Interpretation of some integrable systems via multiple orthogonal polynomials

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Contents

The Toda lattice

We study the construction of some solutions $\{\widetilde{\alpha}_n(t)\,,\,\widetilde{\lambda}_n(t)\}\,,\ n \in \mathbb{Z}$, of the Toda complex lattice

$$\left.\begin{array}{rcl} \dot{\alpha}_n(t) & = & \lambda_{n+1}^2(t) - \lambda_n^2(t) \\ \dot{\lambda}_{n+1}(t) & = & \dfrac{\lambda_{n+1}(t)}{2}\left[\alpha_{n+1}(t) - \alpha_n(t)\right] \end{array}\right\}, \quad n \in \mathbb{S}\,, \qquad (1)$$

from another given solution $\{\alpha_n(t)\,,\,\lambda_n(t)\}\,,\ n \in \mathbb{Z}$.
We consider:

1. the semi-infinite problem: $\mathbb{S} = \mathbb{N}$, $\lambda_1 = 0$,
2. the infinite problem: $\mathbb{S} = \mathbb{Z}$,

In [P] the semi-infinite complex problem was analyzed. In the real, infinite case, sufficient conditions for the existence of a new solution were given in [GHSZ].
The problem: obtain a similar result to the complex infinite Toda lattice.

The generalized Toda lattice

In a more general way, when $\mathbb{S} = \mathbb{N}$ we consider the generalized Toda lattice of order $p \in \mathbb{N}$ (see [AB]),

$$\left.\begin{array}{rcl} \dot{J}_{nn}(t) & = & J_{n,n+1}(t)J^{p}_{n,n+1}(t) - J_{n-1,n}(t)J^{p}_{n-1,n}(t) \\ \dot{J}_{n,n+1}(t) & = & \frac{1}{2}J_{n,n+1}(t)\left[J^{p}_{n+1,n+1}(t) - J^{p}_{n,n}(t)\right] \end{array}\right\} \qquad (2)$$

where we denote by $J_{i,j}(t)$ (respectively $J^{p}_{i,j}(t)$) the entry in the $(i+1)$-row and $(j+1)$-column of matrix $J(t)$ (respectively $(J(t))^{p}$,

$$J(t) = \begin{pmatrix} \alpha_1(t) & \lambda_2(t) & \\ \lambda_2(t) & \alpha_2(t) & \ddots \\ & \ddots & \ddots \end{pmatrix}, \quad t \in \mathbb{R}\,.$$

The generalized Toda lattice admits a Lax pair representation, i.e. a formulation in terms of the commutator of two operators,

$$\dot{J}(t) = [J(t), K(t)] = J(t)K(t) - K(t)J(t)\,,$$

The generalized Toda lattice (cont.)

where for $t \in \mathbb{R}$

$$K(t) = \frac{1}{2}\begin{pmatrix} 0 & -J^p_{01}(t) & \cdots & -J^p_{0p}(t) & 0 & \cdots \\ J^p_{01}(t) & 0 & -J^p_{12}(t) & \cdots & \ddots & \\ \vdots & \ddots & \ddots & \ddots & & \\ J^p_{0p}(t) & & & & & \\ 0 & J^p_{1,p+1}(t) & \ddots & & & \\ & & & & & \\ \vdots & 0 & \ddots & & & \end{pmatrix}.$$

In [Theorem 1.1, ABM], given a solution $J(t)$ of (2), for each $C \in \mathbb{C}$ verifying

$$\det(J_n(t) - C\mathrm{I}_n) \neq 0\,, \quad n \in \mathbb{N}\,, \tag{3}$$

we prove the existence of

The generalized Toda lattice (cont.)

$$\widetilde{J}(t) = \begin{pmatrix} \widetilde{\alpha}_1(t) & \widetilde{\lambda}_2(t) & \\ \widetilde{\lambda}_2(t) & \widetilde{\alpha}_2(t) & \ddots \\ & \ddots & \ddots \end{pmatrix} \Gamma(t) = \begin{pmatrix} 0 & \gamma_2(t) & \\ \gamma_2(t) & 0 & \ddots \\ & \ddots & \ddots \end{pmatrix}$$

verifying

$$\left.\begin{array}{ll} \lambda_{n+1}^2(t) = \gamma_{2n}^2(t)\gamma_{2n+1}^2(t)\,, & \alpha_n(t) = \gamma_{2n-1}^2(t) + \gamma_{2n}^2(t) + C \\ \widetilde{\lambda}_{n+1}^2(t) = \gamma_{2n+1}^2(t)\gamma_{2n+2}^2(t)\,, & \widetilde{\alpha}_n(t) = \gamma_{2n}^2(t) + \gamma_{2n+1}^2(t) + C \end{array}\right\}$$

such that $\widetilde{J}(t)$ is another solution of (2), and $\Gamma(t)$ is a solution of the Volterra lattice:

$$\dot{\Gamma}_{n-1,n}(t) = \frac{1}{2}\Gamma_{n-1,n}(t)\left[(\Gamma^2(t) + C\mathrm{I})_{nn}^p - (\Gamma^2(t) + C\mathrm{I})_{n-1,n-1}^p\right].$$

The new solutions and the Darboux transformation

The matrix $J(t)$ t defines the sequence of polynomials given by

$$\left.\begin{array}{r} P_n(t,z) = (z-\alpha_n(t))P_{n-1}(t,z) - \lambda_n^2(t)P_{n-2}(t,z), \quad n \in \mathbb{N}\,, \\ P_{-1}(t,z) \equiv 0,\; P_0(t,z) \equiv 1\,. \end{array}\right\}$$

The main tools in the proof of [Theorem 1.3, ABM]:

a. We have established the dynamic behavior of $P_n(t,z)$,

$$\dot{P}_n(t,z) = -\sum_{j=1}^{p} J^p_{n,n-j}(t)\lambda_{n-j+2}(t)\ldots\lambda_{n+1}(t)P_{n-j}(t,z)\,,$$

b. As was proposed in [P], we use the *kernel polynomials* (cf. [C])

$$Q_n^{(C)}(t,z) = \frac{P_{n+1}(t,z) - \frac{P_{n+1}(t,C)}{P_n(t,C)}P_n(t,z)}{z-C}\,.$$

where $C \in \mathbb{C}$ verifies (3). The sequence $Q_n^{(C)}(t,C)$ satisfies a three-term recurrence relation whose coefficients define the new generalized solution $\widetilde{J}(t) = \widetilde{J}(t,C)$

The new solutions and the Darboux transformation

If we define $J^{(1)}(t) := \begin{pmatrix} \alpha_1(t) & \lambda_2(t)^2 & & \\ 1 & \alpha_2(t) & \lambda_3(t)^2 & \\ & 1 & \alpha_3(t) & \ddots \\ & & \ddots & \ddots \end{pmatrix}$ and $C \in \mathbb{C}$ verifies (3), then there exist $L(t) = \begin{pmatrix} \gamma_2^2(t) & & \\ 1 & \gamma_4^2(t) & \\ & \ddots & \ddots \end{pmatrix}$, $U(t) = \begin{pmatrix} 1 & \gamma_3^2(t) & & \\ & 1 & \gamma_5^2(t) & \\ & & \ddots & \ddots \end{pmatrix}$ such that $J^{(1)}(t) - C\mathrm{I} = L(t)U(t)$. The new solution is defined by the Darboux transformation of $J^{(1)}(t) - C\,\mathrm{I} = U(t)L(t)$, where

$$\widetilde{J}^{(1)}(t) := \begin{pmatrix} \widetilde{\alpha}_1(t) & \widetilde{\lambda}_2(t)^2 & & \\ 1 & \widetilde{\alpha}_2(t) & \widetilde{\lambda}_3(t)^2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

The infinite Toda lattice

Let us consider (1) with $\mathbb{S} = \mathbb{Z}$ and take the infinite matrix

$$J = \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & \alpha_{-1}(t) & \lambda_0(t) & & \\ & \lambda_0(t) & \alpha_0(t) & \lambda_1(t) & \\ & & \lambda_1(t) & \alpha_1(t) & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

The infinite Toda lattice admits also a Lax pair representation. Taking $\mathcal{R}_n := \begin{pmatrix} f_n & f_{-n+1} \end{pmatrix}^T$, $n \in \mathbb{N}$, it is possible to change the infinite recurrence relation for $n \in \mathbb{Z}$

$\lambda_{n+1}(t) f_{n-1}(t,z) + (\alpha_{n+1} - z) f_n(t,z) + \lambda_{n+2}(t) f_{n+1}(t,z) = 0\,,$

to a semi-infinite recurrence relation for $n \in \mathbb{N}$

$E_n(t)\mathcal{R}_{n-1}(t,z) + (V_n(t) - zI_2)\,\mathcal{R}_n(t,z) + E_{n+1}(t)\mathcal{R}_{n+1}(t,z) = 0\,,$

where E_m, V_m, $m \in \mathbb{N}$, are 2×2-finite matrices.

The infinite Toda lattice (cont.)

In this way, we can study the infinite case as a semi-infinite vectorial case. The vectors $\mathcal{R}_n$ are not polynomials, but we can prove $\mathcal{R}_n = (E_2 \cdots E_n)^{-1}\, C_n \mathcal{R}_1\,,$
where the sequence $\{C_n\}$ of 2×2 matrices verifies for all $n \in \mathbb{N}$

$$\left.\begin{array}{r} E_n^2 C_{n-1} + (V_n(t) - zI_2)\, C_n + C_{n+1} = 0 \\ C_0 = O_2\,,\ \ C_1 = I_2 \end{array}\right\}$$

i.e., $C_n = \begin{pmatrix} c_{n1}(t,z) & c_{n2}(t,z) \\ c_{n3}(t,z) & c_{n4}(t,z) \end{pmatrix}$
and for each $i = 1,2,3,4$, c_{ni} is a polynomial in z, $\deg c_{ni} \leq n-1$.
Taking $I_{-1} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $W_n := I_{-1} V_n$, $n \in \mathbb{N}$, we can show

$$\left.\begin{array}{rcl} \dot{W}_n & = & E_{n+1}^2 - E_n^2 \\ \dot{E}_{n+1} & = & \frac{1}{2} E_{n+1}(W_{n+1} - W_n) \end{array}\right\}, \ n = 2,3,\ldots \tag{4}$$

This is, $\{W_n, E_n\}$ is a solution of a semi-infinite matricial Toda lattice, like (1).

The infinite Toda lattice and the Darboux transformation

We define

$$J^{(B)} := \begin{pmatrix} V_1 & E_2^2 & & & \\ I_2 & V_2 & E_3^2 & & \\ & I_2 & V_3 & \ddots & \\ & & \ddots & \ddots & \end{pmatrix}.$$

Let $C \in \mathbb{C}$ be such that $\det\left(J_{2n}^{(B)}(t) - CI_{2n}\right) \neq 0\,, \quad t \in \mathbb{R}\,,\ n \in \mathbb{N}\,.$ Then, we know (see [IB]) that there exist two blocked matrices

$$L^{(B)} := \begin{pmatrix} A_1 & & & & \\ I_2 & A_2 & & & \\ & I_2 & A_3 & & \\ & & \ddots & \ddots & \end{pmatrix}, \quad U^{(B)} := \begin{pmatrix} I_2 & \Gamma_1 & & \\ & I_2 & \Gamma_2 & \\ & & I_2 & \ddots \\ & & & \ddots \end{pmatrix}$$

such that $J^{(B)} - CI = L^{(B)}U^{(B)}$.

The infinite Toda lattice and the Darboux transformation (cont.)

We define the blocked Darboux transformation of $J^{(B)} - CI$ as $\widetilde{J}^{(B)} - CI := U^{(B)} L^{(B)} =$

$$\begin{pmatrix} \widetilde{V}_1 - CI_2 & \widetilde{E}_2^2 & & \\ I_2 & \widetilde{V}_2 - CI_2 & \widetilde{E}_3^2 & \\ & I_2 & \widetilde{V}_3 - CI_2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}.$$

We are researching the two following questions:

1. Can we construct a vectorial solution of the Toda lattice, like (4), from $\widetilde{J}^{(B)} - CI$?
2. Are the (scalar) entries of $\widetilde{J}^{(B)}$ a new solution of the Toda lattice (1)?

Bogoyavlenskii lattice

Goal:

Characterization of solutions of some integrable systems by using matrical moments

Bogoyavlenskii lattice: Systems is given by
$\dot{J} = [J, M] = JM - MJ$, with:

$$J = \begin{pmatrix} 0 & 1 & & & & & & \\ \vdots & \ddots & \ddots & & & & & \\ 0 & \cdots & 0 & 1 & & & & \\ a_1 & 0 & \cdots & 0 & 1 & & & \\ & a_2 & 0 & \cdots & 0 & 1 & & \\ & & \ddots & \ddots & & \ddots & \ddots \end{pmatrix}, \quad M = (\gamma_{ij}) \,, \quad \gamma_{ij} = \begin{cases} 0 & , \quad i \leq j \\ \beta_{ij} & , \quad i > j \end{cases}$$

where $J^{p+1} = (\beta_{ij})$

Introduction

We study the Bogoyavlenskii lattice

$$\dot{a}_n(t) = a_n(t)\left[\sum_{i=1}^{p} a_{n+i}(t) - \sum_{i=1}^{p} a_{n-i}(t)\right] \tag{5}$$

$\Leftrightarrow \dot{J} = [J, M] = JM - MJ\,,\ J, M$ given above.

- We analyze the relationship between the solutions of (5) and the dynamic behavior of $(z\mathcal{I} - J(t))^{-1}$.
- We use, as a main tool, the sequence $\{P_n\}$ of polynomials given by the recurrence relation
$$\left.\begin{array}{l} zP_n(z) = P_{n+1}(z) + a_{n-p+1}P_{n-p}(z)\,, \quad n = p, p+1, \ldots \\ P_i(z) = z^i\,, \quad i = 0, 1, \ldots, p \end{array}\right\} \tag{6}$$
- The method of investigation is based on the analysis of the moments for J. We study the dynamic behavior of the moments.

Vector orthogonality

From the recurrence relation (6) we have

$$\begin{cases} zP_{mp}(z) = P_{mp+1}(z) + a_{(m-1)p+1}P_{(m-1)p}(z) \\ \qquad\qquad \vdots \\ zP_{(m+1)p-1}(z) = P_{(m+1)p}(z) + a_{mp}P_{mp-1}(z)\,. \end{cases}$$

Then, denoting $\mathcal{B}_m(z) = \left(P_{mp}(z), P_{mp+1}(z), \ldots, P_{(m+1)p-1}(z)\right)^T$, we can rewrite (6) as

$$\left.\begin{array}{l} z\mathcal{B}_m(z) = A\mathcal{B}_{m+1}(z) + B\mathcal{B}_m(z) + C_m\mathcal{B}_{m-1}(z),\ m \in \mathbb{N}\,, \\ \mathcal{B}_{-1} = (0, \ldots, 0)^T\,,\ \mathcal{B}_0(z) = \left(1, z, \ldots, z^{p-1}\right)^T \end{array}\right\} \qquad (7)$$

where $\quad C_m = \operatorname{diag}\{a_{(m-1)p+1}, a_{(m-1)p+2}, \ldots, a_{mp}\}$,

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \ldots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}$$

Vector orthogonality

Let $\mathcal{P}$ be the space of polynomials. We know (see Theorem 3.2 in [$*$]) that there exist p linear moment functionals $u^1, \ldots, u^p$ from $\mathcal{P}$ to $\mathbb{C}$ such that for each $s \in \{0, 1, \ldots, p-1\}$ the following orthogonality relations are satisfied

$$u^i[z^j P_{mp+s}(z)] = 0 \text{ for } \begin{cases} j = 0, 1, \ldots, m, i = 1, \ldots, s \\ j = 0, 1, \ldots, m-1, i = s+1, \ldots, p \end{cases} \tag{8}$$

[$*$] J. Van Iseghem, *Vector orthogonal relations. Vector QD-algorithm*, J. Comput. Appl. Math. **19** (1987), 141-150.

Vector orthogonality

We consider $\mathcal{P}^p := \{(q_1, \ldots q_p)^T : q_i \text{ polynomial}, \, i = 1, \ldots, p\}$,
$\mathcal{M}_{p\times p} \equiv (p \times p)$-matrices with complex entries. We define

$$\mathcal{W} : \mathcal{P}^p \to \mathcal{M}_{p\times p}\,, \quad \mathcal{W}\begin{pmatrix} q_1 \\ \vdots \\ q_p \end{pmatrix} = \begin{pmatrix} u^1[q_1] & \ldots & u^p[q_1] \\ \vdots & \ddots & \vdots \\ u^1[q_p] & \ldots & u^p[q_p] \end{pmatrix}.$$

In particular, for $m, j \in \{0, 1, \ldots\}$ we have

$$\mathcal{W}\left(z^j \mathcal{B}_m\right) = \begin{pmatrix} u^1[z^j P_{mp}(z)] & \ldots & u^p[z^j P_{mp}(z)] \\ \vdots & \ddots & \vdots \\ u^1[z^j P_{(m+1)p-1}(z)] & \ldots & u^p[z^j P_{(m+1)p-1}(z)] \end{pmatrix}.$$

Then, the orthogonality conditions (8) can be reinterpreted as
$\mathcal{W}(z^j \mathcal{B}_m) = 0\,, \quad j = 0, 1, \ldots, m-1\,.$

Vector orthogonality

For a fixed $M \in \mathcal{M}_{p\times p}$ we define the function

$$\mathcal{U}_M : \mathcal{P}^p \longrightarrow \mathcal{M}_{p\times p}\,, \quad \mathcal{U}_M \begin{pmatrix} q_1 \\ \vdots \\ q_p \end{pmatrix} = \mathcal{W} \begin{pmatrix} q_1 \\ \vdots \\ q_p \end{pmatrix} M\,.$$

(Briefly, we write $\mathcal{U}_M = \mathcal{W} M\,$.) For any $M \in \mathcal{M}_{p\times p}$, from $\mathcal{W}(z^j \mathcal{B}_m) = 0\,,\ j = 0, 1, \ldots, m-1\,,$ we have

$$\mathcal{U}_M(z^j \mathcal{B}_m) = 0\,, \quad j = 0, 1, \ldots, m-1\,. \tag{9}$$

Definition

We say that $\mathcal{U}_M$, verifying (9), is a vector of functionals defined by the sequence $\{\mathcal{B}_n\}$. Also, we say that $\{\mathcal{B}_n\}$ is a sequence of vectorial polynomials orthogonal with respect to $\mathcal{U}_M$.

Vector orthogonality

More generally, let $\{v^1, \dots, v^p\}$ be a set of linear functionals.

Definition

The function $\mathcal{V} : \mathcal{P}^p \longrightarrow \mathcal{M}_{p\times p}$ given by

$$\mathcal{V}\begin{pmatrix} q_1 \\ \vdots \\ q_p \end{pmatrix} = \begin{pmatrix} v^1[q_1] & \dots & v^p[q_1] \\ \vdots & \ddots & \vdots \\ v^1[q_p] & \dots & v^p[q_p] \end{pmatrix} M_{\mathcal{V}}$$

for each $(q_1, \dots, q_p)^T \in \mathcal{P}^p$ is called *vector of functionals* associated with the linear functionals $v^1, \dots, v^p$ and with the regular matrix $M_{\mathcal{V}} \in \mathcal{M}_{p\times p}$.

It is easy to see that, for any vector of functionals $\mathcal{V}$, we have

$$\mathcal{V}(\mathcal{Q}_1 + \mathcal{Q}_2) = \mathcal{V}(\mathcal{Q}_1) + \mathcal{V}(\mathcal{Q}_2)\,, \text{ for } \mathcal{Q}_1, \mathcal{Q}_2 \in \mathcal{P}^p, \tag{10}$$

$$\mathcal{V}(M\,\mathcal{Q}) = M\,\mathcal{V}(\mathcal{Q})\,, \text{ for } \mathcal{Q} \in \mathcal{P}^p \text{ and } M \in \mathcal{M}_{p\times p}\,. \tag{11}$$

Vector orthogonality

As a consequence of (10)-(11), if $\mathcal{U}_M$ is a vector of functional defined by the sequence $\{\mathcal{B}_n\}$, using the recurrence relation (7) and the orthogonality we have:

$$\begin{aligned}
&\mathcal{U}_M\left(z^m\mathcal{B}_m\right) = \mathcal{U}_M\left(z^{m-1}A\mathcal{B}_{m+1} + z^{m-1}B\mathcal{B}_m + z^{m-1}C_m\mathcal{B}_{m-1}\right)\\
&= A\,\mathcal{U}_M\left(z^{m-1}\,\mathcal{B}_{m+1}\right) + B\,\mathcal{U}_M\left(z^{m-1}\,\mathcal{B}_m\right) + C_m\mathcal{U}_M\left(z^{m-1}\,\mathcal{B}_{m-1}\right)\\
&= C_m\mathcal{U}_M\left(z^{m-1}\,\mathcal{B}_{m-1}\right) = (\text{ iterating }) = C_mC_{m-1}\cdots C_1\,\mathcal{U}_M\left(\mathcal{B}_0\right) .
\end{aligned}$$

In the sequel we assume $\mathcal{W}\left(\mathcal{B}_0\right)$ a regular matrix, $\mathcal{U} := \mathcal{U}_M$ for $M = \left(\mathcal{W}(\mathcal{B}_0)\right)^{-1}$. Then, $\mathcal{U}$ is the vector of functionals determined by the conditions

$$\left.\begin{array}{l}
\mathcal{U}\left(z^j\,\mathcal{B}_m\right) = \Delta_m\delta_{mj}\,, \qquad m = 1,2,\ldots, \qquad j = 0,1,\ldots,m\,,\\
\Delta_m = C_mC_{m-1}\cdots C_1\,, \quad \mathcal{U}\left(\mathcal{B}_0\right) = \mathcal{I}_p\,.
\end{array}\right\}$$

Vectorial moments

We will use the vectorial polynomials

$$\mathcal{P}_n = \mathcal{P}_n(z) = \left(z^{np}, z^{np+1}, \ldots, z^{(n+1)p-1}\right)^T, \quad n = 0, 1, \ldots .$$

(In particular, $\mathcal{P}_0 = \mathcal{B}_0$.)

Definition

Given a vector of functionals $\mathcal{V}$, for each $m = 0, 1, \ldots$, the matrix $\mathcal{V}\left(z^m \mathcal{P}_0\right)$ is called moment of order m for $\mathcal{V}$.

We are going to use the moments associated with the vector of functionals $\mathcal{U}$.

Lemma 1

For each $n = 0, 1, \ldots$ we have
$\mathcal{U}\left(z^n \mathcal{P}_0\right) = J_{11}^n$, where J_{11}^n is the finite matrix formed by the first p rows and columns of J^n.

Connection with operator theory

We assume $J = J(t)$ be a bounded operator. Then we know:

$$(\zeta\mathcal{I} - J)^{-1} = \sum_{n\geq 0} \frac{J^n}{\zeta^{n+1}}, \quad |\zeta| > \|J\|.$$

We take

$$\mathcal{R}_J(\zeta) := (\zeta\mathcal{I} - J)_{11}^{-1} = \sum_{n\geq 0} \frac{J_{11}^n}{\zeta^{n+1}}, \quad |\zeta| > \|J\|,$$

where $(\zeta\mathcal{I} - J)_{11}^{-1}$ denotes the finite matrix given by the first p rows and columns of $(\zeta\mathcal{I} - J)^{-1}$.
We are interested in studying the evolution of $\mathcal{R}_J(\zeta)$. In the sequel, we assume

$$a_n(t) \neq 0, \, |a_n(t)| \leq M, \text{ for all } n \in \mathbb{N} \text{ and } t \in \mathbb{R}.$$

The main results

Theorem 1

In the above conditions, the following statements are equivalent:

(a) $\dot{a}_n(t) = a_n(t)\left[\sum_{i=1}^{p} a_{n+i}(t) - \sum_{i=1}^{p} a_{n-i}(t)\right], \quad n \in \mathbb{N}.$

(b) For each $m, k = 0, 1, \ldots,$ we have

$$\frac{d}{dt}\mathcal{U}\left(z^k\,\mathcal{P}_m\right) = \mathcal{U}\left(z^{k+1}\,\mathcal{P}_{m+1}\right) - \mathcal{U}\left(z^k\,\mathcal{P}_m\right)\mathcal{U}\left(z\,\mathcal{P}_1\right).$$

(c) We have

$$\frac{d}{dt}\mathcal{R}_J(\zeta) = \mathcal{R}_J(\zeta)\left[\zeta^{p+1}\mathcal{I}_p - \mathcal{U}(z\,\mathcal{P}_1)\right] - \sum_{k=0}^{p}\zeta^{p-k}\,\mathcal{U}\left(z^k\,\mathcal{P}_0\right)$$

for all $\zeta \in \mathbb{C}$ such that $|\zeta| > \|J\|$.

The main results

We can obtain explicitly the resolvent function in a neighborhood of $\zeta = \infty$. Let $S(\zeta) = (s_{ij}(\zeta))$ be the $(p \times p)$-matrix with entries

$$s_{ij}(\zeta) := \sum_{k=0}^{p} \zeta^{p-k} \int \left(J_{11}^{k} \right)_{ij} e^{-\zeta^{p+1} t} e^{\int \left(J_{11}^{p+1} \right)_{jj} dt} dt \,,$$

$i, j = 1, \ldots, p$, where $(J_{11}^{n})_{ij}$ is the entry corresponding to the row i and the column j in the $(p \times p)$-block J_{11}^{n}.

We have:

Theorem 2

Under the conditions of Theorem 1, if (a) holds, then

$$\mathcal{R}_J(\zeta) = -e^{\zeta^{p+1} t} S(\zeta) e^{-\int J_{11}^{p+1} dt}$$

for each $\zeta \in \mathbb{C}$ such that $|\zeta| > \|J\|$.

Full Kostant-Toda lattice

Goal:

Characterization of solutions of some integrable systems by using matrical moments

Full Kostant-Toda lattice: Systems is given by
$\dot{J} = [J, M] = JM - MJ$, with:

$$J = \begin{pmatrix} a_1 & 1 & & \\ b_1 & a_2 & 1 & \\ c_1 & b_2 & a_3 & \ddots \\ 0 & c_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad M = \begin{pmatrix} 0 & & & \\ b_1 & 0 & & \\ c_1 & b_2 & 0 & \\ 0 & c_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

The full Kostant-Toda lattice

We consider the system

$$\left.\begin{array}{rcl} \dot{a}_n & = & b_n - b_{n-1} \\ \dot{b}_n & = & b_n(a_{n+1} - a_n) + c_n - c_{n-1} \\ \dot{c}_n & = & c_n(a_{n+2} - a_n) \end{array}\right\}, \quad n \in \mathbb{N}. \tag{12}$$

We assume $b_0 \equiv 0$, $c_n \neq 0$. We can write (12) as $\dot{J} = JJ_- - J_-J$, where

$$J = \begin{pmatrix} a_1 & 1 & & \\ b_1 & a_2 & 1 & \\ c_1 & b_2 & a_3 & \ddots \\ 0 & c_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad J_- = \begin{pmatrix} 0 & & & \\ b_1 & 0 & & \\ c_1 & b_2 & 0 & \\ 0 & c_2 & b_3 & \ddots \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

Notation

We use a similar notation as before. We consider the sequence of polynomials $\{P_n\}$ given by

$$\left.\begin{array}{r} c_{n-1}P_{n-2} + b_n P_{n-1} + (a_{n+1} - z)P_n + P_{n+1} = 0,\ n = 0, 1, \ldots \\ P_0 = 1\,, \quad P_{-1} = P_{-2} = 0 \end{array}\right\} \quad (13)$$

Taking $\mathcal{B}_m = (P_{2m}, P_{2m+1})^T$, we can rewrite (13) as

$$\left.\begin{array}{r} C_n\mathcal{B}_{n-1} + (B_{n+1} - zI_2)\mathcal{B}_n + A\mathcal{B}_{n+1} = 0\,, \quad n = 0, 1, \ldots \\ \mathcal{B}_{-1} = 0\,, \quad \mathcal{B}_0 = (1, z - a_1)^T \end{array}\right\}$$

where, for $n \in \mathbb{N}$,

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ C_n = \begin{pmatrix} c_{2n-1} & b_{2n} \\ 0 & c_{2n} \end{pmatrix},\ B_n = \begin{pmatrix} a_{2n-1} & 1 \\ b_{2n-1} & a_{2n} \end{pmatrix}$$

and C_0 is an arbitrary 2×2 matrix.

Main results: Theorem 3

We want to study the solutions of the full Kostant-Toda system in terms of J and the polynomials $\{P_n\}$, $\{\mathcal{B}_n\}$.

Theorem 3

Assume $K \in \mathbb{R}_+$ such that máx$\{|a_n(t)|, |b_n(t)|, |c_n(t)|\} \leq M$ for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$. Then, the following conditions are equivalent:

(a) $\{a_n, b_n, c_n\}$ is a solution of the full Kostant-Toda system.

(b) $\frac{d}{dt} J_{11}^n = J_{11}^{n+1} - J_{11}^n B_1 + [J_{11}^n, (J_-)_{11}]$, $n = 0, 1, \ldots$.

(c) $\dot{\mathcal{R}}_J(\zeta) = \mathcal{R}_J(\zeta)(\zeta \mathcal{I}_2 - B_1) - \mathcal{I}_2 + [\mathcal{R}_J(\zeta), (J_-)_{11}]$, $|\zeta| > \|J\|$.

(d) $\dot{\mathcal{B}}_n = -C_n \mathcal{B}_{n-1} - D_n \mathcal{B}_n$, where $D_n = \begin{pmatrix} 0 & 0 \\ b_{2n+1} & 0 \end{pmatrix}$.

Vector orthogonality

From the recurrence relation for $\{P_n\}$ we know: There exist linear functionals u^1, u^2 such that

$$\begin{cases} u^i[z^j P_{2m}] = u^i[z^j P_{2m+1}] = 0, j = 0, 1, \dots, m-1, i = 1, 2, \\ u^1[z^m P_{2m+1}] = 0. \end{cases} \tag{14}$$

Definition

If the functionals u^1, u^2 verify (14), then we say that the function $\mathcal{W} : \mathcal{P}^2 \to \mathcal{M}_{2\times 2}$ given by

$$\mathcal{W}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} u^1[q_1] & u^2[q_1] \\ u^1[q_2] & u^2[q_2] \end{pmatrix}$$

is a *vector of functionals associated with* $\{P_n\}$.

Vector orthogonality

$\mathcal{W}$ is a vector of functionals associated with $\{P_n\}$
$$\Rightarrow \mathcal{W}(z^j \mathcal{B}_m) = 0\,, \quad j = 0, 1, \ldots, m-1\,. \tag{15}$$

Definition

A function $\mathcal{W} : \mathcal{P}^2 \to \mathcal{M}_{2\times 2}$ verifying (15) is called *orthogonality vector of functionals* for the sequence $\{\mathcal{B}_n\}$.

If $\mathcal{W}$ is a vector of functionals associated with $\{P_n\}$
$\Rightarrow \mathcal{W}$ is an orthogonality vector of functionals associated with $\{\mathcal{B}_n\}$
$\Rightarrow \mathcal{W}_M \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} := \mathcal{W} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} M$ is an orthogonality vector of functionals associated with $\{\mathcal{B}_n\}$.
We assume $\mathcal{W}$ a fixed vector of functionals associated with $\{P_n\}$ such that $\mathcal{W}(\mathcal{B}_0)$ is an invertible matrix.

Vector orthogonality

In the sequel we take

$$C_0 = \begin{pmatrix} 1 & 0 \\ -a_1 & 1 \end{pmatrix}, \quad M = (\mathcal{W}(\mathcal{B}_0))^{-1} C_0 , \quad \mathcal{U} = \mathcal{W}_M$$

$$\Rightarrow \mathcal{U}(\mathcal{B}_0) = C_0 . \tag{16}$$

From the recurrence relation for $\{\mathcal{B}_n\}$,

$$\mathcal{U}(z^m \mathcal{B}_m) = C_m \mathcal{U}(z^{m-1} \mathcal{B}_{m-1}) , \quad m \in \mathbb{N} . \tag{17}$$

Using (16) and (17)

$$\mathcal{U}(z^j \mathcal{B}_m) = \begin{cases} 0 & , \quad j = 0, 1, \ldots, m-1 \\ C_m C_{m-1} \cdots C_0 & , \quad j = m . \end{cases}$$

Matrical moments

We use the vectors $\mathcal{P}_m = \mathcal{P}_m(z) = \left(z^{2m}, z^{2m+1}\right)^T$.

Definition

For each $m = 0, 1, \ldots$, the matrix $\mathcal{U}\left(z^m \mathcal{P}_0\right)$ is called *moment of order* m for the vector of functionals $\mathcal{U}$.

In particular: $\mathcal{B}_0 = C_0 \mathcal{P}_0 \Rightarrow \mathcal{U}\left(\mathcal{P}_0\right) = \mathcal{I}_2$.

We define the derivative of $\mathcal{U} = \mathcal{U}_t$ as usual,

$$\frac{d\mathcal{U}}{dt}(\mathcal{B}) = \text{lím}_{\Delta t \to 0} \frac{\mathcal{U}\{t+\Delta t\}(\mathcal{B}) - \mathcal{U}\{t\}(\mathcal{B})}{\Delta t}$$
$$\Rightarrow \frac{d}{dt}\left(\mathcal{U}(\mathcal{B})\right) = \frac{d\mathcal{U}}{dt}(\mathcal{B}) + \mathcal{U}(\dot{\mathcal{B}}), \quad \forall \mathcal{B} \in \mathcal{P}^2 .$$

We define the *function of the moments* as
$\mathcal{F}_J(\zeta) = C_0^{-1} \mathcal{R}_J(\zeta) C_0 , \quad |\zeta| > \|J\| .$

Main results: Theorem 4

We will see that Theorem 3 is a direct consequence of the following result:

Theorem 4

In the conditions of Theorem 3, assume $\dot{a}_1 = b_1$. Then, the following assertions are equivalent:

(e) $\{a_n\,,\,b_n\,,\,c_n\}\,,\,n\in\mathbb{N}$, is a solution of the full Kostant-Toda system.

(f) $\dfrac{d}{dt}\mathcal{U}\left(z^n\mathcal{P}_0\right)=\mathcal{U}\left(z^{n+1}\mathcal{P}_0\right)-\mathcal{U}\left(z^n\mathcal{P}_0\right)\mathcal{U}\left(z\mathcal{P}_0\right)\,,\,n=0,1,\ldots\,.$

(g) $\dot{\mathcal{F}}_J(\zeta)=\mathcal{F}_J(\zeta)\left(\zeta\mathcal{I}_2-\mathcal{U}(z\mathcal{P}_0)\right)-\mathcal{I}_2\,,\quad|\zeta|>\|J\|\,.$

(h) $\left(\dfrac{d}{dt}\mathcal{U}\right)(\mathcal{B})=\mathcal{U}(z\mathcal{B})-\mathcal{U}(\mathcal{B})\mathcal{U}(z\mathcal{P}_0)\,,\,\mathcal{B}\in\mathcal{P}^2\,.$

(i) $\dot{\mathcal{B}}_n=-C_n\mathcal{B}_{n-1}-D_n\mathcal{B}_n\,,\,n=0,1,\ldots\,.$

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