
	*		*			*	
*				*			
*		*					
			*		*		

Structural invariants

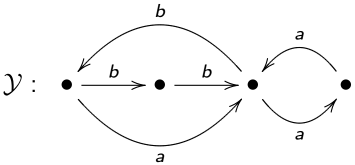
Given two $\mathcal{D}$-classes D_1 and D_2, let $D_1 \prec D_2$ if the principal ideal generated by D_1 is contained in that generated by D_2. The relation $\prec$ is a pre-order (i.e. reflexive and transitive). If the semigroup is finite, then it is a partial order (i.e. reflexive, transitive, and anti-symmetric).

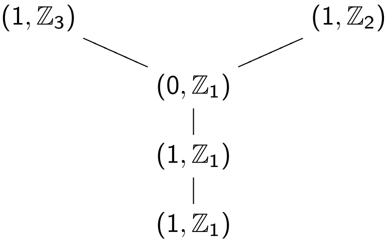
- Let $LU(\mathcal{X})$ be the set of *local units* of $S(\mathcal{X})$, that is, of elements s of $S(\mathcal{X})$ such that $s = esf$ for some idempotents e and f.
- Let $(D(\mathcal{X}), \prec)$ be the partial pre-ordered set of the D-classes of $S(\mathcal{X})$ contained in $D(\mathcal{X})$.
- Label each element D of $D(\mathcal{X})$ with the pair (ε, H), where $\varepsilon = 1$ if D contains an idempotent, $\varepsilon = 0$ if not, and H is the Schützenberger group of D.

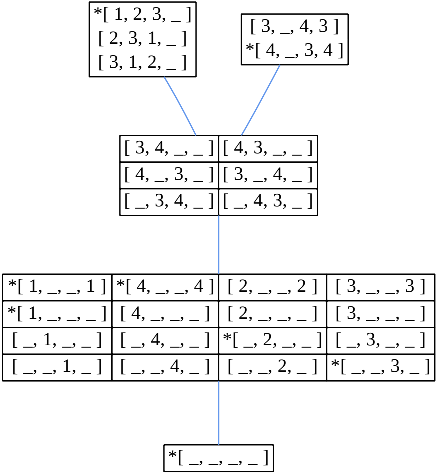
Theorem (AC, 2006 + AC & B. Steinberg, ongoing)

The labeled pre-ordered set $D(\mathcal{X})$ is a conjugacy invariant.

Example


$\mathcal{Y}$:
b
b
b
a
a
a


$(1, \mathbb{Z}_3)$
$(1, \mathbb{Z}_2)$
$(0, \mathbb{Z}_1)$
$(1, \mathbb{Z}_1)$
$(1, \mathbb{Z}_1)$


*[1, 2, 3, _]
[2, 3, 1, _]
[3, 1, 2, _]
[3, _, 4, 3]
*[4, _, 3, 4]
[3, 4, _, _]
[4, 3, _, _]
[4, _, 3, _]
[3, _, 4, _]
[_, 3, 4, _]
[_, 4, 3, _]
*[1, _, _, 1]
*[4, _, _, 4]
[2, _, _, 2]
[3, _, _, 3]
*[1, _, _, _]
[4, _, _, _]
[2, _, _, _]
[3, _, _, _]
[_, 1, _, _]
[_, 4, _, _]
*[_, 2, _, _]
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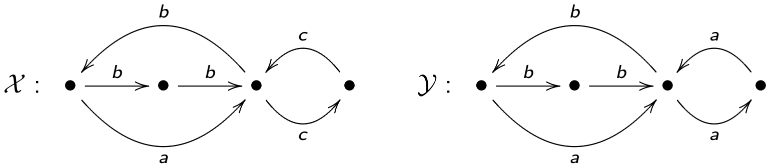
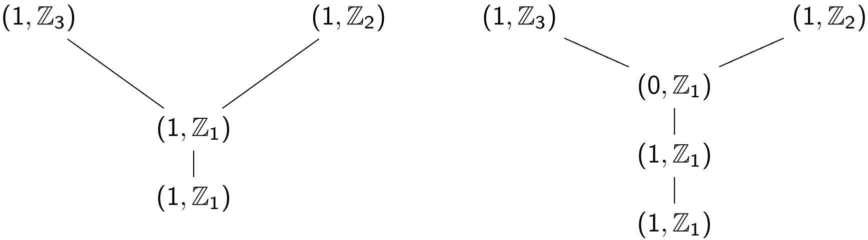

$\mathcal{X}$:
b
b
b
c
c
a
$\mathcal{Y}$:
b
b
b
a
a
a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$(1, \mathbb{Z}_1)$
$(1, \mathbb{Z}_1)$

Key idea

To code words as we code bi-infinite sequences...

... and to see the effect in the syntactic congruence...

Often, it suffices to consider 1-conjugacies. The coding of words is then just a homomorphism between free semigroups.

Working with the idea

Lemma

Let $\Phi = \phi^{[0,0]} \colon \mathcal{X} \to \mathcal{Y}$ be a conjugacy. Suppose Φ^{-1} has memory and anticipation k. Let u, v be words of length greater or equal than $2k$ such that

$$\mathrm{i}_{2k}(u) = \mathrm{i}_{2k}(v), \quad \mathrm{t}_{2k}(u) = \mathrm{t}_{2k}(v).$$

Suppose $v \in L(\mathcal{X})$. If $C_{L(\mathcal{Y})}(\phi(u)) \subseteq C_{L(\mathcal{Y})}(\phi(v))$ then $C_{L(\mathcal{X})}(u) \subseteq C_{L(\mathcal{X})}(v)$.

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- Let $(x, y) \in C_{L(\mathcal{X})}(u)$; this means $xuy \in L(\mathcal{X})$;
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Free profinite semigroup generated by A

A semigroup S is profinite if it is compact and residually finite as a topological semigroup.

The latter condition means that there is a continuous homomorphism $\varphi\colon S \to F$ onto a finite semigroup (endowed with the discrete topology) such that $\varphi(s) \neq \varphi(t)$ whenever $s \neq t$.

For every profinite semigroup S,

- $A^+ \subseteq \overline{\Omega}_A S$
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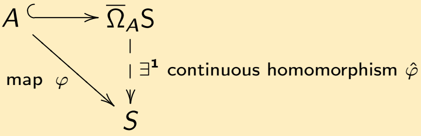

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Taking the topological closure

Let $\mathcal{X}$ be a subshift of $A^{\mathbb{Z}}$. We can consider the topological closure of $L(\mathcal{X})$ in $\overline{\Omega}_A S$, denoted $\overline{L(\mathcal{X})}$.

One-to-one mappings

$\mathcal{X} \mapsto L(\mathcal{X}) \mapsto \overline{L(\mathcal{X})} \mapsto \overline{L(\mathcal{X})} \setminus A^+$

The $\mathcal{J}$-class associated to $\mathcal{X}$

Irreducible subshifts:

$$u, v \in L(\mathcal{X}) \Rightarrow \exists w : uwv \in L(\mathcal{X})$$

As a subset of $\overline{\Omega}_A S$, the set $\overline{L(\mathcal{X})}$ contains a unique $\mathcal{J}$-minimal class, denoted $\mathcal{J}(\mathcal{X})$, which is regular.

One-to-one mapping

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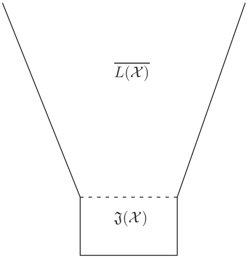
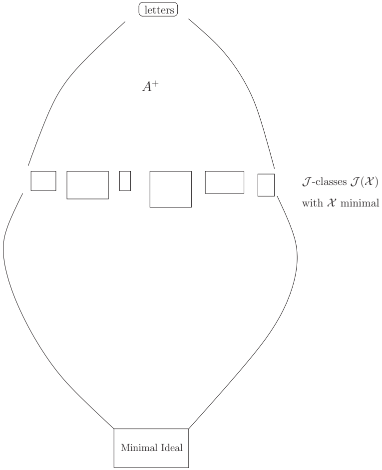



letters
A^+
$\mathcal{J}$-classes $\mathcal{J}(\mathcal{X})$
with $\mathcal{X}$ minimal
Minimal Ideal

The maximal subgroup

Theorem (AC (2006) but announced by J. Almeida (2003))

The maximal subgroup $G(\mathcal{X})$ of $\mathcal{J}(\mathcal{X})$ is a conjugacy invariant.

The group $G(\mathcal{X})$ was determined for several classes of minimal shifts:

- **(J. Almeida, 2005)** If $\mathcal{X}$ is *Arnoux-Rauzy of degree* $k \in \mathbb{N}$ then $G(\mathcal{X})$ is free profinite of finite rank k.
- **(J. Almeida, 2005)** Examples were given such that $G(\mathcal{X})$ is not free profinite, and...
- **(J. Almeida & AC, 2010)**... a presentation was given in some of these cases (e.g. Prouhet-Thue-Morse shift)

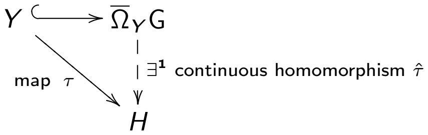
Sofic case

Theorem (AC & B. Steinberg, 2010)

If $\mathcal{X}$ is an irreducible non-periodic sofic subshift then $G(\mathcal{X})$ is a free profinite group of countable rank.

The proof of this result relies on a refinement of arguments used in the proof by B. Steinberg of the particular case concerning the maximal subgroup of the minimal ideal.

- A subset Y of a profinite group G converges to the identity if each neighborhood of the identity contains all but finitely many elements of Y.
- A free profinite group on a subset Y converging to the identity is a profinite group $F := \overline{\Omega}_Y \mathrm{G}$ generated by Y (with the further demand that Y is converging to the identity), such that every continuous map τ from Y into a profinite group H such that $\tau(Y)$ converges to the identity can be extended to a unique continuous group homomorphism $\hat{\tau} : \overline{\Omega}_Y \mathrm{G} \to H$.
- $rank(\overline{\Omega}_Y \mathrm{F}) = |Y|$



To prove a metrizable profinite group G is a free of countable rank:

For every finite group H, and every α, φ continuous onto homomorphisms, there is a continuous homomorphism $\tilde{\varphi}$...

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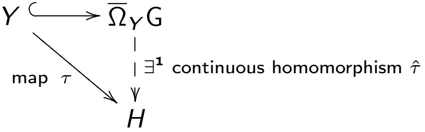

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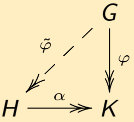

Reduction on the type of subshift to be considered

Lemma

Let $\mathcal{Y}$ be an irreducible sofic subshift of $B^{\mathbb{Z}}$. Then there is a conjugate irreducible sofic subshift $\mathcal{X}$ of $A^{\mathbb{Z}}$, an idempotent $e \in J(\mathcal{X})$ and a word z so that $e = z^{\omega}e$ and $\text{alphabet}(z) \subsetneq \text{alphabet}(\mathcal{X})$.

Since $G(\mathcal{X})$ is a conjugacy invariant:

We can suppose

$$\text{alphabet}(z) \subsetneq \text{alphabet}(\mathcal{X})$$

Open problems and bibliography

- To investigate the dynamical meaning of the semigroup invariants.
- To compute the profinite group $G(\mathcal{X})$ for more subshifts. When is it decidable?

Bibliography

- D. Lind and B. Marcus, *An introduction to symbolic dynamics and coding*, Cambridge University Press, Cambridge, 1996.
- J. Almeida, *Profinite semigroups and applications*, Structural Theory of Automata, Semigroups and Universal Algebra (New York) (V. B. Kudryavtsev and I. G. Rosenberg, eds.), Springer, 2005, pp. 1–45.
- Rhodes and Steinberg:2009, *The q-theory of finite semigroups*, Springer Monographs in Mathematics, Springer, New York, 2009.
- A. Costa, *Conjugacy invariants of subshifts: an approach from profinite semigroup theory*, Int. J. Algebra Comput. **16** (2006), no. 4, 629–655.
- A. Costa and B. Steinberg, *Profinite groups associated to sofic shifts are free*, Proc. London Math. Soc., (2010); doi: 10.1112/plms/pdq024.
- J. Almeida and A. Costa, *Presentations of schützenberger groups of minimal subshifts*, arXiv:1001.1475v1 [math.GR].