Introduction to geometric control theory - controllability and Lie bracket -

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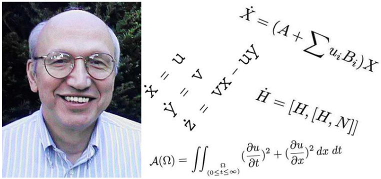
Nonlinear Geometric Control

- Control theory is a theory that deals with influencing the behavior (controlling) of dynamical systems.
- Many processes in industries like robotics and aerospace industry have strong nonlinear dynamics.
- The configuration spaces of many mechanical systems are smooth manifolds (Lie groups, symmetric spaces,...). Techniques from differential and Riemannian geometry are fundamental in modern control theory.

The evolution ...

- Beginning - 60's

Roger Brockett
- the father -



- Adultness - mid 70's

Agrachev, Bloch, Crouch, Nijmeijer, Jurdjevic, Krener, Sachkov, Sontag, Sussmann, Van der Schaft, ...

- The steam of publications has grown sharply in recent years and gives every indication of continuing to grow...

What is a dynamical system?

A dynamical system is a differential equation

$$\dot{x} = f(x), \quad x \in M.$$

- M is a smooth manifold.
- f is a (smooth) vector field on M:

$$x \in M \mapsto f(x) \in T_x M \text{ (tangent space to } M \text{ at } x).$$

γ is a solution of $\dot{x} = f(x) \Leftrightarrow \gamma$ is an integral curve of f

A dynamical system is a vector field

Vector fields and flows

Assume that the vector field f is complete (i.e., for all $x_0 \in M$ the solution $x(t, x_0)$ of the Cauchy problem $\dot{x} = f(x)$, $x(0) = x_0$, is defined for all $t \in \mathbb{R}$).

Flow generated by the vector field f:

$$t \;\to\; \exp(t\,f) : \begin{array}{ccc} M & \to & M \\ x_0 & \mapsto & x(t, x_0) \end{array} \qquad , \; t \in \mathbb{R}$$

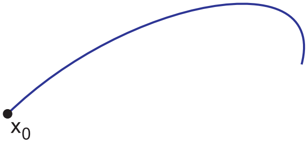
If $\dot{x} = f(x)$ describes the dynamics of a moving fluid in M, then $\exp(t\,f)$ takes any particle of the fluid from a position x_0 and moves it for a time $t \in \mathbb{R}$ to the position $\exp(t\,f)(x_0) = x(t, x_0)$.

(If f not complete, flow is local)

Dynamical systems

A dynamical system evolving on a smooth manifold M is a vector field $f(.)$ on M

$$\dot{x}(t) = f(x(t)), \quad x(t) \in M.$$



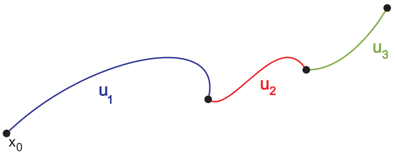
The dynamics of this system is determined by the flow of one vector field only. The future $x(t, x_0)$ is completely determined by the present state x_0.

In order to affect (control) the dynamics we must consider a family of vector fields.

What is a control system?

A control system evolving on M is a family of vector fields $f(., u)$ on M, parameterized by the controls u.

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t) \in M, \quad u(t) \in U \subset \mathbb{R}^m$$



x is the state of the system, M is the state space

u is the input or control of the system

In control theory we can change the dynamics of the control system at any moment of time by changing the control u.

Technical assumptions

$$\dot{x} = f(x(t), u(t)), \quad x(t) \in M, \quad u(t) \in U \subset \mathbb{R}^m$$

The controls belong to a class $\mathcal{U}$ of admissible controls. Its choice depends on what the control system is modeling.

- On the set of admissible controls:

$\mathcal{U}$ contains all the piecewise constant functions with values in U, which are piecewise continuous from the right.

> *Results on the continuity of solutions guarantee that if a more general control function is approximated by piecewise constant functions, the solution of the control system for this class of admissible controls approximates the solution of the original system.*

- On the vector fields: For each $x_0 \in M$ and $u \in \mathcal{U}$, the ODE

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0$$

has a solution for all $t \in [0, \infty[$.

Lie brackets

The set of all smooth vector fields on M forms a Lie algebra.

We can perform linear combinations and Lie brackets.

In terms of coordinates, vector fields may be identified with column matrices.

Lie bracket of two vector fields

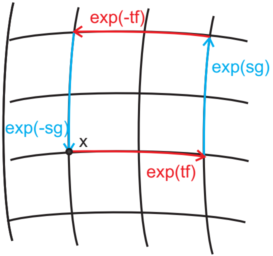
$$[f, g](x) = \frac{\partial g}{\partial x}(x)f(x) - \frac{\partial f}{\partial x}(x)g(x).$$

($\frac{\partial g}{\partial x}$ is the Jacobian matrix of g).

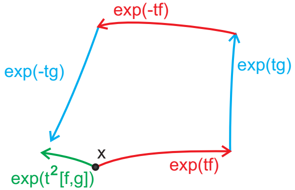
Lie brackets (cont.)

The Lie bracket of 2 vector fields measures non-commutativity of the corresponding flows

$$[f, g] \equiv 0 \;\Leftrightarrow\; \exp(t\,f) \circ \exp(sg) = \exp(sg) \circ \exp(t\,f),\ \forall s, t \in \mathbb{R}.$$



f and *g* commute



f and *g* don't commute

The power of Lie brackets

The Lie bracket allows studying interconnections between different dynamical systems.

For control theory it is particularly important that

$$[f, g] \notin \operatorname{span}\{f, g\}.$$

Reachable sets

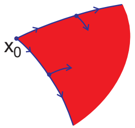
$$f_u(.) := f(., u), \quad \mathcal{F} = \{f_u\}_{u \in \mathcal{U}}$$

Reachable set of $\mathcal{F}$ from a point $x_0 \in M$

$$\mathcal{R}(x_0) = \{exp(t_k f_{u_k}) \cdots exp(t_1 f_{u_1})(x_0) \,|\, k \in \mathbb{N},\ f_{u_i} \in \mathcal{F},\ t_i \geq 0\}$$

The reachable set characterize the states that can be reached from a given initial state $x_0 \in M$ in positive time, by choosing various controls and switching from one to another from time to time.

Only forward-in-time motions allowed!



Reachable set

Controllability

Controllability is the ability to steer a system from a given initial state to any final state, in finite time, using the available controls.

A system is said to be **controllable** if $\mathcal{R}(x) = M, \ \forall x \in M$.

Affine control systems

$$\dot{x} = \underbrace{g_0(x) + \sum_{i=1}^{m} u_i g_i(x)}_{f(x,u)}$$

• g_0 is the drift vector field - specifies the dynamics in the absence of controls.

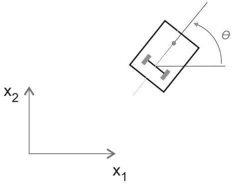
• $g_i,\ i = 1, \cdots m$, are called the control vector fields

Assumptions:

• On the set of admissible controls:
$\mathcal{U}$ consists of all the piecewise constant functions with values in U, which are piecewise continuous from the right.

• On the vector fields:
$g_0, g_1, \cdots, g_m$ are smooth (of class $\mathcal{C}^\infty$). $m \leq n = \dim(M)$.

The car



The state of the car:

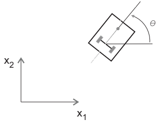
- position of its center of mass $(x_1, x_2) \in \mathbb{R}^2$
- orientation angle $\theta \in S^1$ (relative to the positive direction of the axis x_1)

The state space:

$$M = \{x = (x_1, x_2, \theta) \,|\, x_1, x_2 \in \mathbb{R},\ \theta \in S^1\} = \mathbb{R}^2 \times S^1.$$

The car (cont.)

Possible kinds of motion:



Linear motion: drive the car forward and backward with some fixed linear velocity $u_1 = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}$

$$\begin{cases} \dot{x}_1 & = & u_1 \cos\theta \\ \dot{x}_2 & = & u_1 \sin\theta \\ \dot{\theta} & = & 0 \end{cases}$$

(dynamical system for linear motion)

Rotational motion: turn the car around its center of mass with with some fixed angular velocity $u_2 = \dot{\theta}$

$$\begin{cases} \dot{x}_1 & = & 0 \\ \dot{x}_2 & = & 0 \\ \dot{\theta} & = & u_2 \end{cases}$$

(dynamical system for rotational motion)

The car (cont.)

In vector form:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \theta \end{bmatrix} \in \mathbb{R}^2 \times S^1, \quad g_1(x) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \quad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

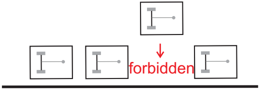
Combining both kinds of motion in an admissible way:

$$\dot{x} = \underbrace{u_1 g_1(x)}_{\text{linear motion}} + \underbrace{u_2 g_2(x)}_{\text{rotational motion}}$$

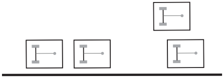
Affine control system, underactuated.

The control $u = (u_1, u_2)$ can take any value in $U \subset \mathbb{R}^2$

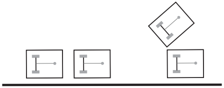
Typical maneuver in parking a car



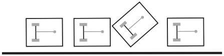
Four motions with the same amplitude perform forbidden motion:



1. motion forward



2. rotation counterclockwise



3. motion backward

4. rotation clockwise

The Lie bracket does your job!

$$g_1(x) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \qquad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

$$[g_1, g_2](x) = -\begin{bmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix}.$$

- The vector field g_1 generates the forward/backward motion.
- The vector field g_2 generates the clockwise/counterclockwise rotation.
- The vector field $[g_1, g_2]$ generates the motion in the direction perpendicular to the orientation of the car.

Distributions and integrability

• A distribution Δ on the manifold M is a map which assigns to each point in M a subspace of the tangent space at this point:

$$M \ni x \mapsto \Delta(x) \subset T_x M.$$

$\dim(\Delta) = k$ if $\dim(\Delta(x)) = k$, $\forall x \in M$.

Example: $M = \mathbb{R}^3 \backslash \{0\}$, $\qquad \Delta(x) = \{v \in \mathbb{R}^3 \,|\, v^\top x = 0\}$

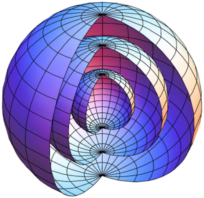
$\Delta(x)$ is the tangent space at x to the sphere centered at 0 passing through x.
This distribution Δ has a special property: for every $x \in M$, there exists a smooth 2-dim submanifold N_x of M (the sphere centered at 0 passing through x) which is everywhere tangent to Δ.

This property is called integrability

• An integral manifold of a distribution Δ is a submanifold N of M satisfying

$$T_x N = \Delta(x), \quad \forall x \in N.$$

• A distribution Δ is integrable if, for every $x \in M$, there exists a (maximal) integral manifold, $N(x)$, of Δ through every point $x \in M$, or equivalently, there exists a (integral) foliation on M whose tangent bundle is Δ.



An integral foliation on $M = \mathbb{R}^3 \backslash \{0\}$

Involutive distribution

The integrability of a distribution depends on its involutivity.

- A distribution Δ on M is said to be involutive if, $\forall x \in M$,

$$f(x), g(x) \in \Delta(x) \;\Rightarrow\; [f, g](x) \in \Delta(x).$$

Frobenius theorem

Suppose a distribution Δ has constant dimension. Then,

$$\Delta \;\text{ is integrable if and only if }\; \Delta \;\text{ is involutive.}$$

Control systems without drift

$$\dot{x} = \sum_{i=1}^{m} u_i g_i(x), \quad x \in M, \qquad \text{(unconstrained inputs).}$$

Control distribution

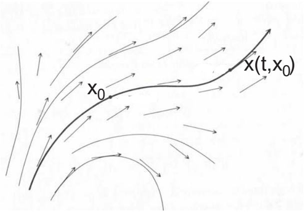
$$\Delta(x) = \mathrm{span}\{g_1(x), \cdots, g_m(x)\} \subset T_x M.$$

Control distribution

Control distribution: $\Delta(x) = \text{span}\{g_1(x), \cdots, g_m(x)\}$

Example: $M = \mathbb{R}^2,\ m = 1,\ g_1(x) \neq 0$, for all x.

The control distribution is 1-dimensional. Through each point $x_0 \in \mathbb{R}^2$ passes a curve $\gamma(x_0) = x(t, x_0)$ which is everywhere tangent to Δ. Δ is integrable.



What is the reachable set from x_0?

$$\boxed{\mathcal{R}(x_0) = \gamma(x_0) \neq \mathbb{R}^2}$$

The system is not controllable!

Control distribution (example)

$$\dot{x} = u_1 \begin{bmatrix} x_2 \\ -x_1 \\ 0 \end{bmatrix} + u_2 \begin{bmatrix} 0 \\ x_3 \\ -x_2 \end{bmatrix} + u_3 \begin{bmatrix} x_3 \\ 0 \\ -x_1 \end{bmatrix}, \quad x \in \mathbb{R}^3 \backslash \{0\}.$$

The control distribution Δ is 2-dimensional.

$$[g_1, g_2] = -g_3 \quad [g_2, g_3] = -g_1 \quad [g_3, g_1] = -g_2.$$

Δ is involutive, so Δ is integrable.

$$2x(t)^\top \dot{x}(t) = \frac{d}{dt}\left[x(t)^\top x(t)\right] = 0.$$

Consequently, the maximal integral manifold of Δ, at a given $x \in M$, is the sphere centered at the origin, passing through x.

The reachable set from $x \in M$ is contained in a 2-dimensional sphere. The system is not controllable.

Integrability of control distribution rules out controllability!?

Control distribution (example)

Back to the car model

$$g_1(x) = \begin{bmatrix} \cos\theta \\ \sin\theta \\ 0 \end{bmatrix}, \qquad g_2(x) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

The control distribution Δ is 2-dimensional.

$$[g_1, g_2](x) = \begin{bmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{bmatrix} \notin \Delta(x).$$

The control distribution is not involutive. So, Δ is not integrable.

The reachable sets are not restricted to 2-dimensional submanifolds.

Is the system controllable? Our experience says yes!

Bracket generating property

If the system

$$\dot{x} = \sum_{i=1}^{m} u_i g_i(x), \quad x \in M$$

is controllable, its control distribution

$$\Delta = \mathrm{span}\{g_1, \cdots, g_m\}$$

should satisfy a property that is intuitively opposite to integrability.

The distribution $\Delta = \mathrm{span}\{g_1, \cdots, g_m\}$ on M is said to be bracket generating if the iterated Lie brackets

$$g_i, [g_i, g_j], [g_i, [g_j, g_k]], \cdots, 1 \leq i, j, k \leq m,$$

span the tangent space of M at every point.

Rashevsky-Chow Theorem

In other words:

$\Delta = \mathrm{span}\{g_1, \cdots, g_m\}$ is bracket generating iff

$$\mathrm{Lie}_x(\mathcal{F}) = T_x M, \text{ for every } x \in M.$$

Theorem (Rashevsky-Chow)

Assume that M is connected.
If the control distribution $\Delta = \mathrm{span}\{g_1, \cdots, g_m\}$ is bracket generating, then the (drift free) system

$$\dot{x} = \sum_{i=1}^{m} u_i g_i(x), \quad x \in M$$

is controllable.

$$\dot{x} = \underbrace{g_0(x)}_{\text{drift}} + \sum_{i=1}^{m} u_i g_i(x), \quad x \in M$$

The presence of drift significantly complicates the question of controllability!

- It is possible that the trajectories of a system can't be restricted to a lower dimensional sub-manifold, and yet the system is uncontrollable, as this example shows.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix} + u \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad M = \mathbb{R}^2.$$

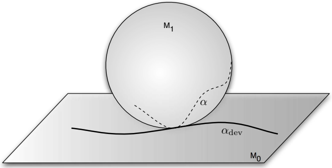
The reachable set from a point $z \in \mathbb{R}^2$ is:

$$\mathcal{R}(z) = \left\{ w \in \mathbb{R}^2 \middle| \, w_1 > z_1 \right\} \cup \{z\}.$$

Thus, the system is accessible but not controllable!

The rolling sphere

The rolling sphere consists of a sphere in 3-space, rolling without slip or twist over the tangent space at a point.



This rigid motion is described by the action of $SE(3)$ (the special Euclidean group), but has 2 types of constraints:

- Holonomic constraints (sphere keeps tangent to the plane during motion)
- Nonholonomic constraints (sphere can't twist or slip)
  - No twist (performing spins not allowed!)
  - No slip (performing sliding not allowed!)

Kinematic equations for the rolling sphere

The motion of S^2, when rolling over the south tangent plane is described by the following right-invariant control system evolving on $SE(3) = \mathbb{R}^3 \ltimes SO(3)$:

$$\dot{s} = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix} \qquad \text{(translational velocity)}$$

$$\dot{R} = \begin{bmatrix} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{bmatrix} R \qquad \text{(rotational velocity)}$$

This system is controllable.

The rolling sphere is a complete nonholonomic system.

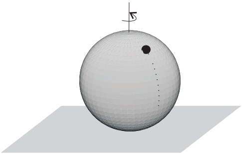
Forbidden motions

Question:

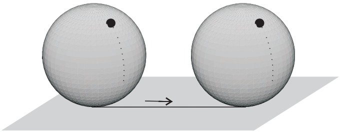
How to steer the rolling sphere from one initial configuration to any other admissible configuration, without violating the nonholonomic constraints (i.e, avoiding forbidden motions)?

Forbidden motions

Twists



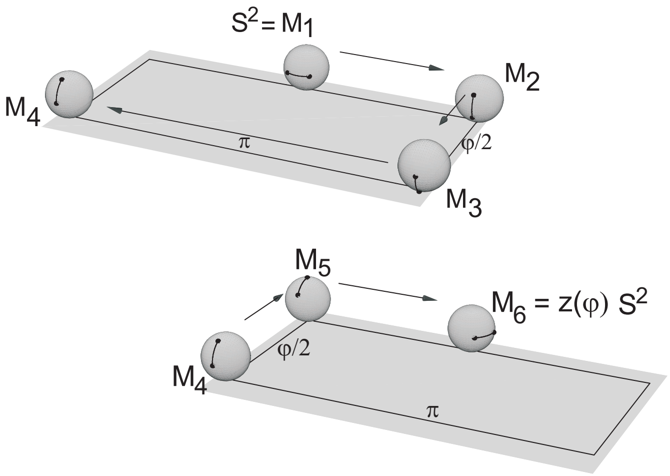
Slips



Answer: **Realizing the forbidden motions by rolling the sphere without slip or twist!**

Realizing a twist

Realizing a rotation of an angle φ around the z-axis:



Realizing a twist

A twist is a rotation around the z-axis.

$$z(\varphi) = e^{-\varphi A_{12}} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Euler's theorem guarantees that any rotation around the z-axis decomposes as rotations around the x-axis and the y-axis.

But to perform a twist, such decomposition has to be carefully chosen, so that the angles of rotation around these 2 orthogonal axis add up to zero.

Decomposition corresponding to the previous picture:

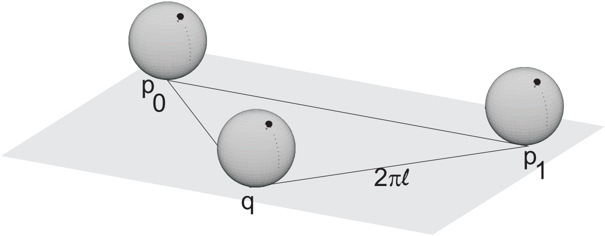
$$z(\varphi) = x(\tfrac{\pi}{2})\ y(\tfrac{\varphi}{2})\ x(-\pi)\ y(-\tfrac{\varphi}{2})\ x(\tfrac{\pi}{2})$$

Realizing a slip

A slip is a pure translation.

- $d(p_0, p_1) =$ multiple of 2π
  Roll along the segment $\overline{p_0 p_1}$.
- $d(p_0, p_1) \neq$ multiple of 2π
  Roll along the sides of the isosceles triangle in the picture.

l is the smallest integer satisfying $2\pi l > d(p_0, p_1)$.



Optimal control

Controllability doesn't care about the quality of the trajectory between two states, neither the amount of control effort!

Optimal control

What is the optimal way to control the system? We may require smooth trajectories, minimizing costs, ...

Rolling optimally

Given 2 admissible configurations, roll the sphere upon the tangent plane from the first configuration to the second, so that the curve traced in the plane by the contact point be the shortest possible.

$J(u) = \frac{1}{2}\int_0^{t_1}(u_1^2 + u_2^2)\,dt \to \min$ (cost functional)

subject to:

$$\dot{s}(t) = \begin{bmatrix} u_1 \\ u_2 \\ 0 \end{bmatrix}$$

(control system)

$$\dot{R}(t) = R(t)\begin{bmatrix} 0 & 0 & u_1 \\ 0 & 0 & u_2 \\ -u_1 & -u_2 & 0 \end{bmatrix}$$

$X(0) = X_0 = (s_0, R_0) \quad X(t_1) = X_1 = (s_1, R_1)$ (boundary cond.)

The rolling sphere meets Euler

To solve optimal control problems one needs to understand the Pontryagin Maximum Principal (Hamiltonian equations, symplectic geometry,...).

The rolling sphere meets Euler

The point of contact of the sphere rolling optimally traces Euler elastica on the plane!

Elastic curves of Euler (1744)

An elastic rod is a 1-dimensional object which is flexible (bendable but not stretchable), which looks like a portion of a straight line in its natural state.

Which form takes an elastic rod when subject to external forces applied to its ends?

Jacob Bernoulli posed this problem in 1691 and showed that the elastic energy of a deformed elastic rod is proportional to $\int \kappa^2(t)\, dt$, where $\kappa(t)$ is the geodesic curvature.

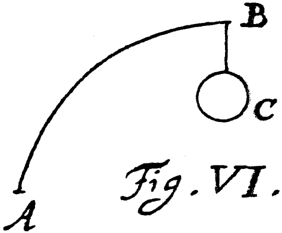


Fig. VI.

Elastic curves of Euler (1744)

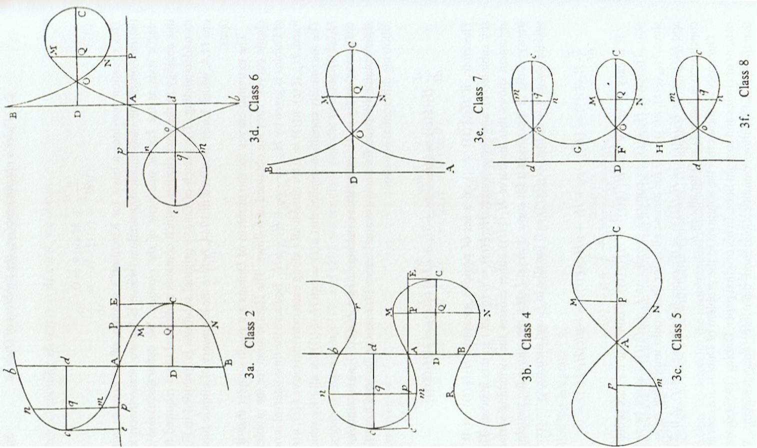
Euler (1744) also studied the variational principle

$$\int \kappa^2 \, dt \quad \rightarrow \text{min}$$

that gives rise to the shape that minimizes the elastic energy.

Euler’s Elastica

Euler also sketched these beautiful curves even before the discover of the elliptic functions (Carl Jacobi) was borne!



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