Profinite Algebra

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DEFINITION

A *uniformity* on a set X is a set $\mathscr{U}$ of reflexive binary relations on X such that the following conditions hold:

1. if $R_1 \in \mathscr{U}$ and $R_1 \subseteq R_2$, then $R_2 \in \mathscr{U}$;
2. if $R_1, R_2 \in \mathscr{U}$, then there exists $R_3 \in \mathscr{U}$ such that $R_3 \subseteq R_1 \cap R_2$;
3. if $R \in \mathscr{U}$, then there exists $R' \in \mathscr{U}$ such that $R' \circ R' \subseteq R$;
4. if $R \in \mathscr{U}$, then $R^{-1} \in \mathscr{U}$.

An element of a uniformity is called an *entourage*. A *uniform space* is a set endowed with a uniformity, which is usually understood and not mentioned explicitly.
A *uniformity basis* on a set X is a set $\mathscr{U}$ of reflexive binary relations on X satisfying the above conditions (ii)–(iv). The *uniformity generated* by $\mathscr{U}$ consists of all binary relations on X that contain some member of $\mathscr{U}$.
A uniformity $\mathscr{U}$ is *transitive* if it admits a basis consisting of transitive relations.

Suppose that S is an algebra and $\mathbf{V}$ is a pseudovariety.
The *pro-$\mathbf{V}$ uniformity* on S, denoted $\mathscr{U}_{\mathbf{V}}$, is generated by the basis consisting of all congruences θ such that $S/\theta \in \mathbf{V}$.
Note that it is indeed a uniformity on S, which is transitive.
In case $\mathbf{V}$ consists of all finite algebras, we also call the pro-$\mathbf{V}$ uniformity the *profinite uniformity*.

PROPOSITION

Suppose that $\mathbf{V}$ is a pseudovariety and S is an algebra.

1. *The Hausdorff completion of S under $\mathscr{U}_{\mathbf{V}}$ is compact.*
2. *A subset L of S is $\mathbf{V}$-recognizable if and only if the syntactic congruence θ_L belongs to $\mathscr{U}_{\mathbf{V}}$.*
3. *The $\mathbf{V}$-recognizable subsets of S are clopen and constitute a basis of the topology of S. In particular, S is zero-dimensional and a subset L of S is $\mathbf{V}$-open if and only if L is a union of $\mathbf{V}$-recognizable sets.*

For a pseudovariety $\mathbf{V}$ and an algebra S, we define two functions on $S \times S$ as follows.

- For $s, t \in S$, $r_{\mathbf{V}}(s, t)$ is the minimum of the cardinalities of algebras P from $\mathbf{V}$ for which there is some homomorphism $\varphi : S \to P$ such that $\varphi(s) \neq \varphi(t)$, where we set $\min \emptyset = \infty$.
- We then put $d_{\mathbf{V}}(s, t) = 2^{-r_{\mathbf{V}}(s,t)}$ with the convention that $2^{-\infty} = 0$.
- One can easily check that $d_{\mathbf{V}}$ is a pseudo-ultrametric on S, which is called the *pro-$\mathbf{V}$ pseudo-ultrametric* on S.

A *pseudometric* on a set X is a function d from $X \times X$ to the non-negative reals such that the following conditions hold:

1. $d(x, x) = 0$ for every $x \in X$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

In case, additionally, $d(x, y) = 0$ implies $x = y$, then we say that d is a (classical) *metric* on X. If, instead of the triangle inequality, we impose the stronger

4. (ultrametric inequality) $d(x, z) \leq \max\{d(x, y), d(y, z)\}$ for all $x, y, z \in X$,

then we refer respectively to a *pseudo-ultrametric* and an *ultrametric*.

PROPOSITION

Suppose that σ is a finite signature. For a pseudovariety $\mathbf{V}$ *and an algebra S, the following conditions are equivalent:*

1. *the pro-*$\mathbf{V}$ *uniformity on S is defined by the pro-*$\mathbf{V}$ *pseudo-ultrametric on S;*
2. *the pro-*$\mathbf{V}$ *uniformity on S is defined by some pseudo-ultrametric on S;*
3. *there are at most countably many* $\mathbf{V}$*-recognizable subsets of S;*
4. *for every algebra P from* $\mathbf{V}$*, there are at most countably many homomorphisms* $S \to P$*.*

In particular, all these conditions hold in case S is finitely generated. Moreover, if $\mathbf{V}$ *contains nontrivial algebras then, for the free algebra* $F_A\mathcal{V}$ *over the variety generated by* $\mathbf{V}$*, the pro-*$\mathbf{V}$ *uniformity is defined by the pro-*$\mathbf{V}$ *pseudo-ultrametric if and only if A is finite.*

Let **V** and **W** be two pseudovarieties.

- We say that a function $\varphi : S \to T$ between two algebras is $(\mathbf{V}, \mathbf{W})$*-uniformly continuous* if it is uniformly continuous with respect to the uniformities $\mathscr{U}_{\mathbf{V}}$, on S, and $\mathscr{U}_{\mathbf{W}}$, on T.
- Similarly, we say that φ is $(\mathbf{V}, \mathbf{W})$*-continuous* if it is continuous with respect to the topologies defined by the uniformities $\mathscr{U}_{\mathbf{V}}$, on S, and $\mathscr{U}_{\mathbf{W}}$, on T.

PROPOSITION

Let S and T be two algebras, and $\varphi : S \to T$ an arbitrary function.

1. *The function φ is $(\mathbf{V}, \mathbf{W})$-uniformly continuous if and only if, for every $\mathbf{W}$-recognizable subset L of T, $\varphi^{-1}L$ is a $\mathbf{V}$-recognizable subset of S.*
2. *The function φ is $(\mathbf{V}, \mathbf{W})$-continuous if and only if, for every $\mathbf{W}$-recognizable subset L of T, $\varphi^{-1}L$ is a union of $\mathbf{V}$-recognizable subsets of S.*

Proposition

Let S be an algebra and $\mathbf{V}$ a pseudovariety.

1. *The pro-$\mathbf{V}$ uniformity of S is the smallest uniformity $\mathscr{U}$ on S for which all homomorphisms from S into members of V are uniformly continuous.*
2. *The pro-$\mathbf{V}$ topology of S is the smallest topology $\mathscr{T}$ on S for which all homomorphisms from S into members of V are continuous.*
3. *The algebra S is a uniform algebra with respect to its pro-$\mathbf{V}$ uniformity. In particular, it is a topological algebra for its pro-$\mathbf{V}$ topology.*

- A *topological algebra* is a an algebra with a topology for which the basic operations are continuous. In case the topology is compact, the algebra is said to be a *compact algebra*.
- For a pseudovariety $\mathbf{V}$, a *pro-$\mathbf{V}$ algebra* is a compact algebra S which is *residually in* $\mathbf{V}$ in the sense that, for any pair of distinct points $s, t \in S$, there is a continuous homomorphism $\varphi : S \to P$, with $P \in \mathbf{V}$ such that $\varphi(s) \neq \varphi(t)$.
- A *profinite algebra* is a pro-$\mathbf{V}$ algebra for the class $\mathbf{V}$ of all finite algebras.

Proposition

Let $\mathbf{V}$ be a pseudovariety and let $\bar{\mathbf{V}}$ denote the class of all pro-$\mathbf{V}$ algebras.
Then $\bar{\mathbf{V}}$ consists of all inverse systems of algebras from $\mathbf{V}$ and it is the smallest class of topological algebras containing $\mathbf{V}$ that is closed under taking isomorphic algebras, closed subalgebras, and arbitrary direct products.
The class $\bar{\mathbf{V}}$ is additionally closed under taking profinite continuous homomorphic images, but not under taking continuous homomorphic images.

Theorem

Let S be a compact algebra and consider the following conditions:

1. *S is profinite;*
2. *S is an inverse limit of an inverse system of finite algebras;*
3. *S is isomorphic to a closed subalgebra of a direct product of finite algebras;*
4. *S is a compact zero-dimensional algebra.*

Then the implications (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) *always hold, while* (4) $\Rightarrow$ (3) *also holds in case the syntactic congruence of S is determined by a finite number of terms.*

Proposition

Let $\mathbf{V}$ be a pseudovariety and let S be a pro-$\mathbf{V}$ algebra. Then the following conditions are equivalent for an arbitrary subset L of S:

1. *the set L is clopen in S;*
2. *the subset L of S is $\mathbf{V}$-recognizable;*
3. *the subset L of S is recognizable.*

In particular, the topology of S is the smallest topology for which all continuous homomorphisms from S into algebras from $\mathbf{V}$ (or, alternatively, into finite algebras) are continuous with respect to it.

Corollary

Let $\mathbf{V}$ be a pseudovariety, S a pro-$\mathbf{V}$ algebra, and $\varphi : S \to T$ a continuous homomorphism onto a finite algebra. Then T belongs to $\mathbf{V}$.

A way of constructing profinite algebras is through Hausdorff completion of an arbitrary algebra S with respect to its pro-$\mathbf{V}$ uniformity. We denote this completion by $C_{\mathbf{V}}(S)$.

Proposition

Let S be an algebra and $\mathbf{V}$ a pseudovariety. Then $C_{\mathbf{V}}(S)$ is a pro-$\mathbf{V}$ algebra.

Theorem

Let $\mathbf{V}$ be a pseudovariety and let S be an arbitrary algebra. Then the following are equivalent for a subset L of S:

1. *the set L is $\mathbf{V}$-recognizable;*
2. *the set L is of the form $K \cap S$ for some clopen subset K of $C_{\mathbf{V}}(S)$;*
3. *the closure $\overline{L}$ of L in $C_{\mathbf{V}}(S)$ is open and $\overline{L} \cap S = L$.*

In case the pro-$\mathbf{V}$ topology of S is discrete, a further equivalent condition is that $\overline{L}$ is open.

- Compact zero-dimensional (or, equivalently, totally disconnected) spaces are also known as *Boolean spaces*.
- The reason for this is that there is a duality between such spaces and Boolean algebras, known as *Stone duality*.
- The **V**-recognizable subsets of an algebra S constitute a Boolean subalgebra of the Boolean algebra of all subsets of S.
- The preceding theorem implies that $C_{\mathbf{V}}(S)$ is the Stone dual of the Boolean algebra of **V**-recognizable subsets of S.

Proposition

For every pseudovariety $\mathbf{V}$ and every set A, there exists a free pro-$\mathbf{V}$ algebra over A. Up to isomorphism respecting the choice of free generators, it is unique.

Proposition

Let $\mathbf{V}$ be a pseudovariety and let A be a set. Let $\mathcal{V}$ be the variety generated by $\mathbf{V}$. Then the pro-$\mathbf{V}$ Hausdorff completion of the free algebra $F_A\mathcal{V}$ is a free pro-$\mathbf{V}$ algebra over A.

We say that a profinite algebra S is *self-free* with *basis* A if A is a generating subset of S such that every mapping $A \to S$ extends uniquely to a continuous endomorphism of S.

THEOREM

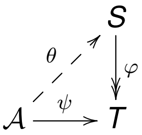
The following conditions are equivalent for a profinite algebra S:

1. *the topological algebra S is self-free with basis A;*
2. *there is a pseudovariety $\mathbf{V}$ such that S is a free pro-$\mathbf{V}$ algebra over A.*

In other words, the relatively free profinite algebras are exactly the self-free profinite algebras.

Let $\mathcal{A} = (A, \otimes)$ be a partial magma, consisting of a set A with a partial binary operation $\otimes$ on A.

- By the *diagram* of $\mathcal{A}$ we mean the set of all equations of the form $ab = a \otimes b$, with $a, b \in A$, which we denote $\Delta(\mathcal{A})$.
  (In such equations, the elements of A are viewed as the variables, and ab means the product of the two variables in the structure where the system is to be solved, while $a \otimes b$ stands for their product in the magma.)
- A semigroup S is *Henselian with respect to $\Delta(\mathcal{A})$* if every diagram of the following form may be completed, where arrows represent morphisms of appropriate type:



- An example of a partial magma is obtained by taking the set of morphisms of a small category under composition.
- Recall that a *groupoid* is a small category in which all morphisms are isomorphisms.
- A category is trivial if, for any objects c and d, there is at most one morphism $c \to d$.

The following result is inspired by a non-commutative version of Hensel's lemma in Ring Theory due to Zassenhaus.

THEOREM

Let S be a profinite semigroup and let $\mathcal{G}$ be a finite trivial groupoid. Then S is Henselian with respect to $\Delta(\mathcal{G})$.