Symplectic Geometry

versus

Riemannian Geometry

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necessarily very incomplete

↑

The aim of this talk is to give an overview of Symplectic Geometry

(there is no intention of giving a list of recent results or open problems)

Notation: throughout the talk:

- M - real, finite-dim[al], differentiable manifold without boundary

- $C^\infty(M)=\{f:M\to\mathbb{R}: f \text{ is smooth}\}$

- $\chi(M)=\{X:M\to TM : X \text{ is a vector field}\}$

- $\Omega^k(M)=\{\omega:TM\times\cdots\times TM\to\mathbb{R}\}$

$\omega(p;v_1,\ldots,v_k)\in\mathbb{R}$

or $\boxed{\Omega^k(M)=\{\omega:\chi(M)\times\cdots\times\chi(M)\to C^\infty(M)\}}$

$\omega(X_1,\ldots,X_k)\in C^\infty(M)$ given by: $\omega(X_1,\ldots,X_k)(p)=\omega\left(p;X_{1_p},\ldots,X_{k_p}\right)$

1. Symplectic Manifolds

Def: Symplectic manifold is a pair (M,ω), where:

(a) $\omega \in \Omega^2(M)$ i.e.,

$$\omega(Y,X) = -\omega(X,Y)$$

$$\omega(fX+gY,Z) = f\omega(X,Z) + g\omega(Y,Z)$$

(b) ω is nondegenerate, i.e.:

$$\omega(X,Y)=0,\ \forall X \in \chi(M) \quad \Leftrightarrow \quad Y=0$$

(c) ω is closed, i.e. $d\omega = 0$

We call ω a symplectic form.

which manifolds "qualify" for being symplectic?

Necessary conditions:

(N1) $\dim M = 2n$

consider local coordinates $(x_1,\ldots,x_m)$ and build the matrix A with entries:

1. Riemannian Manifolds

Def: Riemannian manifold is a pair $(M,<,>)$, where:

(a) $<,>: \chi(M)\times\chi(M) \to C^\infty(M)$ satisfies:

$$<Y,X> \ = \ <X,Y>$$

$$<fX+gY,Z> \ = f<X,Z>+g<Y,Z>$$

(b) $<\,,>$ is positive definite.

Consequence: $<\,,>$ is nondegenerate.

which manifolds "qualify" for being Riemannian?

all smooth manifolds!

$$a_{ij} = \omega\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_i}\right)$$

then (a) and (b) imply:

$$A^T = -A \quad \text{and} \quad \det(A) \neq 0$$

$$\Downarrow$$

$$m = 2n$$

(N2) M is oriented

consider the nth exterior power:

$$\omega^{(n)} = \omega \wedge \ldots \wedge \omega \in \Omega^{2n}(M)$$

then (b) implies $\omega^{(n)}$ is a volume form on M.

$\omega^{(n)}$ is called symplectic volume.

(N3) if M is compact then

$$H^2_{DR}(M, \mathbb{R}) \neq 0$$

$$\omega = d\alpha \Rightarrow \omega^{(n)} = d(\alpha \wedge \omega \wedge \ldots \wedge \omega)$$

$$\Downarrow$$

$$\mathrm{vol}(M) = \int_M \omega^{(n)} = \int_M d\beta = \int_{\partial M} \beta = 0$$

S^{2n} is not symplectic, for any $n > 1$

(S^2 is symplectic)

2. Examples

example 1: $M = \mathbb{R}^{2n}$ with coords:

$$\{x_1,\ldots,x_n,y_1,\ldots,y_n\}$$

and symplectic structure given by:

$$\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i = dx \wedge dy$$

example 2: $M = T^*N$ with symplectic form:

$$\omega = d\lambda$$

(λ is the Liouville 1-form on M :

$$\lambda_{(p,\alpha)}(X) = < \alpha, d\pi_{(p,\alpha)}(X) > \quad)$$

- The importance of example 1 will be clear soon.
- example 2 is behind the Hamiltonian formulation of *Conservative Mechanics*.

3. Special vector fields

- Nondegeneracy of ω implies that the following is an isomorphism:

$$I:\chi(M)\to\Omega^1(M)$$
$$X\to i_X\omega=\omega(X,\cdot)$$

Def: Given $f\in C^\infty(M)$ its Hamiltonian vector field is:

$$X_f=I^{-1}(df)$$

(in other words: $\omega(X_f,\cdot)=df(\cdot)$)

Lemma

X_f is tangent to the level surface:

$$\Sigma_C=\{p\in M:f(p)=C\}$$

(equivalently f is constant on the flow of X_f)

note that: $v\in T\Sigma_C \Leftrightarrow df(v)=0$ and:

$$df(X_f)=\omega(X_f,X_f)=0$$

3. Special vector fields

- Nondegeneracy of $<,>$ implies that the following is an isomorphism:

$$I:\chi(M)\to\Omega^1(M)$$
$$X\to<X,\cdot>$$

Def: Given $f\in C^\infty(M)$ its gradient vector field is:

$$\nabla f=I^{-1}(df)$$

(in other words: $<\nabla f,\cdot>=df(\cdot)$)

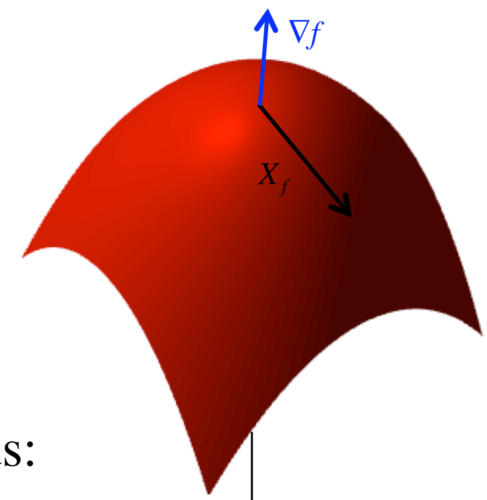
Lemma

∇f is normal to the level surface:

$$\Sigma_C=\{p\in M:f(p)=C\}$$

if $v\in T\Sigma_C$ then:

$$<\nabla f,v>=df(v)=0$$



Other important vector fields:

<u>Def</u>: A vector field X is said to be symplectic if $I(X)$ is closed. In other words:

$$d i_X \omega = 0$$

- Hamiltonian vector fields are symplectic, since ddf=0.
- Condition (c) implies that the flow of any symplectic vector field "preserves" ω:

$$L_X \omega = \underbrace{d i_X \omega}_{=0} + i_X \underbrace{d\omega}_{=0 \text{ because of (c)}} = 0$$

4. Poisson bracket

One can use Hamiltonian vector fields to define an "operation" between smooth functions:

Def: Poisson bracket on M is:

$$\{,\}: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$$

$$(f,g) \to \omega\left(X_f, X_g\right) = df\left(X_g\right)$$

- Condition (c) implies this bracket satisfies Jacobi's identity:

$$\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$$

which, together with obvious properties of $\{,\}$, implies that:

$$\left(C^\infty(M),\{,\}\right)$$

is an infinite-dimensional Lie algebra.

5. Equivalence

Def: Two symplectic manifolds (M,ω) and (M',ω') are symplectomorphic if there exists a C^1 map:

$$\varphi : M \to M'$$

satisfying:

$$\varphi^* \omega' = \omega$$

i.e.,

$$\omega'_{\varphi(p)}\left(d\varphi_p(X), d\varphi_p(Y)\right) = \omega_p(X,Y)$$

φ is called a symplectic map and necessarily $d\varphi_p$ is injective, for all p so:

$$\boxed{\dim M \leq \dim M'}$$

- A symplectomorphism is a symplectic diffeomorphism of M. Symplectomorphisms form an (infinite dimensional) subgroup of the group *diff(M)*.

5. Equivalence

Def: Two Riemannian manifolds $(M,<,>)$ and $(M',<,>')$ are isometric if there exists a C^1 map:

$$\varphi : M \to M'$$

satisfying:

$$< d\varphi_p(X), d\varphi_p(Y) >'_{\varphi(p)} \;=\; < X, Y >_p$$

φ is called an isometry and necessarily $d\varphi_p$ is injective, for all p so:

$$\boxed{\dim M \leq \dim M'}$$

Darboux-Weinstein theorem

Let p be any point on a symplectic manifold of dimension $2n$. Then there exist local coordinates $(x_1,\ldots,x_n,y_1,\ldots,y_n)$ (in U) such that:

$$\omega\big|_U = \sum_{i=1}^{n} dx_i \wedge dy_i$$

Therefore all symplectic manifolds are (locally) symplectomorphic to example 1. Consequence:

there are no local invariants (apart from dimension) in Symplectic Geometry

curvature is a local invariant in Riemannian Geometry

6. Global Invariants

As seen before, a symplectic manifold carries a symplectic volume:

$$\omega^{(n)} = \omega \wedge \ldots \wedge \omega$$

If $\varphi : M \to M'$ is a symplectic map, then it preserves (symplectic) volumes:

$$\varphi^* \omega'^{(n)} = \varphi^* (\omega' \wedge \ldots \wedge \omega') = \varphi^* (\omega') \wedge \ldots \wedge \varphi^* (\omega') = \omega \wedge \ldots \wedge \omega = \omega^{(n)}$$

but the converse is not true for $n > 1$.

example: take $(M,\omega) = \left(\mathbb{R}^4, \sum_{i=1}^{2} dx_i \wedge dy_i \right)$ and consider the map:

$$\varphi(x_1, x_2, y_1, y_2) = \left(\frac{1}{2} x_1, 2x_2, \frac{1}{2} y_1, 2y_2 \right)$$

This map preserves the symplectic volume $-2dx_1 \wedge dx_2 \wedge dy_1 \wedge dy_2$ but is not symplectic.

so what really characterizes symplectomorphisms?

6.1. Gromov's Nonsqueezing Theorem

The key theorem for characterizing symplectomorphisms (using symplectic invariants) is:

Nonsqueezing theorem - Gromov (1985)

There is a symplectic embedding:

$$\varphi : \left(B^{2n}(r), \omega_0\right) \nearrow \left(Z^{2n}(R), \omega_0\right)$$

if and only if $r \leq R$.

(where:

$$B^{2n}(r) = \left\{(x, y) \in \mathbb{R}^{2n} : \sum_{i=1}^{n} x_i^2 + y_i^2 < r^2\right\}$$ open (symplectic) ball of radius r

$$Z^{2n}(R) = \left\{(x, y) \in \mathbb{R}^{2n} : x_1^2 + y_1^2 < R^2\right\}$$ open symplectic cylinder of radius R)

- a symplectic embedding is just a symplectic map which is also an embedding. It is denoted by: $\varphi : (M, \omega) \nearrow (M', \omega')$.
- if (M, ω) is a symplectic manifold and U is open in M, then $\left(U, \omega|_U\right)$ is also symplectic.

- The theorem is not valid if “symplectic cylinder” is replaced by “cylinder”:

$$C^{2n}(R) = \left\{ (x,y) \in \mathbb{R}^{2n} : x_1^{\ 2} + x_2^{\ 2} < R^2 \right\}$$

example: the following is a symplectic embedding from $B^4(2)$ into $C^4(1)$:

$$\boxed{\varphi(x_1, x_2, y_1, y_2) = \left(\frac{1}{2} x_1, \frac{1}{2} x_2, 2y_1, 2y_2 \right)}$$

6.2. Symplectic Invariants - Capacities

Using Gromov's nonsqueezing theorem, we will construct a symplectic invariant: Gromov's width. This is one of many symplectic invariants known as symplectic capacities.

Let *M(2n)* denote the set of all symplectic manifolds of dimension $2n$.

Def Symplectic capacity is a map:

$$c : M(2n) \to \mathbb{R}_0^+ \cup +\infty$$

satisfying all three properties:

(1) monotonicity - if there is a symplectic embedding:

$$\varphi : (M_1, \omega_1) \nearrow (M_2, \omega_2)$$

then $c(M_1, \omega_1) \leq c(M_2, \omega_2)$

(2) conformality - $c(M, \lambda\omega) = |\lambda| c(M, \omega) \quad \forall \lambda \neq 0$

(3) (strong) nontriviality - $c(B^{2n}(1), \omega_0) = \pi = c(Z^{2n}(1), \omega_0)$

- If $n = 1$ then "(absolute value of) total volume of M" is a symplectic capacity.
- If $n > 1$ then "(absolute value of) total volume of M" is not a symplectic capacity (nontriviality fails).

Theorem

Any symplectic capacity is a symplectic invariant, i.e., if there is a symplectomorphism:

$$\varphi : (M_1, \omega_1) \leftrightarrow (M_2, \omega_2)$$

then $c(M_1, \omega_1) = c(M_2, \omega_2)$.

(proof: monotonicity in both ways)

Lemma

Any symplectic capacity satisfies:

$$c(B^{2n}(r), \omega_0) = \pi r^2 = c(Z^{2n}(r), \omega_0).$$

(proof: previous theorem+conformality+nontriviality)

Theorem
The existence of a symplectic capacity is equivalent to **Gromov's nonsqueezing theorem**.

⇒ suppose c exists and that:

$$\varphi : \left(B^{2n}(r), \omega_0\right) \nearrow \left(Z^{2n}(R), \omega_0\right)$$

is a symplectic embedding. Then monotonicity+previous lemma imply:

$$\pi r^2 = c\left(B^{2n}(r), \omega_0\right) \leq c\left(Z^{2n}(R), \omega_0\right) = \pi R^2$$

so $r \leq R$.

⇐ define the Gromov's width of a symplectic manifold:

$$W_G(M, \omega) = \sup_{r>0} \left\{ \pi r^2 : \exists \varphi : \left(B^{2n}(r), \omega_0\right) \nearrow (M, \omega) \right\}$$

(the area of the disk of the bigger ball one can symplectically-embed on (M, ω)*)*

If Gromov's nonsqueezing theorem holds (it does!) then Gromov's width is a symplectic capacity.

Theorem

Gromov's width W_G is the smallest of all capacities:

$$W_G(M,\omega) \leq c(M,\omega), \text{ for any capacity } c \text{ and any } (M,\omega).$$

(proof: let c be any capacity and fix r such that there is an embedding:

$$\varphi : \left(B^{2n}(r), \omega_0\right) \nearrow (M,\omega)$$

Then monotonicity of c implies:

$$\pi r^2 = c\left(B^{2n}(r), \omega_0\right) \leq c(M,\omega)$$

Since this holds for all r and $c(M,\omega)$ is independent of r:

$$\sup_{r>0}\left\{\pi r^2 : \exists \varphi : \left(B^{2n}(r), \omega_0\right) \nearrow (M,\omega)\right\} \leq c(M,\omega)$$

proving the result).

6.3. Back to Symplectomorphisms

We go back to the question:

so what really characterizes symplectomorphisms?

It turns out that symplectomorphisms of $\mathbb{R}^{2n}$ are (almost) characterized by the property of "preserving capacity of ellipsoids[(1)]":

Theorem (Eliashberg, 1987) (Hofer, 1990)
Let $\varphi:\left(\mathbb{R}^{2n},\omega_0\right)\to\left(\mathbb{R}^{2n},\omega_0\right)$ be a diffeomorphism and c a capacity. Then:

$$c\left(\varphi\left(E,\omega_0\right)\right)=c\left(E,\omega_0\right) \text{ for any ellipsoid } E\subset\mathbb{R}^{2n}$$

if and only if φ is symplectic or anti-symplectic[(2)].

(1) ellipsoid is the image of a ball by a linear/affine diffeomorphism

(2) meaning that $\varphi^*\omega_0=-\omega_0$