Littlewood-Richardson Miscellany

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0. Symmetric functions

Let $(x_1, x_2, \cdots)$ be a (infinite) list of indeterminates, and let $n \in \mathbb{N}$. A homogeneous symmetric function of degree n over $\mathbb{Q}$ is a formal power series

$$f(x) = \sum_{\alpha} c_\alpha x^\alpha,$$

where

- (i) α ranges over all nonnegative integer vectors $\alpha = (\alpha_1, \alpha_2, \cdots)$ whose sum of the entries is n;
- (ii) $c_\alpha \in \mathbb{Q}$;
- (iii) x^α stands for the monomial $x_1^{\alpha_1} x_2^{\alpha_2} \cdots$; and
- (iv) $f(x_1, x_2, \cdots) = f(x_{\omega(1)}, x_{\omega(2)}, \cdots)$ for every permutation ω.

The set of all homogeneous symmetric functions of degree n form a vector space over $\mathbb{Q}$.
The space of symmetric functions has several bases. Schur functions constitute one of the most important bases. Most of their importance arises from their relationship with other areas of mathematics such as representation theory, algebraic geometry and combinatorics.

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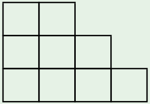
Example

$\lambda = (4, 3, 2)$, $|\lambda| = 9$, $l(\lambda) = 3$

$\lambda \quad =$ 

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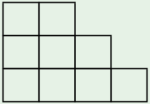
$\lambda \quad =$  $\lambda' = (3, 3, 2, 1)$.

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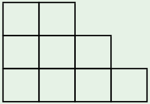
$\lambda \quad =$  $\lambda' = (3, 3, 2, 1).$

- The partition λ' *conjugate* of λ is such that its Ferrers shape is obtained from λ by interchanging rows and columns.

Young Tableaux

- $n \geq r$, $\lambda = (\lambda_1, \ldots, \lambda_r)$, $l(\lambda) = r$.
- A *semistandard tableau* T of shape λ is a filling of the boxes of the Ferrer diagram λ with elements i in $\{1, \ldots, n\}$ which is
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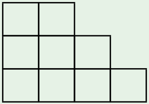
$\lambda = (4, 3, 2)$, $l(\lambda) = 3$, $n = 6$



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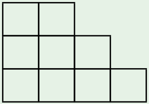

$$T = \begin{array}{|c|c|c|c|} \cline{1-2} 5 & 6 \\ \cline{1-3} 4 & 4 & 6 \\ \hline 2 & 3 & 4 & 6 \\ \hline \end{array}$$

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Schur functions

Example

$n = 7$

$$T \quad = \quad \begin{array}{|c|c|c|c|}\cline{1-2} 5 & 6 \\ \cline{1-3} 4 & 4 & 6 \\ \hline 2 & 3 & 4 & 6 \\ \hline \end{array} \qquad\qquad x^{\alpha(T)} = x_1^0 x_2 x_3 x_4^3 x_5 x_6^3 x_7^0$$

$\alpha(T)$=(0,1,1,3,1,3,0)

Schur functions continued

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$$s_\lambda(\mathbf{x}) = \sum_T X^{\alpha(T)}$$

  where T runs over all semistandard tableaux of shape λ on the alphabet $\{1, \ldots, n\}$.

Example

$\lambda = (2, 1)$, $|\lambda| = 3$

- $n = 3$

$$\begin{array}{|c|c} \hline 2 & \\ \hline 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 & \\ \hline 1 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 & \\ \hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 2 & \\ \hline 1 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 & \\ \hline 1 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 & \\ \hline 2 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c} \hline 3 & \\ \hline 2 & 3 \\ \hline \end{array}.$$

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$$s_\lambda(x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

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- Kostka number $K_{\lambda,\alpha}$ is the number of semistandard tableaux of shape λ and type α.
- The Schur function on the variables $x_1, \ldots, x_n$

$$s_n(\lambda, \mathbf{x}) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} K_{\lambda,\alpha} x^{\alpha},$$

with $\alpha_1 + \cdots + \alpha_n = |\lambda|$.

- $K_{\lambda\,\beta} = K_{\lambda\,\alpha}$, with β any permutation of α.

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Corollary

- *The Schur function* $s(\lambda, \mathbf{x}) = \sum_{\alpha \text{ weak composition of } |\lambda|} K_{\lambda,\,\alpha} x^{\alpha}$, *is a homogeneous symmetric function in* $x_1, \ldots, x_n$.

Product of Schur functions

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- $c^\lambda_{\mu\,\nu}$ are the famous Littlewood-Richardson coefficients.

Skew Schur functions

Given $\mu \subseteq \lambda$, consider the skew shape λ/μ. The skew Schur function $s_{\lambda/\mu}$ in the variables $x = (x_1, x_2, \cdots)$ is the formal power series

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- The tensor product of two irreducible polynomial representations V_μ and V_ν of the general linear group $GL_d(\mathbb{C})$ decomposes into irreducible representations of $GL_d(\mathbb{C})$

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- Schubert classes σ_λ form a linear basis for $H^*(G(d,n))$, the cohomology ring of the Grassmannian $G(d,n)$ of complex d-dimensional linear subspaces of $\mathbb{C}^n$,

$$\sigma_\mu \sigma_\nu = \sum_{\lambda\subseteq d\times(n-d)} c_{\mu\,\nu}^{\lambda} \sigma_\lambda.$$

- There exist $n \times n$ non singular matrices A, B and C, over a *local principal ideal domain*, with Smith invariants $\mu = (\mu_1, \ldots, \mu_n)$, $\nu = (\nu_1, \ldots, \nu_n)$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ respectively, such that $AB = C$ if and only if $c_{\mu\,\nu}^{\lambda} > 0$.

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- There exist $n \times n$ Hermitian matrices A, B and C, with integer eigenvalues arranged in weakly decreasing order $\mu = (\mu_1, \dots, \mu_n)$, $\nu = (\nu_1, \dots, \nu_n)$ and $\lambda = (\lambda_1, \dots, \lambda_n)$ respectively, such that $C = A + B$ if and only if $c^{\lambda}_{\mu,\nu} > 0$.

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$$(\mu, \nu) \longrightarrow (\mu, \nu)^\star = (\lambda, \rho), \quad |\mu| + |\nu| = |\lambda| + |\rho|.$$

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Equivalently,

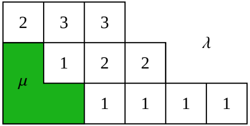
$$\text{is} \quad s_\lambda s_\rho - s_\mu s_\nu \quad \text{Schur positive?}$$

3. What do $c^{\lambda}_{\mu\nu}$ count?

Littlewood-Richardson rule

- $c^{\lambda}_{\mu\nu}$ is the number of tableaux with shape λ/μ and content ν satisfying
  - If one reads the labeled entries in reverse reading order, that is, from right to left across rows taken in turn from bottom to top,

    at any stage, the number of i's encountered is at least as large as the number of $(i+1)$'s encountered, $\#1's \geq \#2's \ldots$.



$\nu = (5, 3, 2)$

Knutson-Tao-Woodward Puzzles (04)

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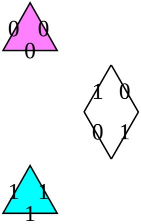

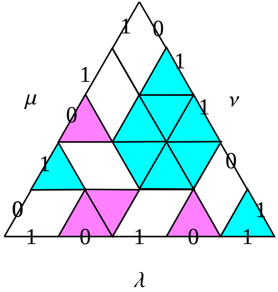

(Knutson-Tao-Woodward) $c_{\mu\,\nu\,\lambda}$ is the number of puzzles with μ, ν and λ appearing clockwise as 01-strings along the boundary.