A lax algebraic study of non-Archimedean approach spaces

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In this work we start from the well-known description of the category App of approach spaces and contractions as lax algebras by using the ultrafilter monad and extending it to numerical relations, i.e. $\mathsf{App} \cong (\beta, \mathsf{P}_+) - \mathsf{Cat}$ [2].

Now we interchange the quantale P_+ for $\mathsf{P}_\vee = ([0,\infty]^{op}, \vee, 0)$. Analogously to the P_+-situation, the extension of the ultrafiltermonad β is a flat and associative lax extension to $\mathsf{P}_\vee - \mathsf{Rel}$. The lax-homomorphism $\varphi : \mathsf{P}_\vee \to \mathsf{P}_+$ is compatible with the lax- extensions of the ultrafilter monad, hence it induces a change-of-base functor

$$(\beta, \mathsf{P}_\vee) - \mathsf{Cat} \to (\beta, \mathsf{P}_+) - \mathsf{Cat},$$

which is an embedding.

We identify the category $(\beta, \mathsf{P}_\vee) - \mathsf{Cat}$ as the full subcategory of App consisting of all non-Archimedean approach spaces, i.e. approach spaces (X, λ) where the limit operator $\lambda : \beta X \to \mathsf{P}_\vee^X$ satisfies the strong triangular inequality

For any set J, for any $\psi : J \to X$, for any $\sigma : J \to \beta X$ and for any $\mathcal{U} \in \beta J$

$$\lambda \Sigma \sigma(\mathcal{U}) \leq \lambda \psi(\mathcal{U}) \vee \sup_{U \in \mathcal{U}} \inf_{j \in U} \lambda \sigma(j)\big(\psi(j)\big).$$

To the equivalent descriptions of non-Archimedean approach spaces by limit operators, distances and towers, introduced in [1], we add a new one using the gauge.

We investigate topological properties in $(\beta, \mathsf{P}_\vee) - \mathsf{Cat}$, following the relational calculus developed in [3] for $(\mathbb{T}, \mathcal{V})$-properties. We introduce low separation properties, Hausdorffness, compactness, regularity and normality as an application of this theory to $(\beta, \mathsf{P}_\vee) - \mathsf{Cat}$. On the other hand, we make use of the well known meaning of these properties in the setting of Top. For a non-Archimedean approach space X with tower of topologies $(\mathcal{T}_\varepsilon)_{\varepsilon \in \mathbb{R}^+}$ we compare the properties $(\beta, \mathsf{P}_\vee) - p$ to the properties 'X has p', meaning that the topological coreflection $\mathrm{T}X$ has p in Top, and 'X strongly has p', meaning that $(X, \mathcal{T}_\varepsilon)$ has p in Top for every $\varepsilon \in \mathbb{R}^+$.

*Joint work with Eva Colebunders

References

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