Rolling Pseudo-Riemannian Manifolds

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Contents

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Rolling the Hyperbolic *n*-sphere over its affine tangent space at a point, both embedded in the generalized Minkowski space $\mathbb{R}_1^{n+1}$:

- Geodesics
- The Kinematic Equations
- Paralell Transport

Rolling Motions in Euclidean Space

Some Notations

- SO_N (Special Orthogonal Group) $\rightsquigarrow$ connected component of the orthogonal group O_N containing the identity matrix.
- $\mathrm{SE}_N = \mathrm{SO}_N \ltimes \mathbb{R}^N$ (Special Euclidean Group) $\rightsquigarrow$ group of isometries preserving orientations, also called rigid motions (these include rotations, translations and combinations of them).

Action of SE_N on $\mathbb{R}^N$

$(R, s) \in \mathrm{SE}_N$:

- $(R, s) \circ p = Rp + s \rightsquigarrow$ action of (R, s) on points $p \in \mathbb{R}^N$

- The action of SE_N on $\mathbb{R}^N$ induces a linear map between tangent spaces, sending η to $R\eta$

Rolling Map - Euclidean Case

M_1 and M_0 $\rightsquigarrow$ manifolds of the same dimension embedded in $(\mathbb{R}^N, \langle .,. \rangle)$

A rolling map of M_1 on M_0, without slipping or twisting over a curve $\alpha : [0,\tau] \to M_1$ $(\tau > 0)$ is a smooth map:

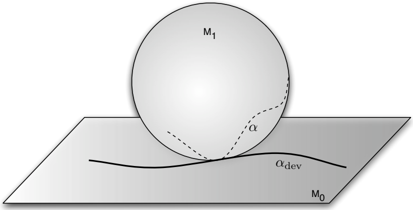
$$\begin{array}{rccc} h: & [0,\tau] & \to & SE_N = SO_N \ltimes \mathbb{R}^N \\ & t & \mapsto & h(t) = (R(t), s(t)) \end{array}$$

satisfying, for each $t \in [0,\tau]$, the following (RMC) conditions:

1. Rolling conditions
2. No-slip condition
3. No-twist conditions

- $h(t) \circ \alpha(t) =: \alpha_{\mathrm{dev}}(t) \in M_0$
- $T_{h(t)\circ\alpha(t)}(h(t) \circ M_1) = T_{h(t)\circ\alpha(t)} M_0$
  - α is called **rolling curve on** M_1
  - α_{dev} is called **development of** α **on** M_0

Example: S^2



No-slip condition

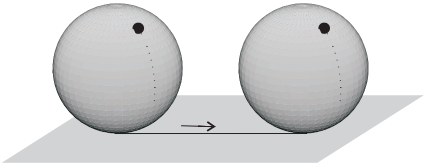
$h(t) \circ M_1$ and M_0 have the same velocity at the contact point, that is:

$$h(t) \circ \dot{\alpha}(t) = \dot{\alpha}_{\mathrm{dev}}(t)$$

$$\Updownarrow$$

$$\dot{R}(t)R^\top(t)(\alpha_{\mathrm{dev}}(t) - s(t)) + \dot{s}(t) = 0$$

Example: S^2



(slip)

No-twist conditions

Tangential Part

Any tangent vector field $X(t)$ along $\alpha(t)$ is parallel along $\alpha(t)$ iff $h(t) \circ X(t)$ is parallel along $\alpha_{\mathrm{dev}}(t)$.

$$\Updownarrow$$

$$\dot{R}(t)R^{\top}(t)T_{\alpha_{\mathrm{dev}}(t)}M_0 \subset (T_{\alpha_{\mathrm{dev}}(t)}M_0)^{\perp}$$

Normal Part

Any normal vector field $Z(t)$ along $\alpha(t)$ is normal parallel along $\alpha(t)$ iff $h(t) \circ Z(t)$ is normal parallel vector field along $\alpha_{\mathrm{dev}}(t)$.

$$\Updownarrow$$

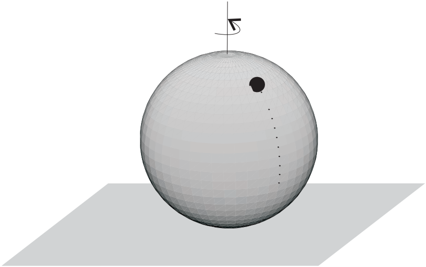
$$\dot{R}(t)R^{\top}(t)(T_{\alpha_{\mathrm{dev}}(t)}M_0)^{\perp} \subset T_{\alpha_{\mathrm{dev}}(t)}M_0$$

No-twist conditions

Remarks:

- Tangential Part is always satisfied for manifolds of dimension 1
- Normal Part is always satisfied for manifolds of codimension 1 (such as Euclidean sphere S^n, hyperbolic sphere H^n)

Example: S^2



(twist)

Our Goal Now

Is to generalize the concept of rolling for pseudo-Riemannian manifolds

$$\Downarrow$$

the metric fails to be positive definite

Pseudo-Riemannian Manifold

Smooth manifold $\overline{M}$ furnished with a metric tensor $\overline{g}$ (a symmetric nondegenerate (0,2) tensor field on $\overline{M}$ of constant index.)

If $(\overline{M}, \overline{g})$ is a pseudo-Riemannian manifold and $v \in T_p\overline{M}$, then:

- v is spacelike if $\overline{g}(v, v) > 0$ or $v = 0$
- v is timelike if $\overline{g}(v, v) < 0$
- v is lightlike if $\overline{g}(v, v) = 0$ and $v \neq 0$

Rolling Map – Pseudo-Riemannian Case

M_1 and M_0 $\rightsquigarrow$ manifolds of the same dimension embedded in a pseudo-Riemannian space $(\overline{M}, \overline{g})$

A rolling motion of M_1 over M_0, without slipping or twisting is described by a smooth mapping:

$$\begin{array}{rccc} h: & [0,\tau] & \longrightarrow & \overline{G} \\ & t & \longmapsto & h(t) \end{array}$$

that satisfies the (RMC) conditions, where:

- $\overline{G}$ is now the connected group of orientation preserving isometries of $\overline{M}$
- orthogonality being taken with respect to the pseudo-Riem. metric $\overline{g}$
- $\circ$ being the action of $\overline{G}$ on our manifolds

$N = n + 1$

$\overline{M} = \mathbb{R}_1^{n+1} \rightsquigarrow$ denotes $\mathbb{R}^{n+1}$ equipped with the pseudo-Riemannian metric:

$$\overline{g}(x, y) := \langle x, y \rangle_J = \langle x, Jy \rangle = x^\top Jy, \text{ with } J = \text{diag}(I_n, -1)$$

$M_1 := H^n \rightsquigarrow$ n-dimensional hyperbolic sphere (connected component)

$$H^n = \{p \in \mathbb{R}^{n+1} : \langle p, p \rangle_J = -1 \quad \text{and} \quad p_{n+1} > 0\}$$

$$T_{p_0} H^n = \{v \in \mathbb{R}^{n+1} : v = \Omega p_0, \;\; \Omega \in \mathfrak{so}(n,1)\}$$

$$\mathfrak{so}(n,1) = \{A \in \mathfrak{gl}(n+1) : A^\top J = -JA\}$$

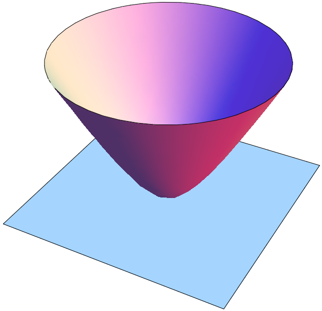
$$\text{SO}(n,1) = \{X \in \text{GL}(n+1) : X^\top JX = J \quad \text{and} \quad \det(X) = 1\}$$

$$M_0 := T_{p_0}^{\text{aff}} H^n = \{x \in \mathbb{R}^{n+1} : x = p_0 + \Omega p_0, \ \ \Omega \in \mathfrak{so}(n,1)\}$$

$(T_{p_0} H^n)^{\perp} = \mathbb{R} p_0$ (codimension 1)

Example: $n = 2$

$M_1 = H^2$
$M_0 = T_{p_0}^{\text{aff}} H^2$



$SE(n+1) \rightsquigarrow$ is the connected group of orientation preserving isometries of the Euclidean space $\mathbb{R}^{n+1}$

Question

What is the connected group of orientation preserving isometries of the Minkowski space $\mathbb{R}_1^{n+1}$?

Answer:

$$\overline{G} = SO_o(n,1) \ltimes \mathbb{R}^{n+1}$$

where

$$SO_o(n,1) = \{R \in GL(n+1) : R^\top J R = J, \quad R_{n+1,n+1} > 0\}$$

Rolling the Hyperbolic n-sphere - Remarks

Elements in $\overline{G}$ are pairs (R, s), $R \in SO_o(n, 1)$, $s \in \mathbb{R}_1^{n+1}$

Group Operations

$(I, 0)$ is the identity and:

- $(R_1, s_1)(R_2, s_2) := (R_1 R_2, R_1 s_2 + s_1)$
- $(R, s)^{-1} := (R^{-1}, -R^{-1} s)$

Action of $\overline{G}$ on $\mathbb{R}^{n+1}$

$(R, s) \circ x = Rx + s$

$\downarrow$

this induces a linear map between $T_x\mathbb{R}^{n+1}$ and $T_{Rx+s}\mathbb{R}^{n+1}$, sending every η to $R\eta$

Rolling the Hyperbolic n-sphere - Remarks

- The Lie algebra of $SO_o(n,1)$ is $\mathfrak{so}(n,1)$
- $SO_o(n,1)$ is a Lie subgroup of $SO(n,1)$ and

$$R \in SO_o(n,1) \text{ and } p \in H^n \Rightarrow Rp \in H^n$$

$$\downarrow$$

the "rotational" part of the rolling map maintains H^n invariant

The restriction of $\langle .,. \rangle_J$ to T_pH^n at any point $p \in H^n$ is positive definite

$$\downarrow$$

Although H^n is embedded in a pseudo-Riemannian manifold $\mathbb{R}^{n+1}_1$, it is indeed a Riemannian manifold (all tangent vectors are spacelike)

Rolling the Hyperbolic n-sphere - Geodesics

Geodesics on H^n, with respect to the Riemannian metric $\langle ., . \rangle_J$, can be written explicitly:

Let $p \in H^n$ and $v \in T_p H^n$ with $\langle v, v \rangle_J = 1$. Then,

$$t \mapsto \gamma(t) = p \cosh t + v \sinh t$$

is the geodesic in H^n satisfying $\gamma(0) = p$, $\dot{\gamma}(0) = v$.

Let $p, q \in H^n$. Then,

$$t \mapsto \gamma(t) = p \left(\cosh t - \frac{\cosh \theta}{\sinh \theta} \sinh t \right) + q \frac{\sinh t}{\sinh \theta},$$

where θ is defined by $\cosh \theta = -\langle p, q \rangle_J$ is the geodesic in H^n that joins the point p (at $t = 0$) to the point q (at $t = \theta$).

Rolling the Hyperbolic n-sphere - Kinematic Equations

$M_1 := H^n \qquad M_0 := T^{\text{aff}}_{p_0} H^n = p_0 + T_{p_0} H^n \qquad$ rolling curve α s.t. $\alpha(0) = p_0$

$H^n \cap T^{\text{aff}}_{p_0} H^n = \{p_0\}$

$t \mapsto u(t) \in \mathbb{R}^{n+1}$ a piecewise smooth function s.t. $\langle u(t), p_0 \rangle_J = 0$

Kinematic Equations (KE)

$$\begin{cases} \dot{s}(t) = u(t) \\ \dot{R}(t) = R(t)\left(-u(t)p_0^\top + p_0 u^\top(t)\right) J \end{cases}$$

If $(R, s) \in \overline{G}$ is the solution of (KE) satisfying $s(0) = 0$, $R(0) = I$, then:

- $t \mapsto h(t) = (R^{-1}(t), s(t)) \in \overline{G} \rightsquigarrow$ rolling map of H^n over $T^{\text{aff}}_{p_0} H^n$
- $t \mapsto \alpha(t) = R(t)p_0 \rightsquigarrow$ rolling curve
- $t \mapsto \alpha_{\text{dev}}(t) = p_0 + s(t) \in T^{\text{aff}}_{p_0} H^n \rightsquigarrow$ development curve

Rolling the Hyperbolic n-sphere - Kinematic Equations

Definition

$$\begin{array}{rccl} A: & \mathbb{R} & \rightarrow & \mathfrak{so}(n,1) \\ & t & \mapsto & A(t) = \left(-u(t)p_0^\top + p_0 u^\top(t)\right) J \end{array}$$

Assumptions: $u^\top(t)Jp_0 = 0$ $\quad$ $p_0^\top Jp_0 = -1$

$$A^{2j-1} = \left(u^\top(t)Ju(t)\right)^{j-1} A(t); \qquad A^{2j} = \left(u^\top(t)Ju(t)\right)^{j-1} A^2(t)$$

If u(t)=u (constant), then

$$e^{At} = I + \frac{\cosh \rho t}{\rho^2} A^2 + \frac{\sinh \rho t}{\rho} A,$$

where $A = \left(-up_0^\top + p_0 u^\top\right) J \in \mathfrak{so}(n,1)$ and $\rho := (u^\top Ju)^{\frac{1}{2}}$

Rolling the Hyperbolic n-sphere - Kinematic Equations

Example 1

$u(t) = u$ (constant) s.t. $u^\top J p_0 = 0$ and $(u^\top J u)^{\frac{1}{2}} = 1$

Solution of KE, with initial conditions $s(0) = 0$, $R(0) = I$

$$\begin{cases} s(t) = ut \\ R(t) = e^{At} \end{cases}$$

- $\alpha(t) = e^{At} p_0 = p_0 \cosh t + u \sinh t$

  (geodesic in H^n passing through p_0 at $t = 0$ with initial velocity u)

- $\alpha_{\mathrm{dev}}(t) = p_0 + ut$

  (geodesic in $T^{\mathrm{aff}}_{p_0} H^n$)

Rolling the Hyperbolic n-sphere - Kinematic Equations

Example 2

$p_0 = e_{n+1} = [\begin{array}{cccccc} 0 & 0 & \dots & 0 & 1 \end{array}]^\top$, then we must have
$u = [\begin{array}{ccccc} u_1 & u_2 & \dots & u_n & 0 \end{array}]^\top$

Solution of KE

$$\begin{cases} \dot{s}(t) = u(t) \\ \dot{R}(t) = R(t) \left(\sum_{i=1}^{n} u_i(t)(E_{i,n+1} + E_{n+1,i}) \right) \end{cases}$$

where the matrices $E_{i,j}$ have all entries equal to zero except the entry (i,j) which is equal to 1.

- These (KE) can be rewritten as a right-invariant control system evolving on $\mathrm{SO}_o(n,1) \ltimes \mathbb{R}^{n+1}$.
- For $n \geq 2$, this system is controllable.

Idea:

The tangent (resp. normal) parallel transport of a vector Y_0, tangent (resp. normal) to a manifold at a point p_0, along a curve α, s.t. $\alpha(0) = p_0$, can be accomplished by rolling (without slip or twist) along that curve.

Rolling the Hyperbolic n-sphere - Parallel Transport

Let $h(t) = (R^{-1}(t), s(t)) \in \overline{G}$ be a rolling map for H^n, with rolling curve α satisfying $\alpha(0) = p_0$.

Tangent Parallel Transport

If $\Omega p_0 \in T_{p_0} H^n$, then

$$Y(t) = R(t)\Omega p_0$$

defines the unique tangent parallel vector field along α, satisfying $Y(0) = \Omega p_0$.

Normal Parallel Transport

If $Z_0 \in T^{\perp}_{p_0} H^n$, then

$$Z(t) = R(t) Z_0$$

defines the unique normal parallel vector field along α, satisfying $Z(0) = Z_0$.

Final Remarks

- Lorentzian manifolds - their applications to theory of general relativity
- Stiefel Manifolds
- S^n, SO_n and Grassmann Manifolds
- Quadratic Lie groups
- Riemannian manifolds
- Controllability

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