The Fundamental Group and The Van Kampen Theorem

Ronald Alberto Zúñiga Rojas

Universidade de Coimbra
Departamento de Matemática
Topologia Algébrica

Contents

Contents

- **Homotopy Between Paths**
  If $\gamma : [0,1] \to U$ and $\delta : [0,1] \to U$ are paths with the same initial and final points and $U \subseteq X$ is an open subset of a topological space X a *homotopy* from γ to δ is a continuous mapping $H : [0,1] \times [0,1] \to U$ such that

$$H(t,0) = \gamma(t) \qquad \text{and} \qquad H(t,1) = \delta(t) \qquad \forall t \in [0,1]$$

$$H(0,s) = \gamma(0) = \delta(0) \quad \text{and} \quad H(1,s) = \gamma(1) = \delta(1) \quad \forall s \in [0,1]$$

  Such γ and δ are called homotopic paths.
- **Lemma**
  $\gamma \simeq \delta \iff \gamma$ and δ are homotopic paths is an equivalence relation.
- **Notation**
  Denote $[\gamma]$ as the equivalence class of the path γ.

- **Product Path**
  If σ is a path from a point x_0 to a point x_1 and τ is a path from x_1 to another point x_2, there is a *product* path denoted $\sigma \cdot \tau$, which is a path from x_0 to x_2. It first traverses σ and then τ, but it must do so at the double speed to complete the trip in the same unit time:

$$\sigma \cdot \tau(t) = \begin{cases} \sigma(2t) & \textit{if} \quad 0 \leqslant t \leqslant \frac{1}{2} \\ \tau(2t-1) & \textit{if} \quad \frac{1}{2} \leqslant t \leqslant 1 \end{cases}$$

- **Inverse Path**
  If σ is a path from a point x_0 to a point x_1, there is an *inverse* path σ^{-1} from x_1 to x_0 given by:

$$\sigma^{-1}(t) = \sigma(1-t), \quad 0 \leqslant t \leqslant 1$$

- **Constant Path**
  For any $x \in X$, let ε_x be the *constant* path at x:

$$\varepsilon_x(t) = x, \quad 0 \leqslant t \leqslant 1$$

- **Results**

1. $\varepsilon_x \cdot \sigma \simeq \sigma$ for every σ that starts at x
2. $\sigma \cdot \varepsilon_x \simeq \sigma$ for every σ that ends at x
3. $\varepsilon_x \simeq \sigma \cdot \sigma^{-1}$ for every σ that starts at x
4. $\varepsilon_x \simeq \sigma^{-1} \cdot \sigma$ for every σ that ends at x
5. $\sigma \cdot \tau \cdot \gamma \simeq \sigma \cdot (\tau \cdot \gamma) \simeq (\sigma \cdot \tau) \cdot \gamma$ where $\sigma \cdot \tau \cdot \gamma$ is defined by:

$$\sigma \cdot \tau \cdot \gamma(t) = \begin{cases} \sigma(3t) & \textit{if} \quad 0 \leqslant t \leqslant \frac{1}{3} \\ \tau(3t-1) & \textit{if} \quad \frac{1}{3} \leqslant t \leqslant \frac{2}{3} \\ \gamma(3t-2) & \textit{if} \quad \frac{2}{3} \leqslant t \leqslant 1 \end{cases}$$

6. If $\sigma_1 \simeq \sigma_2$ and $\tau_1 \simeq \tau_2$, then: $\sigma_1 \cdot \tau_1 \simeq \sigma_2 \cdot \tau_2$ when the products defined.

- **Loop**
  For a point $x \in X$, a *loop* at x is a closed path that starts and ends at x.
- **Fundamental Group**
  The *Fundamental Group of X with base point x*, denoted $\pi_1(X, x)$, is the set of equivalence classes by homotopy of the loops at x.
  The *identity* is the class $e = [\varepsilon_x]$.
  The *product* is defined by $[\sigma] \cdot [\tau] = [\sigma \cdot \tau]$.

- **Results**

  1. The product $[\sigma] \cdot [\tau] = [\sigma \cdot \tau]$ is well defined.
  2. $[\sigma] \cdot ([\tau] \cdot [\gamma]) = ([\sigma] \cdot [\tau]) \cdot [\gamma]$
  3. $e \cdot [\sigma] = [\sigma] = [\sigma] \cdot e$
  4. $[\sigma] \cdot [\sigma^{-1}] = e = [\sigma^{-1}] \cdot [\sigma]$

Therefore, $\pi_1(X, x)$ is a group.

- **Some Definitions**
  1. A path-connected space X is called *simply connected* if its fundamental group is the trivial group.
  2. A path-connected space X is called *locally simply connected* if every neighborhood of a point contains a neighborhood that is simply connected.
  3. A path-connected space X is called *semilocally simply connected* if every point has a neighborhood such that every loop in the neighborhood is homotopic in X to a constant path.

- **Lemma**
  If $f : X \to Y$ is a continuous function and $f(x) = y$, then f determines a group homomorphism

  $$f_* : \pi_1(X, x) \to \pi_1(Y, y)$$

  which takes $[\sigma]$ to $[f \circ \sigma]$.

Some Interesting Results

- **Proposition**
  $\pi_1(S^1, p) \cong \mathbb{Z}$ where $p = (1, 0)$.
- **Proposition**
  $\pi_1(X \times Y, (x, y)) \cong \pi_1(X, x) \times \pi_1(Y, y)$.
- **Corollary**
  $\pi_1(T, (p, p)) \cong \mathbb{Z} \times \mathbb{Z}$ where $p = (1, 0)$.

- **Homotopy Between Continuous Maps**
  In a more general way, suppose $f : X \to Y$ and $g : X \to Y$ are continuous maps, where X and Y are topological spaces. They are called *homotopic maps* if there is a continuous mapping $H : X \times [0,1] \to Y$ such that

$$H(x,0) = f(x) \qquad \text{and} \qquad H(x,1) = g(x) \qquad \forall x \in X$$

  Such H is called a *homotopy* from f to g.

- **Proposition**
  Considering f, g and H as before, x to be a base point of X, and $y_0 = f(x) \in Y$ and $y_1 = g(x) \in Y$; then, the mapping $\tau(t) = H(x, t)$ is a path from y_0 to y_1, and the following diagram commutes:

$$\begin{array}{ccc} & & \pi_1(Y, y_0) \\ & \overset{f_*}{\nearrow} & \\ \pi_1(X, x) & & \Big\downarrow \tau_\# \\ & \underset{g_*}{\searrow} & \\ & & \pi_1(Y, y_1) \end{array}$$

$$i.e. \quad \tau_\# \circ f_* = g_*$$

- **Corollary**
  With $f,\ g$ and H as before,
  $H(x,s) = f(x) = g(x) = y \quad \forall s \in [0,1]$
  then
  $$f_* = g_* : \pi_1(X,x) \to \pi_1(Y,y)$$
- **Definition**
  Two spaces X and Y are said to *have the same homotopy type* if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. The map f is called a *homotopy equivalence* if there is such a g.

- **Group Action**

  An action of a group G on a space Y is a mapping $G \times Y \to Y, \quad (g, y) \mapsto g \cdot y$ such that:

  1. $g \cdot (h \cdot y) = (g \cdot h) \cdot y \qquad \forall g, h \in G, \quad \forall y \in Y$
  2. $e \cdot y = y \qquad \forall y \in Y$ where $e \in G$
  3. $Y \to Y, \quad y \mapsto g \cdot y$ is a homeomorphism of $Y \quad \forall g \in G$

  Two points $y, y' \in Y$ are in the same orbit if $\exists g \in G$ such that $y' = g \cdot y$. Since G is a group, this is an equivalence relation.

  G acts *evenly* on Y if $\forall y \in Y \quad \exists V_y$ neighborhood of y such that $g \cdot V_y \cap h \cdot V_y = \emptyset \quad \forall g, h \in G, \; g \neq h$

Contents

- **Covering**
  If X and Y are topological spaces, a *covering map* is a continuous mapping $p : Y \to X$ with the property that $\forall x \in X$ there is an open neighborhood N_x such that $p^{-1}(N_x)$ is a disjoint union of open sets, each of which is mapped homeomorphically by p onto N_x. Such a covering map is called a *covering of* X.

- **Some Examples**

1. The mapping $p : \mathbb{R} \to S^1$ given by:

$$p(t) = (\cos(t), \sin(t))$$

2. The *Polar Coordinate Mapping*

$$p : \{(r, \theta) \in \mathbb{R}^2 : r > 0\} \to \mathbb{R}^2 \setminus \{(0, 0)\}$$

given by:

$$p(r, \theta) = (r \cos(\theta), r \sin(\theta))$$

3. Another example is the mapping $p_n : S^1 \to S^1$, for any integer $n \geqslant 1$, given by:

$$p(cos(2\pi t), sin(2\pi t)) = (cos(2\pi nt), sin(2\pi nt))$$

or in terms of complex numbers:

$$p(z) = z^n$$

- **Isomorphism Between Coverings**
  Let $p : Y \to X$ and $p' : Y' \to X$ be a pair of coverings of X. A homeomorphism $\varphi : Y \to Y'$ such that $p' \circ \varphi = p$ is called an isomorphism between coverings.
- **Trivial Covering**
  A covering is called *trivial* if it is isomorphic to the projection of a product $\pi : X \times T \to X$
  with $\pi(x, t) = x$ where T is any set with the discrete topology. So any covering is locally trivial.

- **Interesting Application**
  A covering of $\mathbb{R}^2 \setminus \{(0,0)\}$ can be realized as the right half plane, via the polar coordinate mapping

$$(r,\theta) \mapsto (r\cos(\theta), r\sin(\theta))$$

  and another covering could be the entire complex plane $\mathbb{C}$ via the mapping

$$z \mapsto exp(z)$$

  One can find an isomorphism between these coverings.

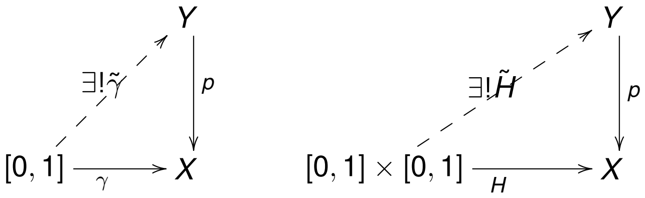
- **G-Coverings**

1. **Lemma**
   If a group G acts evenly on a topological space Y, then the projection $\pi : Y \to Y/G$ is a covering map.
2. **Definition**
   A covering $p : Y \to X$ that arises from an even action of a group G on Y, is called a *G-covering*.
3. **Definition**
   An *isomorphism* of G-coverings is an isomorphism of coverings that commutes with the action of G; i.e. an isomorphism of the G-covering $p : Y \to X$ with the G-covering $p' : Y' \to X$ is a homeomorphism $\varphi : Y \to Y'$ such that $p' \circ \varphi = p$ and $\varphi(g \cdot y) = g \cdot \varphi(y)$.
4. **Definition**
   The trivial G-covering of X is the product $G \times X \to X$ where G acts on X by left multiplication.

- **Examples of $\mathbb{Z}_n$-Coverings**
  1. The mapping $p : \mathbb{R} \to S^1$ given by:
  $$p(t) = (\cos(t), \sin(t))$$
  2. The *Polar Coordinate Mapping*
  $$p : \{(r, \theta) \in \mathbb{R}^2 : r > 0\} \to \mathbb{R}^2 \setminus \{(0, 0)\}$$
  given by:
  $$p(r, \theta) = (r\cos(\theta), r\sin(\theta))$$
  3. Another example is the mapping $p_n : S^1 \to S^1$, for any integer $n \geqslant 1$, given by:
  $$p(cos(2\pi t), sin(2\pi t)) = (cos(2\pi nt), sin(2\pi nt))$$
  or in terms of complex numbers:
  $$p(z) = z^n$$

- **Liftings**
  1. **Proposition** (Path Lifting)
     Let $p: Y \to X$ be a covering, and let $\gamma : [0,1] \to X$ be a continuous path in X. Let $y \in Y$ such that $p(y) = \gamma(0)$. Then $\exists! \tilde{\gamma} : [0,1] \to Y$ continuous path such that $\tilde{\gamma}(0) = y$ and $p \circ \tilde{\gamma}(t) = \gamma(t) \quad \forall t \in [0,1]$.
  2. **Proposition** (Homotopy Lifting)
     Let $p: Y \to X$ be a covering, and let $H : [0,1] \times [0,1] \to X$ be a homotopy of paths in X with $\gamma_0(t) = H(t,0)$ as initial path. Let $\tilde{\gamma}_0 : [0,1] \to Y$ be a lifting of γ_0. Then $\exists! \tilde{H} : [0,1] \times [0,1] \to Y$ homotopy of paths in Y, lifting of H such that $\tilde{H}(t,0) = \tilde{\gamma}_0(t) \quad \forall t \in [0,1]$ and $p \circ \tilde{H} = H$.

- **Liftings**


Y
$\exists!\tilde{\gamma}$
p
$[0,1]$
γ
X
Y
$\exists!\tilde{H}$
p
$[0,1]\times[0,1]$
H
X

- **Next Goal!**
  Now, we are looking how to relate the fundamental group of X with the automorphism group of a covering of X.
- **Theorem**
  Let $p : Y \to X$ be a covering, with Y connected and X locally path connected, and let $p(y) = x$. If $p_*(\pi_1(Y, y)) \trianglelefteq \pi_1(X, x)$, then there is a canonical isomorphism

$$\pi_1(X, x)/p_*(\pi_1(Y, y)) \xrightarrow{\cong} Aut(Y/X)$$

  Actually, the covering is a G-covering, where $G = \pi_1(X, x)/p_*(\pi_1(Y, y))$.
- **Definition**
  A covering $p : Y \to X$ is called *regular* if $p_*(\pi_1(Y, y)) \trianglelefteq \pi_1(X, x)$.

- **Corollary**
  If $p : Y \to X$ is a covering, with Y simply connected and X locally path connected,then $\pi_1(X, x) \cong Aut(Y/X)$.
- **Corollary**
  If a group G acts evenly on a simply connected and locally path-connected space Y, and $X = Y/G$ is the orbit space, then the fundamental group of X is isomorphic to G.
- **Corollary**
  $\pi_1(S^1, p) \cong \mathbb{Z}$ where $p = (1, 0)$.

- **Definition**
  Assume that X is connected and locally path connected. A covering $p : Y \to X$ is called a *universal covering* if Y is simply connected. Such a covering, if exists, is unique, and unique up to canonical isomorphism if base points are specified.
- **Theorem**
  A connected and locally path-connected space X has a universal covering if and only if X is semilocally simply connected.

Contents

- **Proposition**
  There is a one-to-one correspondence between the set of homomorphisms from $\pi_1(X, x)$ to the group G and the set of G-coverings with base points, up to isomorphism:

$$Hom(\pi_1(X, x), G) \leftrightarrow \{G - \text{coverings}\}/\cong$$

Let X be the union of two open sets U and V, where U, V and $U \cap V$ (and hence, of course X), are path connected. Also assume that X is locally simply connected.
i.e. X, U, V and $U \cap V$ have universal covering spaces.

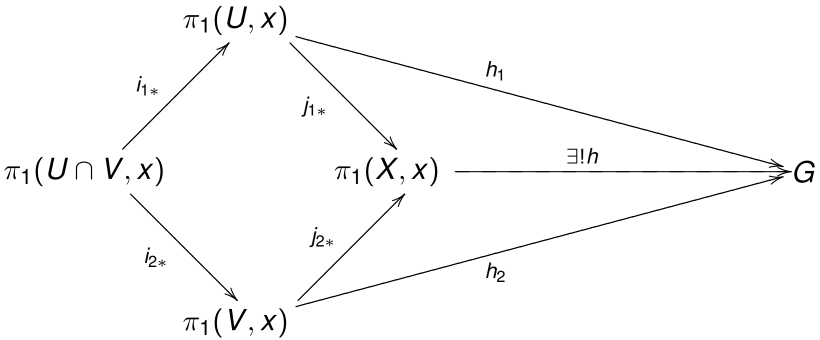
- **Theorem (Seifert-Van Kampen)**
  For any homomorphisms

$$h_1 : \pi_1(U, x) \to G \quad \text{and} \quad h_2 : \pi_1(V, x) \to G,$$

  such that $h_1 \circ i_{1*} = h_2 \circ i_{2*}$, there is a unique homomorphism

$$h : \pi_1(X, x) \to G,$$

  such that $h \circ j_{1*} = h_1 \quad \text{and} \quad h \circ j_{2*} = h_2$.


$\pi_1(U, x)$
i_{1*}
j_{1*}
h_1
$\pi_1(U \cap V, x)$
$\pi_1(X, x)$
$\exists! h$
G
i_{2*}
j_{2*}
h_2
$\pi_1(V, x)$

Contents

Definitions Let G be a group.

- If $\{G_j\}_{j \in J}$ is a family of subgroups of G, these groups *generate* G if every element $x \in G$ can be written as a finite product of elements of the groups G_j. This means that $\exists (x_1, \ ..., x_n)$ a finite sequence of elements of the groups G_j such that $x = x_1 \cdot ... \cdot x_n$. Such a sequence is called a *word* of length $n \in \mathbb{N}$ in the groups G_j that represent the element $x \in G$.
- If $x_i, x_{i+1} \in G_j$, there is a shorter word $(x_1, \ ...\tilde{x}_i, \ ..., x_n)$ of length $n - 1$ with $\tilde{x}_i = x_i \cdot x_{i+1} \in G_i$ that also represents x. Furthermore, if $x_i = 1$ for any i, we can delete x_i from the sequence, obtaining a new shorter word $(y_1, \ ..., y_m)$ that represents x, where no group G_j contains both y_i, y_{i+1} and where $y_i \neq 1 \quad \forall i$. Such a word is called a *reduced word*.

Definitions

- If $\{G_j\}_{j\in J}$ generates G, and $G_i \cap G_j = \{e\}$, whenever $i \neq j$, G is called the *free product* of the groups G_j if for each $x \in G \setminus \{e\}$ there is a unique reduced word in the groups G_j that represent x. In this case, the group G is denoted by:

$$G = \circledast_{j\in J} G_j$$

and

$$G = G_1 * \cdots * G_n$$

in the finite case.

Definitions

- If $\{g_j\}_{j \in J}$ is a family of elements of G, these elements *generate* G if every element $x \in G$ can be written as a product of powers of the elements g_j. If the family $\{g_j\}_{j=1}^{n}$ is finite, G is called *finitely generated*.
- Suppose $\{g_j\}_{j \in J}$ is a family of elements of G such that each g_j generates an infinite cyclic subgroup $G_j \leq G$. If G, is the free product of the groups G_j, then G is said to be a *free group* and $\{g_j\}_{j \in J}$ is called a *system of free generators* for G. In this case, for each $x \in G \setminus \{e\}$, x can be written uniquely as:

$$x = (g_{j_1})^{n_1} \cdots (g_{j_k})^{n_k}$$

where $j_i \neq j_{i+1}$ and $n_i \neq 0$ $\forall i$. Of course, n_i may be negative.

- **Theorem (Rephrasing Seifert-Van Kampen)**
  Let X be the union of two open sets U and V, where U, V and $U \cap V$ (and hence, of course X), are path connected. Also assume that X is locally simply connected. Then

$$\pi_1(X, x) \cong (\pi_1(U, x) * \pi_1(V, x))/N$$

  where N is the least normal subgroup of the free product $\pi_1(U, x) * \pi_1(V, x)$ that contains all elements represented by words of the form

$$(i_{1*}(g)^{-1},\ i_{2*}(g)) \quad \text{for} \quad g \in \pi_1(U \cap V, x)$$

- **Corollary**
  Let X be as above. If $U \cap V$ is simply connected, then

$$\pi_1(U \cup V, x) \cong \pi_1(U, x) * \pi_1(V, x)$$

- **Wedge of Circles**

1. **Definition**
   Let X be the union $X = \bigcup_{i=1}^{n} S_i$ where each subspace S_i is homeomorphic to the unit circle $S^1 \subseteq \mathbb{R}^2$. Assume that there is a point $p \in X$ such that $S_i \cap S_j = \{p\}$ whenever $i \neq j$. Then X is called the wedge of the circles $S_1, \ ..., S_n$.
2. **Theorem**
   Let $X = \bigcup_{i=1}^{n} S_i$ be as above. Then

$$\pi_1(X, p) = \pi_1(S_1, p) * \cdots * \pi_1(S_n, p) = \mathbb{Z} * \cdots * \mathbb{Z}$$

- **M_n: Surface of genus n**
  The surface M_n, sometimes called the *n-fold connected sum of tori*, or the *n-fold torus*, and sometimes also denoted $T\# ...\# T$, is the surface obtained by taking n copies of the torus $T = S^1 \times S^1$, deleting an open disk from two of them and pasting both together along their edges,and repeating this process for the remaining $n-2$ torus.
- **Theorem**
  $\pi_1(M_n, x)$ is isomorphic to the quotient of the free group on the $2n$ generators $\alpha_1,\ ...,\alpha_n,\beta_1,\ ...,\beta_n$ by the least normal subgroup containing the element

$$[\alpha_1,\beta_1][\alpha_2,\beta_2]\cdots[\alpha_n,\beta_n]$$

where $[\alpha,\beta] = \alpha\beta\alpha^{-1}\beta^{-1}$.

Contents

- Fulton, W. ***Algebraic Topology***. Springer-Verlag. New York. 1995.
- Hatcher, A. ***Algebraic Topology***. Cambridge University Press. Cambridge. 2002.
- Bruzzo, U. ***Introduction to Algebraic Topology and Algebraic Geometry***. ISAS Trieste. Genova. 2004.
- Munkres, J.R. ***Topology***. 2nd Edition. Prentice Hall. New Jersey. 2000.
- Gamelin, T.W. Greene, R.E. ***Introduction to Topology***. 2nd Edition. Dover. New York. 1999
- Steen, L.A. Seebach, J.A. ***Counterexamples in Topology***. Dover. New York. 1995
- Dugundji, J. ***Topology***. Allyn and Bacon. Boston. 1978.