Envelopes and sharp embeddings in function spaces

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Outline

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Definition of growth envelope

Decreasing rearrangement of f:

$$f^*(t) := \inf\{s > 0 : |\{x \in \mathbb{R}^n : |f(x)| > s\}| \le t\}, \quad t \ge 0$$

Definition of growth envelope

Let $X \subset L_1^{\mathrm{loc}}$.

(i) Growth envelope function

$$\mathcal{E}_{\mathsf{G}}^{X}(t) := \sup_{\|f|X\| \leq 1} f^*(t), \qquad t > 0.$$

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(ii) Assume $X \not\hookrightarrow L_\infty$. Let $H(t) := -\log \mathcal{E}_{\mathsf{G}}^{X}(t)$, $t \in (0, \varepsilon]$.

$$u_{\mathsf{G}}^{X} := \inf \Big\{ v \in (0, \infty] : \exists\ c > 0 \ \text{ such that}$$

$$\left(\int_0^{\varepsilon} \left(\frac{f^*(t)}{\mathcal{E}_{\mathsf{G}}^{X}(t)} \right)^{v} \mu_H(\mathrm{d}t) \right)^{1/v} \le c\, \|f|X\| \quad \forall f \in X \Big\}.$$

▸ remark

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remark

Growth envelope for the function space X:

$$\mathfrak{E}_{\mathsf{G}}(X) = \left(\mathcal{E}_{\mathsf{G}}^{X}(\cdot), u_{\mathsf{G}}^{X} \right).$$

Outline

Definition of continuity envelope

Modulus of smoothness of f:

$$\omega(f,t) := \sup_{|h|\leq t} \sup_{x\in\mathbb{R}^n} |f(x+h)-f(x)|, \quad t>0$$

Definition of continuity envelope

Let $X \hookrightarrow \mathrm{C}$.

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$$\mathcal{E}_{\mathsf{C}}^{X}(t) := \sup_{\|f|X\| \leq 1} \frac{\omega(f,t)}{t}, \qquad t > 0.$$

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Continuity envelope for the function space X:

$$\mathfrak{E}_{\mathsf{C}}(X) = \left(\mathcal{E}_{\mathsf{C}}^{X}(\cdot), u_{\mathsf{C}}^{X} \right).$$

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Besov spaces

- $\{\varphi_k\}_{k=0}^{\infty}$ smooth dyadic resolution of unity

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  - $\varphi_k \in \mathcal{S}$
  - $\operatorname{supp} \varphi_0$ compact

    $\operatorname{supp} \varphi_k \subset \{x \in \mathbb{R}^n : 2^{k-1} \leq |x| \leq 2^{k+1}\}, \quad k \in \mathbb{N}$
  - $\sup_{k\in\mathbb{N}_0} \sup_{x\in\mathbb{R}^n} 2^{k|\alpha|}|D^\alpha \varphi_k(x)| < \infty, \quad \alpha \in \mathbb{N}_0^n$
  - $\sum_{k=0}^{\infty} \varphi_k(x) = 1, \quad x \in \mathbb{R}^n$

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Definition

Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. The space $B^s_{p,q}$ consists of those $f \in \mathcal{S}'$ such that

$$\|f|B^s_{p,q}\| := \left(\sum_{k=0}^{\infty} 2^{ksq} \|(\varphi_k \hat{f})^\vee | L_p\|^q \right)^{1/q} < \infty.$$

Triebel-Lizorkin spaces

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Special cases:

- $F^0_{p,2} = L_p, \quad 1 < p < \infty$ — Lebesgue spaces
- $F^k_{p,2} = W^k_p, \quad k \in \mathbb{N}_0,\ 1 < p < \infty$ — Sobolev spaces
- $F^s_{p,2} = H^s_p, \quad s > 0,\ 1 < p < \infty$ — Bessel-potencial spaces
- $B^s_{\infty,\infty} = \mathcal{C}^s, \quad s > 0$ — Hölder-Zygmund spaces

Outline

Besov spaces of generalized smoothness

- Slowly varying function Ψ: positive, measurable function on $(0,1]$ with

$$\lim_{t \to 0} \frac{\Psi(st)}{\Psi(t)} = 1, \quad s \in (0,1].$$

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Examples:

$$\Psi(x) = (1 + |\log x|)^a \left(1 + \log\left(1 + |\log x|\right)\right)^b, \quad x \in (0,1],\ a, b \in \mathbb{R},$$

$$\Psi(x) = \exp(|\log x|^c), \qquad x \in (0,1],\ c \in (0,1)$$

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Definition

Let $0 < p, q \leq \infty$, $s \in \mathbb{R}$, Ψ slowly varying function. The space $B^{(s,\Psi)}_{p,q}$ consists of those $f \in \mathcal{S}'$ such that

$$\|f|B^{(s,\Psi)}_{p,q}\| = \left(\sum_{k=0}^{\infty} 2^{ksq}\,\Psi(2^{-k})^q \|(\varphi_k \hat{f})^\vee|L_p\|^q\right)^{1/q} < \infty.$$

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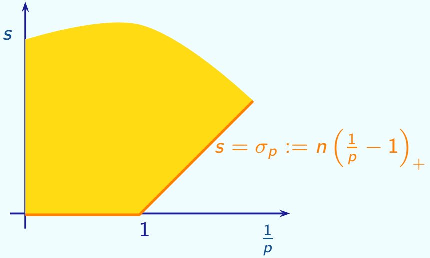
Remark:

$$A^{s+\varepsilon}_{p,q} \hookrightarrow A^{(s,\Psi)}_{p,q} \hookrightarrow A^{s-\varepsilon}_{p,q}, \quad \varepsilon > 0,\ A \in \{B, F\}$$

Outline

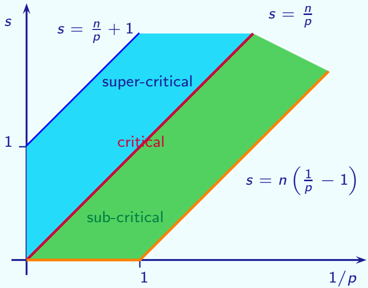
Embeddings in L_∞ and Lip^1



$$s > \sigma_p \;\Rightarrow\; A^{(s,\Psi)}_{p,q} \subset L_1^{\mathrm{loc}}$$

Notation: $$\frac{1}{u'} = \left(1 - \frac{1}{u}\right)_+, \quad u \in (0, \infty]$$

Embeddings in L_∞ and Lip^1



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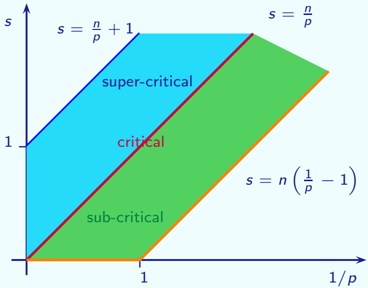

$$B^{(s,\Psi)}_{p,q} \hookrightarrow L_\infty \Leftrightarrow \begin{cases} s > \frac{n}{p} & \text{or} \\ s = \frac{n}{p} & \text{and} \quad \left(\Psi(2^{-j})^{-1}\right)_j \in \ell_{q'} \end{cases}$$

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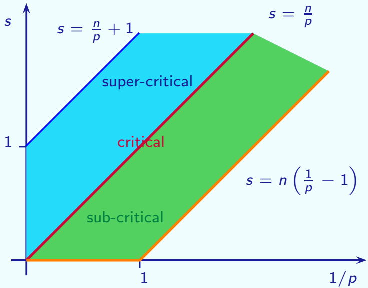

$$B^{(s,\Psi)}_{p,q} \hookrightarrow \mathrm{Lip}^1 \Leftrightarrow \begin{cases} s > \frac{n}{p} + 1 & \text{or} \\ s = \frac{n}{p} + 1 & \text{and} \quad \left(\Psi(2^{-j})^{-1}\right)_j \in \ell_{q'} \end{cases}$$

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Growth envelopes in the subcritical case

Theorem [A. Caetano, S. M., 2004]

Let $0 < p, q \leq \infty$ ($p < \infty$ in F-case), Ψ slowly varying and $\sigma_p < s < \frac{n}{p}$.

$$\text{(i)} \quad \mathfrak{E}_{\mathsf{G}}\left(B^{(s,\Psi)}_{p,q}\right) = \left(t^{-\frac{1}{r}}\,\Psi(t)^{-1}, q\right);$$

$$\text{(ii)} \quad \mathfrak{E}_{\mathsf{G}}\left(F^{(s,\Psi)}_{p,q}\right) = \left(t^{-\frac{1}{r}}\,\Psi(t)^{-1}, p\right).$$

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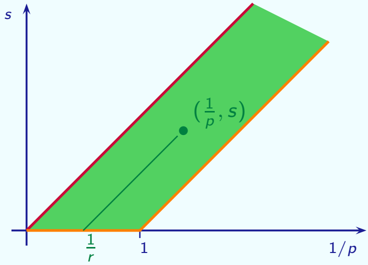

$$\frac{1}{r} = \frac{1}{p} - \frac{s}{n}$$

Growth envelopes in the critical case

Theorem [A. Caetano, S. M., 2004]

Let $0 < p, q \leq \infty$ ($p < \infty$ in F-case), Ψ slowly varying, $s = \frac{n}{p}$ and $\left(\Psi(2^{-j})^{-1}\right)_j \notin \ell_{u'}$ (with $u = q$ for B-spaces and $u = p$ for F-spaces).

(i) $$\mathfrak{E}_{\mathsf{G}}\left(B^{(s,\Psi)}_{p,q}\right) = \left(\left(\int_{t^{1/n}}^{1} \Psi(y)^{-q'}\,\frac{\mathrm{d}t}{t}\right)^{1/q'},\ q\right);$$

(ii) $$\mathfrak{E}_{\mathsf{G}}\left(F^{(s,\Psi)}_{p,q}\right) = \left(\left(\int_{t^{1/n}}^{1} \Psi(y)^{-p'}\,\frac{\mathrm{d}t}{t}\right)^{1/p'},\ p\right).$$

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Theorem [A. Caetano, S. M., 2004]

Let $0 < p, q \leq \infty$ ($p < \infty$ in F-case), Ψ slowly varying, $\sigma_p < s \leq \frac{n}{p}$. In the case $s = \frac{n}{p}$ assume further that $\left(\Psi(2^{-j})^{-1}\right)_j \notin \ell_{u'}$. Let $s - \frac{n}{p} = -\frac{n}{r}$.

(i) $$\mathfrak{E}_{\mathsf{G}}\left(B^{(s,\Psi)}_{p,q}\right) = \left(\left(\int_{t^{1/n}}^{1} y^{-\frac{n}{r}q'} \Psi(y)^{-q'} \, \frac{\mathrm{d}t}{t}\right)^{1/q'}, \, q\right);$$

(ii) $$\mathfrak{E}_{\mathsf{G}}\left(F^{(s,\Psi)}_{p,q}\right) = \left(\left(\int_{t^{1/n}}^{1} y^{-\frac{n}{r}p'} \Psi(y)^{-p'} \, \frac{\mathrm{d}t}{t}\right)^{1/p'}, p\right).$$

Outline

Continuity envelopes in the supercritical case

Theorem [D. Haroske, S. M., 2004]

Let $0 < p, q \leq \infty$, Ψ slowly varying and $0 < s - \frac{n}{p} := \sigma < 1$.

(i) $\mathfrak{E}_C \left(B^{(s,\Psi)}_{p,q} \right) = \left(t^{-(1-\sigma)} \, \Psi(t)^{-1}, q \right)$

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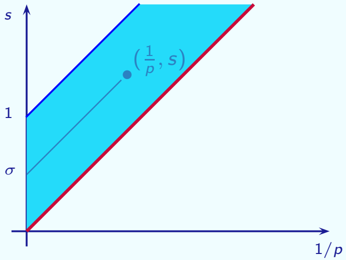

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Theorem [A. Caetano, D. Haroske, 2005]

Let $0 < p, q \leq \infty$, Ψ slowly varying, $s = \frac{n}{p} + 1$ and $\left(\Psi(2^{-j})^{-1}\right)_j \notin \ell_{u'}$.

(i) $\mathfrak{E}_C\left(B^{(s,\Psi)}_{p,q}\right) = \left(\left(\int_t^1 \Psi(y)^{-q'}\,\frac{dy}{y}\right)^{1/q'},\ q\right)$

(ii) $\mathfrak{E}_C\left(F^{(s,\Psi)}_{p,q}\right) = \left(\left(\int_t^1 \Psi(y)^{-p'}\,\frac{dy}{y}\right)^{1/p'},\ p\right)$

Continuity envelopes in the supercritical case

Or, in an unified way...

Theorem [A. Caetano, D. Haroske, 2005]

Let $0 < p, q \leq \infty$, Ψ slowly varying, $\frac{n}{p} < s < \frac{n}{p} + 1$ or $s = \frac{n}{p} + 1$ and $\left(\Psi(2^{-j})^{-1}\right)_j \notin \ell_{u'}$.

$$\text{(i)} \quad \mathfrak{E}_C\left(B^{(s,\Psi)}_{p,q}\right) = \left(\left(\int_t^1 \Psi(y)^{-q'}\, y^{-(1-\sigma)q'}\, \frac{dy}{y}\right)^{1/q'}, q\right)$$

$$\text{(ii)} \quad \mathfrak{E}_C\left(F^{(s,\Psi)}_{p,q}\right) = \left(\left(\int_t^1 \Psi(y)^{-p'}\, y^{-(1-\sigma)p'}\, \frac{dy}{y}\right)^{1/p'}, p\right)$$

Continuity envelopes in the critical case

The case $\boxed{\Psi \sim 1}$

$\frac{n}{p} < s < \frac{n}{p} + 1$

$s = \frac{n}{p} + 1$ and $\begin{cases} 1 < p < \infty, \text{ if } A = F \\ 1 < q \leq \infty, \text{ if } A = B \end{cases}$ [D. Haroske & H. Triebel, 2001]

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Theorem [D. Haroske, 2001]

Let $0 < p, q \leq \infty$ and $s = \frac{n}{p}$.

(i) If $0 < q \leq 1$, then

$$\mathfrak{E}_C\left(B^s_{p,q}\right) = \Big(t^{-1}, \underbrace{?}_{\geq q}\Big)$$

(ii) If $0 < p \leq 1$, then

$$\mathfrak{E}_C\left(F^s_{p,q}\right) = \Big(t^{-1}, \underbrace{?}_{\geq p}\Big).$$

Continuity envelopes in the critical case

Theorem [S. M., J. Neves, M. Piotrowski, 2007]

Let $0 < p, q \leq \infty$, Ψ slowly varying, $s = \frac{n}{p}$ and $\left(\Psi(2^{-j})^{-1}\right)_j \in \ell_{u'}$.

(i) Let $1 < q \leq \infty$. Then

$$\mathfrak{E}_{\mathrm{C}}(B^{(n/p,\Psi)}_{p,q}(\mathbb{R}^n)) = \left(\frac{1}{t} \left(\int_0^t \Psi(s)^{-q'} \, \frac{\mathrm{d}s}{s} \right)^{1/q'}, \infty \right).$$

(ii) Let $1 < p < \infty$. Then

$$\mathfrak{E}_{\mathrm{C}}(F^{(n/p,\Psi)}_{p,q}(\mathbb{R}^n)) = \left(\frac{1}{t} \left(\int_0^t \Psi(s)^{-p'} \, \frac{\mathrm{d}s}{s} \right)^{1/p'}, \infty \right).$$

Continuity envelopes in the critical case

Theorem [S. M., J. Neves, M. Piotrowski, 2007]

Let $0 < p, q \leq \infty$ and Ψ slowly varying with $\left(\Psi(2^{-j})^{-1}\right)_{j\in\mathbb{N}} \in \ell_\infty$.

(i) Let $0 < q \leq 1$ and $\Psi \sim 1$ or Ψ monotonically decreasing with $\lim_{t\to 0^+} \Psi(t) = \infty$. Then

$$\mathfrak{E}_{\mathrm{C}}(B_{p,q}^{(n/p,\Psi)}) = \left(t^{-1}\Psi(t)^{-1}, \infty\right).$$

(ii) Let $0 < p \leq 1$ and $\Psi \sim 1$ or Ψ monotonically decreasing with $\lim_{t\to 0^+} \Psi(t) = \infty$. Then

$$\mathfrak{E}_{\mathrm{C}}(F_{p,q}^{(n/p,\Psi)}) = \left(t^{-1}\Psi(t)^{-1}, \infty\right).$$

Continuity envelopes in the critical case

Corollary

(i) Let $0 < q \leq 1$ and $0 < p \leq \infty$. Then

$$\mathfrak{E}_{\mathrm{C}}(B^{n/p}_{p,q}) = \left(t^{-1}, \infty\right).$$

(ii) Let $0 < p \leq 1$ and $0 < q \leq \infty$. Then

$$\mathfrak{E}_{\mathrm{C}}(F^{n/p}_{p,q}) = \left(t^{-1}, \infty\right).$$

Embedding results

From the continuity envelope, in the critical case

$$B_{p,q}^{(n/p,\Psi)} \hookrightarrow \Lambda_{\infty,\infty}^{tE_C(t)}$$

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Generalized Hölder spaces

Let $0 < r \leq \infty$, $\mu \in \mathcal{L}_r$. The space $\Lambda_{\infty,r}^{\mu(\cdot)}$ consists of those $f \in C_B$ for which

$$||f|\Lambda_{\infty,r}^{\mu(\cdot)}|| := \|f|L_\infty\| + \left(\int_0^1 \left[\frac{\omega(f,t)}{\mu(t)}\right]^r \frac{\mathrm{d}t}{t}\right)^{\frac{1}{r}} < \infty.$$

Embedding results

Theorem [S. M., J. Neves, C. Schneider, 2010]

Let $0 < p \leq \infty$, $0 < q, r \leq \infty$, $\mu \in \mathcal{L}_r$, Ψ a slowly varying function with $\left(\Psi(2^{-j})^{-1}\right)_{j \in \mathbb{N}_0} \in \ell_{q'}$.

(i) If $0 < q \leq r \leq \infty$, then

$$B^{(n/p,\Psi)}_{p,q} \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r},$$

if, and only if,

$$\sup_{N \geq 0} \left(\sum_{j=0}^{N} \int_{2^{-(j+1)}}^{2^{-j}} \mu(t)^{-r} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{r}} \left(\sum_{k=N}^{\infty} \Psi(2^{-k})^{-q'} \right)^{\frac{1}{q'}} < \infty.$$

(ii) If $0 < r < q \leq \infty$, then ...

Theorem (Cont.)

(ii) If $0 < r < q \leq \infty$, then $B^{(n/p,\Psi)}_{p,q}(\mathbb{R}^n) \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}(\mathbb{R}^n)$ if, and only if,

$$\left\{\sum_{N=0}^{\infty}\left(\sum_{j=0}^{N}\int_{2^{-(j+1)}}^{2^{-j}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{u}{q}}\cdot\left(\int_{2^{-(N+1)}}^{2^{-N}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)\cdot\left(\sum_{k=N}^{\infty}\Psi(2^{-k})^{-q'}\right)^{\frac{u}{q'}}\right\}^{\frac{1}{u}}<\infty$$

$$\& \left\{\sum_{N=0}^{\infty}\left(\sum_{j=N}^{\infty}2^{-jr}\int_{2^{-(j+1)}}^{2^{-j}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)^{\frac{u}{q}}\cdot 2^{-Nr}\left(\int_{2^{-(N+1)}}^{2^{-N}}\mu(t)^{-r}\frac{\mathrm{d}t}{t}\right)\cdot\left(\sum_{k=0}^{N}2^{kq'}\Psi(2^{-k})^{-q'}\right)^{\frac{u}{q'}}\right\}^{\frac{1}{u}}<\infty,$$

where $\frac{1}{u} := \frac{1}{r} - \frac{1}{q}$.

Optimal embeddings

Let $1 < q \leq \infty$ and define the weights

$$\lambda_{qr}(t) := \Psi(t)^{\frac{q'}{r}} \left(\int_0^t \Psi(s)^{-q'} \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q'} + \frac{1}{r}}, \qquad t \in (0, 1].$$

Optimal embeddings

Let $1 < q \leq \infty$ and define the weights

$$\lambda_{qr}(t) := \Psi(t)^{\frac{q'}{r}} \left(\int_0^t \Psi(s)^{-q'} \frac{\mathrm{d}s}{s} \right)^{\frac{1}{q'} + \frac{1}{r}}, \qquad t \in (0,1].$$

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- Let $r \in [q,\infty]$. Among the embeddings $(*)$, that one with $\mu = \lambda_{qr}$ is sharp with respect to the parameter μ, i.e. $\Lambda^{\lambda_{qr}(\cdot)}_{\infty,r} \hookrightarrow \Lambda^{\mu(\cdot)}_{\infty,r}$ for any μ such that $(*)$ holds.

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Remark

Monotonicity

$$\sup_{0<t\leq\varepsilon} \frac{g(t)}{\mathcal{E}_{\mathsf{G}}^{X}(t)} \leq c_1 \left(\int_0^{\varepsilon} \left(\frac{g(t)}{\mathcal{E}_{\mathsf{G}}^{X}(t)} \right)^{v_1} \mu_H(\mathrm{d}t) \right)^{1/v_1}$$

$$\leq c_2 \left(\int_0^{\varepsilon} \left(\frac{g(t)}{\mathcal{E}_{\mathsf{G}}^{X}(t)} \right)^{v_0} \mu_H(\mathrm{d}t) \right)^{1/v_0}$$

for $0 < v_0 < v_1 < \infty$ and all $g \searrow$ on $(0, \varepsilon]$.

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