# ON EXTENSIONS OF LAX MONADS

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Abstract. In this paper we construct extensions of **Set**-monads – and, more generally, lax **Rel**-monads – into lax monads of the bicategory Mat($\mathbf{V}$) of generalized $\mathbf{V}$-matrices, whenever $\mathbf{V}$ is a well-behaved lattice equipped with a tensor product. We add some guiding examples.

## Introduction

Extensions of **Set**-monads into lax monads in bicategories of generalized matrices have been used recently to study categories of lax algebras [4, 9, 8], generalizing Barr's [1] description of topological spaces as lax algebras for the ultrafilter monad and Lawvere's [14] description of metric spaces as $\mathbf{V}$-categories for $\mathbf{V}$ the real half-line. The recent interest in this area had its origin in the use of the description of topological spaces via ultrafilter convergence to characterize special classes of continuous maps, such as effective descent morphisms, triquotient maps, exponentiable maps, quotient maps and local homeomorphisms [16, 13, 3, 6, 12, 5].

In this area, one of the difficulties one has to deal with is the construction of lax extensions of **Set**-monads into a larger bicategory. Contrarily to the extensions studied so far, with *ad-hoc* constructions, here we present a uniform construction of an extension of a **Set**-monad, satisfying (BC), into a lax monad of the bicategory Mat($\mathbf{V}$) of generalized $\mathbf{V}$-matrices. This construction consists of three steps: first we apply Barr's extension of the monad into the category **Rel** of relations (in Section 1) and then we extend this into Mat($\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$) and finally into Mat($\mathbf{V}$) (as described in Section 3). This construction includes, for instance, Clementino-Tholen construction of an extension of the ultrafilter monad in case $\mathbf{V}$ is a lattice (Example 5.4). The techniques used here can be used also to extend lax monads from **Rel** into Mat($\mathbf{V}$). We find particularly interesting the presentation of the Hausdorff metric on subsets of a metric space as an extension of the lax powerset monad (Example 6.3).

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## 1. From **Set** to **Rel**

1.1. **The 2-category Rel.** We recall that **Rel** has as objects *sets* and as morphisms $r : X \nrightarrow Y$ *relations* $r \subseteq X \times Y$ (or equivalently $r : X \times Y \to \mathbf{2}$). With the hom-sets $\mathbf{Rel}(X, Y)$ partially ordered by inclusion, **Rel** is a 2-category.

Using its natural involution $(\ )^\circ$, that assigns to each relation $r : X \nrightarrow Y$ its inverse $r^\circ : Y \nrightarrow X$, and the embedding $\mathbf{Set} \hookrightarrow \mathbf{Rel}$, it is easily seen that every relation $r$ can be written as $g \cdot f^\circ$ for suitable maps $f$ and $g$:

$$\begin{tikzcd} X \arrow[rr, "r", "|"{marking}] & & Y \\ & r \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd} \tag{1}$$

where $f$ and $g$ are the projections.

1.2. **Barr's extension.** In order to extend a monad $(T, \eta, \mu)$ in **Set** into **Rel**, Barr [1] defined first $\overline{T}(f^\circ) := (Tf)^\circ$ for any map $f$, and then made use of the factorization (1) of the relation $r : X \nrightarrow Y$ to define

$$\overline{T}r := Tg \cdot \overline{T}f^\circ,$$

that does not depend on the chosen factorization and extends naturally to 2-cells. Hence the following diagram

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "T"] \arrow[d, hook] & \mathbf{Set} \arrow[d, hook] \\ \mathbf{Rel} \arrow[r, "\overline{T}"] & \mathbf{Rel} \end{tikzcd}$$

is commutative.

Barr proved that $\overline{T} : \mathbf{Rel} \to \mathbf{Rel}$ is an *op-lax* functor and that the natural transformations $\eta$ and $\mu$ become *op-lax* in **Rel**; that is:

- $\overline{T}(r \cdot s) \leq \overline{T}r \cdot \overline{T}s$ for any pair of composable relations $r$, $s$;
- for every $r : X \nrightarrow Y$, one has

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "r"', "|"{marking}] \arrow[dr, phantom, "\leq"] & TX \arrow[d, "\overline{T}r", "|"{marking}] \\ Y \arrow[r, "\eta_Y"'] & TY \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "\mu_X"] \arrow[d, "\overline{T}^2r"', "|"{marking}] \arrow[dr, phantom, "\leq"] & TX \arrow[d, "\overline{T}r", "|"{marking}] \\ T^2Y \arrow[r, "\mu_Y"'] & TY. \end{tikzcd}$$

1.3. **The role of the Beck-Chevalley Condition.** As Barr pointed out, this extension may fail to be a functor. The missing inequality depends on the behaviour of the functor $T : \mathbf{Set} \to \mathbf{Set}$: it holds if and only if $T$ satisfies the *Beck-Chevalley Condition (BC)*, that is, if $(Tf)^\circ \cdot Tg = Tk \cdot (Th)^\circ$ for every pullback diagram

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

in **Set**. (Under the Axiom of Choice, (BC) is equivalent to the preservation of weak pullbacks.)

**Theorem.** *For a functor $T : \mathbf{Set} \to \mathbf{Set}$, the following assertions are equivalent:*

(i) *There is a (unique) 2-functor $\overline{T} : \mathbf{Rel} \to \mathbf{Rel}$, preserving the involution, that extends $T$;*

(ii) *$T$ satisfies the Beck-Chevalley Condition.* □

We also have:

**Proposition.** *For functors $S, T : \mathbf{Set} \to \mathbf{Set}$ satisfying (BC) and a natural transformation $\varphi : S \to T$, the following assertions are equivalent:*

(i) *$\overline{\varphi} : \overline{S} \to \overline{T}$ is a natural transformation;*

(ii) *for every map $f : X \to Y$, the **Set**-diagram*

$$\begin{tikzcd} SX \arrow[r, "\varphi_X"] \arrow[d, "Sf"'] & TX \arrow[d, "Tf"] \\ SY \arrow[r, "\varphi_Y"] & TY \end{tikzcd}$$

*satisfies (BC), i.e.* $(Tf)^\circ \cdot \varphi_Y = \varphi_X \cdot (Sf)^\circ$. □

## 2. The extended setting: Mat(V) and lax monads

Throughout we will be concerned with the construction of *lax* extensions of a **Set**-monad to more general 2-categories. In this section we describe the 2-categories as well as the lax axioms for a monad we will deal with.

2.1. **The category of V-matrices.** We consider a *complete and cocomplete lattice* $\mathbf{V}$ as a category and assume that it is *symmetric monoidal-closed*, with tensor product $\otimes$ and unit $k_{\mathbf{V}}$. We denote its initial and terminal objects by $\perp_{\mathbf{V}}$ and $\top_{\mathbf{V}}$, respectively, and assume that $k_{\mathbf{V}} \neq \perp_{\mathbf{V}}$. The 2-category $\mathrm{Mat}(\mathbf{V})$ has as objects *sets* and as 1-cells $r : X \nrightarrow Y$ $\mathbf{V}$-*matrices*, that is, maps $r : X \times Y \to \mathbf{V}$; given $r, s : X \nrightarrow Y$, there is a (unique) 2-cell $r \to s$ if, for every $(x, y) \in X \times Y$, $r(x, y) \leq s(x, y)$ in $\mathbf{V}$. Composition of 1-cells $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ is given by *matrix multiplication*, i.e.

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z),$$

for every $x \in X$ and $z \in Z$. Further information about this category can be found in [17] and [9].

**Rel** is a crucial example of a 2-category of this sort, obtained when $\mathbf{V} = \mathbf{2} = \{\bot, \top\}$, with $\otimes = \wedge$. The monoidal map $\mathbf{2} \hookrightarrow \mathbf{V}$, with $\bot \mapsto \bot_{\mathbf{V}}$ and $\top \mapsto k_{\mathbf{V}}$ gives naturally rise to an embedding $\mathbf{Rel} \hookrightarrow \mathrm{Mat}(\mathbf{V})$. By *relation* in $\mathrm{Mat}(\mathbf{V})$ we mean any $\mathbf{V}$-matrix with entries $\bot_V$ and $k_V$; that is, any image of a relation by this embedding.

2.2. **Lax monads.** Here we consider a definition of lax monad different from Barr's [1]; namely, we assume that the functor is *lax* (and not necessarily op-lax).

By a *lax monad* $(T, \eta, \mu)$ in $\mathrm{Mat}(\mathbf{V})$ we mean:

– a *lax functor* $T : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ (so that $1_{TX} \leq T1_X$ and $Ts \cdot Tr \leq T(s \cdot r)$ for composable $\mathbf{V}$-matrices $r, s$), and
– *op-lax natural transformations* $\eta : 1_{\mathrm{Mat}(\mathbf{V})} \to T$ and $\mu : T^2 \to T$,

such that $\mu T\mu \leq \mu\mu T$, $\mathrm{Id}\, T \leq \mu T\eta$ and $\mu\eta T \leq T\,\mathrm{Id}$; that is, for every set $X$,

$$\begin{tikzcd} T^3X \arrow[r, "\mu_{TX}"] \arrow[d, "T\mu_X"'] \arrow[dr, phantom, "\leq"] & T^2X \arrow[d, "\mu_X"] \\ T^2X \arrow[r, "\mu_X"] & TX \end{tikzcd} \qquad \begin{tikzcd} TX \arrow[r, "T\eta_X"] \arrow[dr, "1_{TX}"'] & T^2X \arrow[d, "\mu_X"] & TX. \arrow[l, "\eta_{TX}"'] \arrow[dl, "T1_X"] \\ & TX & \end{tikzcd} \tag{2}$$

We point out that this definition does not coincide with Bunge's [2], although the only difference is in the right triangle, where we replaced $1_{TX}$ by $T1_X$. In the final section we present an example of a lax monad (in our sense) which is not a lax monad à la Bunge (see Example 6.3).

## 3. The strategy

3.1. **The monoidal closed category $\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}$.** Given another (co)complete lattice $\mathbf{W}$, equipped with a join-preserving tensor product $\otimes$, with unit element $k_{\mathbf{W}}$, it is straightforward to check that the formula:

$$f \otimes g(v) = \bigvee_{v',v'' \,:\, v'\otimes v'' \geq v} f(v') \otimes g(v''), \tag{3}$$

for any $f, g \in \mathbf{W}^{\mathbf{V}^{\mathrm{op}}}$ and $v \in \mathbf{V}$, defines a tensor product in $\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}$ that preserves joins, with unit element

$$\begin{array}{rcl} k : \mathbf{V}^{\mathrm{op}} & \to & \mathbf{W} \\ v & \mapsto & \begin{cases} k_{\mathbf{W}} & \text{if } v \leq k_{\mathbf{V}} \\ \bot_{\mathbf{W}} & \text{elsewhere.} \end{cases} \end{array}$$

(We point out that this tensor product extends Day's *convolution* [10] to this setting.) Symmetry of this tensor is also inherited from symmetry of the tensor product of $\mathbf{W}$ and $\mathbf{V}$, so that we have:

**Proposition.** *Given two symmetric monoidal closed lattices* $\mathbf{W}$ *and* $\mathbf{V}$*, formula* (3) *gives a symmetric closed monoidal structure on* $\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}$*.*

3.2. **$\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}$-matrices versus $\mathbf{V}^{\mathrm{op}}$-indexed $\mathbf{W}$-matrices.** The embedding

$$\begin{array}{rcl} E:\mathbf{W} & \to & \mathbf{W}^{\mathbf{V}^{\mathrm{op}}} \\ w & \mapsto & \begin{array}{rcl} E(w):\mathbf{V}^{\mathrm{op}} & \to & \mathbf{W} \\ v & \mapsto & E(w)(v)=\begin{cases} w & \text{if } v\le k_{\mathbf{V}} \\ \perp_{\mathbf{W}} & \text{elsewhere} \end{cases} \end{array} \end{array}$$

preserves the tensor product, the unit element, infima and suprema. Therefore, as detailed in [9], it induces a 2-functor

$$E:\mathrm{Mat}(\mathbf{W})\to\mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}).$$

Denoting the set of functors from $\mathbf{A}$ to $\mathbf{B}$ by $[\mathbf{A},\mathbf{B}]$, the natural bijections

$$[X\times Y,[\mathbf{V}^{\mathrm{op}},\mathbf{W}]]\cong[X\times Y\times\mathbf{V}^{\mathrm{op}},\mathbf{W}]\cong[V^{\mathrm{op}}\times X\times Y,\mathbf{W}]\cong[\mathbf{V}^{\mathrm{op}},[X\times Y,\mathbf{W}]]$$

assign to any $\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}$-matrix $a:X\times Y\to\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}$ a $\mathbf{V}^{\mathrm{op}}$-indexed family of $\mathbf{W}$-matrices $(a_v:X\times Y\to\mathbf{W})_{v\in\mathbf{V}}$, defined by

$$a_v(x,y)=a(x,y)(v).$$

It is straightforward to prove that:

**Lemma.** *For $a,b\in\mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}})$ and $v,v'\in\mathbf{V}$, one has:*

(a) $v\le v' \;\Rightarrow\; a_v\ge a_{v'}$;
(b) $a\le b \;\Rightarrow\; a_v\le b_v$;
(c) $b_v\cdot a_{v'}\le(b\cdot a)_{v\otimes v'}$.

3.3. **The Yoneda embedding.** We consider now the Yoneda embedding

$$\begin{array}{rcl} \mathsf{Y}:\mathbf{V} & \to & \mathbf{2}^{\mathbf{V}^{\mathrm{op}}} \\ v & \mapsto & \begin{array}{rcl} \mathsf{Y}(v):\mathbf{V}^{\mathrm{op}} & \to & \mathbf{2} \\ u & \mapsto & \begin{cases} \top & \text{if } u\le v \\ \perp & \text{otherwise,} \end{cases} \end{array} \end{array}$$

and its left adjoint

$$\begin{array}{rcl} L:\mathbf{2}^{\mathbf{V}^{\mathrm{op}}} & \to & \mathbf{V} \\ f & \mapsto & \bigvee\{v\in\mathbf{V}\;;\;f(v)=\top\}. \end{array}$$

**Proposition.** *The functors $\mathsf{Y}$ and $L$ are monoidal functors.*

*Proof.* The functor $\mathsf{Y}$ is monoidal: from $\mathsf{Y}(k_V)(v)=\top$ if and only if $v\le k_V$, it follows that $\mathsf{Y}(k_V)=k$; moreover,

$$\begin{array}{rcl} (\mathsf{Y}(v)\otimes\mathsf{Y}(v'))(u)=\top & \Leftrightarrow & \displaystyle\bigvee_{r\otimes s\ge u}\mathsf{Y}(v)(r)\wedge\mathsf{Y}(v')(s)=\top \\ & \Leftrightarrow & \exists r,s\in\mathbf{V}\;:\;r\otimes s\ge u,\; r\le v \text{ and } s\le v' \\ & \Leftrightarrow & u\le v\otimes v' \;\Leftrightarrow\; \mathsf{Y}(v\otimes v')(u)=\top. \end{array}$$

The functor $L$ is monoidal (in fact, a 2-functor), since:

$$L(k)=\bigvee\{v\in\mathbf{V}\;;\;k(v)=\top\}=k_V,\text{ and}$$

$$\begin{aligned} L(f)\otimes L(g) &= \bigvee\{r\in\mathbf{V}\;;\;f(r)=\top\}\otimes\bigvee\{s\in\mathbf{V}\;;\;g(s)=\top\}\\ &= \bigvee\{r\otimes s\;;\;r,s\in\mathbf{V},\;f(r)=\top=g(s)\}\\ &= \bigvee\{v\;;\;(f\otimes g)(v)=\top\}\;=\;L(f\otimes g). \end{aligned}$$

□

These two monoidal functors induce adjoint lax functors

$$\mathrm{Mat}(\mathbf{V}) \underset{L}{\overset{\mathsf{Y}}{\rightleftarrows}} \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}),$$

the latter one being in fact a 2-functor.

3.4. **The use of the embeddings to construct the extension.** The construction of the extension of a lax monad $(T,\eta,\mu)$ in $\mathbf{Rel}=\mathrm{Mat}(\mathbf{2})$ into $\mathrm{Mat}(\mathbf{V})$ we will describe in the next two sections consists of two steps.

First we use the interpretation of a $\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$-matrix as a $\mathbf{V}^{\mathrm{op}}$-indexed family of relations and the embedding described in 3.2, obtaining a commutative diagram

$$\begin{tikzcd} \mathrm{Mat}(\mathbf{2}) \arrow[r, "T"] \arrow[d, "E"'] & \mathrm{Mat}(\mathbf{2}) \arrow[d, "E"] \\ \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "\widehat{T}"] & \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}). \end{tikzcd}$$

Since it does not imply an extra effort, in the next section we will in fact describe this extension in the case $\mathbf{2}$ is replaced by a general lattice $\mathbf{W}$.

Secondly, we use the adjunction $L\dashv\mathsf{Y}$ of Section 3.3 to transfer a lax monad $(S,\delta,\nu)$ in $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$ into $\mathrm{Mat}(\mathbf{V})$, defining $\widetilde{S}:=LS\mathsf{Y}$ and showing that, under some conditions on $\mathbf{V}$, the following diagram

$$\begin{tikzcd} \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "S"] \arrow[d, "L"'] & \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[d, "L"] \\ \mathrm{Mat}(\mathbf{V}) \arrow[r, "\widetilde{S}"] & \mathrm{Mat}(\mathbf{V}) \end{tikzcd}$$

is commutative and $(\widetilde{S},\widetilde{\delta},\widetilde{\nu})$ is a lax monad in $\mathrm{Mat}(\mathbf{V})$.

Finally, gluing these constructions, since $LE$ is the embedding $\mathbf{Rel}\hookrightarrow\mathrm{Mat}(\mathbf{V})$, we obtain a commutative diagram

$$\begin{tikzcd} \mathbf{Rel} \arrow[r, "T"] \arrow[d, hook, "E"'] \arrow[dd, hook, bend right] & \mathbf{Rel} \arrow[d, hook, "E"] \arrow[dd, hook, bend left] \\ \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "\widehat{T}"] \arrow[d, "L"'] & \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[d, "L"] \\ \mathrm{Mat}(\mathbf{V}) \arrow[r, "\widetilde{\widehat{T}}"] & \mathrm{Mat}(\mathbf{V}), \end{tikzcd}$$

and corresponding op-lax natural transformations $\widetilde{\widetilde{\eta}}$ and $\widetilde{\widetilde{\mu}}$.

## 4. From $\mathrm{Mat}(\mathbf{W})$ to $\mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}})$

4.1. **Extension of a lax endofunctor.** Given a lax functor $T : \mathrm{Mat}(\mathbf{W}) \to \mathrm{Mat}(\mathbf{W})$, for each $a : X \nrightarrow Y$ and $(\mathfrak{x}, \mathfrak{y}) \in TX \times TY$, we define

$$\widehat{T}a(\mathfrak{x}, \mathfrak{y})(v) := T(a_v)(\mathfrak{x}, \mathfrak{y}).$$

**Theorem.** *Let $T : \mathrm{Mat}(\mathbf{W}) \to \mathrm{Mat}(\mathbf{W})$ be a lax functor.*

(a) *The assignments $X \mapsto \widehat{T}X := TX$ and $a \mapsto \widehat{T}a$ define a lax functor $\widehat{T} : \mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}) \to \mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}})$ such that*

$$\begin{tikzcd} \mathrm{Mat}(\mathbf{W}) \arrow[r, "T"] \arrow[d, hook, "E"'] \arrow[dr, phantom, "\geq"] & \mathrm{Mat}(\mathbf{W}) \arrow[d, hook, "E"] \\ \mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "\widehat{T}"] & \mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}); \end{tikzcd}$$

(b) *$\widehat{T}$ preserves the involution whenever $T$ does.*

(c) *If $k_V = \top_V$ or $T$ preserves the $\perp$-relation, then $\widehat{T}$ is an extension of $T$, that is, the following diagram commutes*

$$\begin{tikzcd} \mathrm{Mat}(\mathbf{W}) \arrow[r, "T"] \arrow[d, hook, "E"'] & \mathrm{Mat}(\mathbf{W}) \arrow[d, hook, "E"] \\ \mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "\widehat{T}"] & \mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}}). \end{tikzcd}$$

*Proof.* To prove (a), using the Lemma we only have to show that $\widehat{T}(b) \cdot \widehat{T}(a) \leq \widehat{T}(b \cdot a)$, $1_{\widehat{T}X} \leq \widehat{T}1_X$ and that $\widehat{T}E \geq E\widehat{T}$. To show the first inequality, consider $a : X \nrightarrow Y$ and $b : Y \nrightarrow Z$ in $\mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}})$, and $\mathfrak{x} \in TX$, $\mathfrak{z} \in TZ$ and $v \in \mathbf{V}$. Then:

$$\begin{aligned} \widehat{T}b \cdot \widehat{T}a(\mathfrak{x}, \mathfrak{z})(v) &= \left( \bigvee_{\mathfrak{y} \in TY} \widehat{T}a(\mathfrak{x}, \mathfrak{y}) \otimes \widehat{T}b(\mathfrak{y}, \mathfrak{z}) \right)(v) \\ &= \bigvee_{\mathfrak{y} \in TY} \; \bigvee_{v' \otimes v'' \geq v} \widehat{T}a(\mathfrak{x}, \mathfrak{y})(v') \otimes \widehat{T}b(\mathfrak{y}, \mathfrak{z})(v'') \\ &= \bigvee_{v' \otimes v'' \geq v} (Tb_{v''} \cdot Ta_{v'})(\mathfrak{x}, \mathfrak{z}) \\ &\leq \bigvee_{v' \otimes v'' \geq v} T(b \cdot a)_{v' \otimes v''}(\mathfrak{x}, \mathfrak{z}) \;\leq\; T(b \cdot a)_v(\mathfrak{x}, \mathfrak{z}). \end{aligned}$$

Now, for a $\mathbf{W}$-matrix $a : X \nrightarrow Y$,

$$\widehat{T}E(a) = \begin{cases} Ta & \text{if } v \leq k_V \\ T\perp & \text{otherwise} \end{cases} \quad \text{while} \quad E\widehat{T}(a) = \begin{cases} Ta & \text{if } v \leq k_V \\ \perp_W & \text{otherwise,} \end{cases}$$

hence $\widehat{T}E \geq E\widehat{T}$ follows. This inequality implies that $1_{\widehat{T}X} \leq \widehat{T}1_X$, since

$$\widehat{T}1_X = \widehat{T}E1_X \geq ET1_X \geq E1_{TX} = 1_{TX}.$$

The proofs of (b) and (c) are now straightforward. $\square$

Finally we prove some useful results.

**Lemma.** *For $a : X \nrightarrow Y$ in $\mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}})$, $r : Y \nrightarrow Z$, $s : W \nrightarrow X$ in $\mathrm{Mat}(\mathbf{W})$, and $v \in \mathbf{V}$:*

(a) $(\widehat{T}a)_v = Ta_v$;
(b) $(Er \cdot a)_v = Er \cdot a_v$ *and* $(a \cdot Es)_v = a_v \cdot Es$.

*Proof.* (a) is straightforward.

(b): For $x \in X$ and $z \in Z$,

$$\begin{aligned}(Er \cdot a)_v(x,z) &= (Er \cdot a)(x,z)(v) = \bigvee_{y \in Y} (a(x,y) \otimes Er(y,z))(v)\\ &= \bigvee_{v' \otimes v'' \geq v} \bigvee_{y \in Y} (a(x,y)(v') \otimes Er(y,z)(v'')).\end{aligned}$$

In this join it is enough to consider:

- $v'' \leq k_V$, since elsewhere $Er(y,z)(v'') = \bot_W$ and the tensor product is $\bot_W$ as well, and
- $v' = v$, due to monotonicity of $a(x,y)$; hence,

$$(Er \cdot a)_v(x,z) = \bigvee_{y \in Y} a(x,y)(v) \otimes r(y,z) = (Er \cdot a_v)(x,z).$$

The other equality is proved analogously. $\square$

**Proposition.** *If $T : \mathrm{Mat}(\mathbf{W}) \to \mathrm{Mat}(\mathbf{W})$ preserves composition on the left (right) with $a : X \times Y \to \mathbf{W}$, then so does $\widehat{T}$, with $a$ replaced by $Ea$.*

*Proof.* For any $b : Y \nrightarrow Z$ in $\mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}})$, $\mathfrak{x} \in TX$ and $\mathfrak{z} \in TZ$,

$$\begin{aligned}\widehat{T}b \cdot \widehat{T}Ea(\mathfrak{x},\mathfrak{z})(v) &= \bigvee_{\mathfrak{y} \in \widehat{T}Y} (\widehat{T}Ea(\mathfrak{x},\mathfrak{y}) \otimes \widehat{T}b(\mathfrak{y},\mathfrak{z}))(v)\\ &\geq \bigvee_{\mathfrak{y} \in \widehat{T}Y} (ETa(\mathfrak{x},\mathfrak{y}) \otimes \widehat{T}b(\mathfrak{y},\mathfrak{z}))(v)\\ &= \bigvee_{\mathfrak{y} \in \widehat{T}Y} \bigvee_{v' \otimes v'' \geq v} ETa(\mathfrak{x},\mathfrak{y})(v') \otimes \widehat{T}b(\mathfrak{y},\mathfrak{z})(v'')\\ &= \bigvee_{\mathfrak{y} \in \widehat{T}Y} Ta(\mathfrak{x},\mathfrak{y}) \otimes \widehat{T}b(\mathfrak{y},\mathfrak{z})(v)\\ &= (Tb_V \cdot Ta)(\mathfrak{x},\mathfrak{z}) \;=\; T(b_V \cdot a)(\mathfrak{x},\mathfrak{z})\\ &= T(b \cdot Ea)_v(\mathfrak{x},\mathfrak{z}) \;=\; \widehat{T}(b \cdot Ea)(\mathfrak{x},\mathfrak{z})(v).\end{aligned}$$

The stability under composition on the left has an analogous proof. $\square$

4.2. **Extension of a lax monad.**

**Proposition.** *Let* $S, T : \mathrm{Mat}(\mathbf{W}) \to \mathrm{Mat}(\mathbf{W})$ *be lax functors.*

(a) $\widehat{\mathrm{Id}} = \mathrm{Id}$.

(b) $\widehat{S} \cdot \widehat{T} = \widehat{S \cdot T}$.

(c) *If* $\alpha = (\alpha_X) : S \to T$ *is a (lax, op-lax) natural transformation, so is* $\widehat{\alpha} : \widehat{S} \to \widehat{T}$.

*Proof.* Straightforward. $\square$

**Theorem.** *Each lax monad* $(T, \eta, \mu)$ *in* $\mathrm{Mat}(\mathbf{W})$ *gives rise to a lax monad* $(\widehat{T}, \widehat{\eta}, \widehat{\mu})$ *in* $\mathrm{Mat}(\mathbf{W}^{\mathbf{V}^{\mathrm{op}}})$*, provided that* $\widehat{T}E\mu = ET\mu$.

*Proof.* We only have to check diagrams (2). For a set $X$, one always has

$$\widehat{\mu}_X \widehat{T} \widehat{\eta}_X = E\mu_X \widehat{T} E\eta_X \geq E(\mu_X T\eta_X) \geq E1_{TX} = 1_{\widehat{T}X},$$

and

$$\widehat{\mu}_X \widehat{\eta}_{\widehat{T}X} = E\mu_X E\eta_{TX} = E(\mu_X \eta_{TX}) \leq ET1_X \leq \widehat{T}1_X;$$

if, moreover, $\widehat{T}E\mu = ET\mu$, then

$$\widehat{\mu}_X \widehat{T} \widehat{\mu}_X = E\mu_X \widehat{T} E\mu_X = E\mu_X ET\mu_X = E(\mu_X T\mu_X) \leq E(\mu_X \mu_{TX}) = \widehat{\mu}_X \widehat{\mu}_{TX}.$$

$\square$

**Corollary.** *Let* $(T, \eta, \mu)$ *be a lax monad in* $\mathrm{Mat}(\mathbf{W})$*. If* $k_V = \top_V$ *or* $T$ *preserves the* $\bot$*-matrix, then* $(\widehat{T}, \widehat{\eta}, \widehat{\mu})$ *is a lax monad that extends the former one.* $\square$

## 5. From $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$ into $\mathrm{Mat}(\mathbf{V})$

5.1. **Transfer of a lax endofunctor.** Using the monoidal adjunction of 3.3, for a lax endofunctor $S$ in $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$, we define

$$\mathrm{Mat}(\mathbf{V}) \xrightarrow{\widetilde{S}} \mathrm{Mat}(\mathbf{V}) := (\mathrm{Mat}(\mathbf{V}) \xrightarrow{\mathsf{Y}} \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \xrightarrow{S} \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \xrightarrow{L} \mathrm{Mat}(\mathbf{V})).$$

**Proposition.** *Let* $S : \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \to \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$ *be a lax functor. Then* $\widetilde{S} = LS\mathsf{Y}$ *is such that*

$$\begin{tikzcd} \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "S"] \arrow[d, "L"'] \arrow[dr, phantom, "\geq"] & \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[d, "L"] \\ \mathrm{Mat}(\mathbf{V}) \arrow[r, "\widetilde{S}"] & \mathrm{Mat}(\mathbf{V}). \end{tikzcd}$$

*The inequality in the diagram becomes an equality whenever* $S\mathsf{Y}L \leq \mathsf{Y}LS$.

*Proof.* By the adjointness property, $LS \leq LS\mathsf{Y}L$, the required inequality. In addition, if $S\mathsf{Y}L \leq \mathsf{Y}LS$, then $LS\mathsf{Y}L \leq L\mathsf{Y}LS \leq LS$, and the equality follows. $\square$

Analogously to the previous construction, we can easily check that:

**Lemma.** *For* $S : \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \to \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$*, if* $S$ *preserves composition on the left (right) with a matrix* $a : X \nrightarrow Y$*, then so does* $\widetilde{S}$*, with* $a$ *replaced by* $La$. $\square$

5.2. **Transfer of a lax monad.** First we analyse the behaviour of the construction with respect to the composition of functors and natural transformations.

**Lemma.** *Let* $R, S : \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \to \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$ *be lax functors. Then:*

(a) $\widetilde{\mathrm{Id}} = \mathrm{Id}$.
(b) $\widetilde{RS} \le \widetilde{R}\widetilde{S}$, *with equality in case* $R\mathsf{Y}L \le \mathsf{Y}LR$.
(c) *If* $\alpha = (\alpha_X) : R \to S$ *is a (lax, op-lax) natural transformations, so is* $\widetilde{\alpha} = (L\alpha_{\mathsf{Y}X}) = (L\alpha_X) : \widetilde{R} \to \widetilde{S}$.

*Proof.* It is straightforward. □

**Theorem.** *For each lax monad* $(S, \delta, \nu)$ *in* $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$, $(\widetilde{S}, \widetilde{\delta}, \widetilde{\nu})$ *is a lax monad in* $\mathrm{Mat}(\mathbf{V})$ *provided that* $S\mathsf{Y}L \le \mathsf{Y}LS$.

*Proof.* We already know that $\widetilde{S} = LS\mathsf{Y}$ is a lax functor, $\widetilde{\delta} = L\delta : L\mathsf{Y} = \mathrm{Id} \to \widetilde{S}$ and $\widetilde{\nu} = L\nu : \widetilde{SS} = \widetilde{S}\widetilde{S} \to \widetilde{S}$ are op-lax natural transformations. It remains to be shown that they fulfil the conditions of diagram (2): for each set $X$,

$$\begin{aligned}
\widetilde{\nu}_X \cdot \widetilde{S}\widetilde{\nu}_X &= L\nu_X \cdot LS\mathsf{Y}L\nu_X \ \le\ L\nu_X L\mathsf{Y}LS\nu_X \ =\ L(\nu_X S\nu_X) \\
&\le\ L(\nu_X \nu_{SX}) \ =\ L\nu_X L\nu_{SX} \ =\ \widetilde{\nu}_X \widetilde{\nu}_{\widetilde{S}X}; \\
\widetilde{\nu}_X \widetilde{S}\widetilde{\delta}_X &= L\nu_X LS\mathsf{Y}L\delta_X \ \ge\ L\nu_X LS\delta_X \ =\ L(\nu_X S\delta_X) \\
&\ge\ L1_{SX} \ \ge\ 1_{\widetilde{S}X}; \\
\widetilde{\nu}_X \widetilde{\delta}_{\widetilde{S}X} &= L(\nu_X \delta_{SX}) \ \le\ LS1_X \ \le\ LS\mathsf{Y}1_X \ =\ \widetilde{S}1_X.
\end{aligned}$$

□

5.3. **An extra condition on V.** In order to guarantee that $S\mathsf{Y}L \le \mathsf{Y}LS$ we will impose an extra condition on $\mathbf{V}$ which we analyse in the sequel.

**Proposition.** *The following conditions are equivalent:*

(i) *there exists a transitive relation* $\sqsubset$ *on* $\mathbf{V}$ *such that, for every* $v \in \mathbf{V}$,
$$v = \bigvee\{u \in \mathbf{V}\ ;\ \forall S \subseteq \mathbf{V}\ u \sqsubset v \le \bigvee S\ \Rightarrow\ \exists s \in S\ :\ u \le s\}; \tag{4}$$
(ii) *there exists a family* $(A(v))_{v\in\mathbf{V}}$ *of subsets of* $\mathbf{V}$ *such that, for each* $f \in \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$ *and* $v \in \mathbf{V}$,
$$\mathsf{Y}L(f)(v) = \bigwedge_{u\in A(v)} f(u).$$

*Proof.* (i) ⇒ (ii): For each $v \in \mathbf{V}$, let
$$At(v) := \{u \in \mathbf{V}\ ;\ \forall S \subseteq \mathbf{V}\ u \sqsubset v \le \bigvee S\ \Rightarrow\ \exists s \in S\ :\ u \le s\}. \tag{5}$$
Then
$$\begin{aligned}
\mathsf{Y}L(f)(v) = \top\ &\Leftrightarrow\ v \le \bigvee\{w \in \mathbf{V}\ ;\ f(w) = \top\} \\
&\Leftrightarrow\ \forall u \in At(v)\ f(u) = \top.
\end{aligned}$$

(ii) ⇒ (i): Let $\sqsubset$ be defined by
$$u \sqsubset v\ :\Leftrightarrow\ \exists w \in \mathbf{V}\ :\ u \in A(w) \text{ and } w \le v.$$

Hence, for any $u \in A(v)$, $u \sqsubset v$.

To show that $\sqsubset$ is transitive, it is enough to notice that, since $\mathsf{Y}L\mathsf{Y} = \mathsf{Y}$, we have

$$\top = \mathsf{Y}(v)(v) = \mathsf{Y}L\mathsf{Y}(v)(v) \;\Rightarrow\; \bigwedge_{u\in A(v)} \mathsf{Y}(v)(u) = \top \;\Rightarrow\; \forall u \in A(v) \;:\; u \leq v.$$

To show equality (4) we first show that $v = \bigvee A(v)$; in fact, we only have to prove that $v \leq w := \bigvee A(v)$, since the other inequality is shown above:

$$\top = \bigwedge_{u\in A(v)} \mathsf{Y}(w)(u) = \mathsf{Y}L\mathsf{Y}(w)(v) = \mathsf{Y}(w)(v) \;\Rightarrow\; v \leq w.$$

Finally, assume that $v \leq \bigvee S$ for some subset $S$ of $\mathbf{V}$, and consider

$$\begin{array}{rcl} f : \mathbf{V}^{\mathrm{op}} & \to & \mathbf{2} \\ w & \mapsto & \begin{cases} \top & \text{if } \exists s \in S \;:\; w \leq s \\ \bot & \text{elsewhere.} \end{cases} \end{array}$$

Then $\mathsf{Y}L(f)(v) = \top$ since $v \leq L(f) = \bigvee S$, and therefore $f(u) = \top$ for every $u \in A(v)$. □

We remark that the proof of the Proposition shows that the equivalent conditions (i) and (ii) mean exactly that $\mathbf{V}$ is $\sqsubset$-atomic, for a transitive relation $\sqsubset$ on $\mathbf{V}$, in the sense of [9]. From now on, by $\sqsubset$-*atomic* we mean $\sqsubset$-atomic for a transitive relation $\sqsubset$, and we use $At(v)$ as defined in (5).

**Lemma.** *If $\mathbf{V}$ is $\sqsubset$-atomic, for each $\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$-matrix $a : X \nrightarrow Y$ and each element $v$ of $\mathbf{V}$, one has*

$$(\mathsf{Y}La)_v = \bigwedge_{u\in At(v)} a_u.$$

*Proof.* For each $x \in X$ and $y \in Y$, using the proposition above, we have

$$(\mathsf{Y}La)_v(x,y) = \mathsf{Y}L(a(\mathfrak{x},y))(v) = \bigwedge_{u\in At(v)} a(x,y)(u) = \bigwedge_{u\in At(v)} a_u(x,y).$$

□

**Theorem.** *Let $(T,\eta,\mu)$ be a lax monad in* **Rel**. *If $\mathbf{V}$ is $\sqsubset$-atomic, and $k_V = \top_V$ or $\top$ preserves the $\bot$-matrix, then $(\widetilde{\widehat{T}}, \widetilde{\widehat{\eta}}, \widetilde{\widehat{\mu}})$ is a lax monad in* $\mathrm{Mat}(\mathbf{V})$*, that extends the former one.*

*Proof.* Using Theorems 4.1, 4.2, and Proposition 5.1 and Theorem 5.2, it is enough to show that $\widehat{T}\mathsf{Y}L \leq \mathsf{Y}L\widehat{T}$, whenever $\mathbf{V}$ is $\sqsubset$-atomic. For $a : X \nrightarrow Y \in \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$, $v \in \mathbf{V}$, $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$:

$$\begin{array}{rcl} (\widehat{T}\mathsf{Y}L(a))(\mathfrak{x},\mathfrak{y})(v) & = & T(\mathsf{Y}L(a))_v(\mathfrak{x},\mathfrak{y}) \;=\; T(\displaystyle\bigwedge_{u\in At(v)} a_u)(\mathfrak{x},\mathfrak{y}) \\ & \leq & \displaystyle\bigwedge_{u\in At(v)} Ta_u(\mathfrak{x},\mathfrak{y}) \;=\; \mathsf{Y}LTa(\mathfrak{x},\mathfrak{y})(v). \end{array}$$

□

Combining this result with Theorem 1.3, we obtain

**Corollary.** *Let* $(T, \eta, \mu)$ *be a monad in* **Set**. *If* $T$ *satisfies (BC),* $\mathbf{V}$ *is* $\sqsubset$*-atomic, and* $k_V = \top_V$ *or* $\top$ *preserves the* $\bot$*-matrix, then* $(\widetilde{\widehat{\overline{T}}}, \widetilde{\widehat{\overline{\eta}}}, \widetilde{\widehat{\overline{\mu}}})$ *is a lax monad in* $\mathrm{Mat}(\mathbf{V})$*, that extends the given one.* $\square$

**Remark.** In the construction carried out through this section we can easily replace the monoidal adjunction $L \dashv \mathsf{Y}$ by any other such adjunction.

## 6. Examples

In this section we present examples of extensions. Our main examples are based on the category $\mathrm{Mat}(\overline{\mathbb{R}}_+)$, where $([0, \infty], \geq)$ is endowed with the tensor product $+$. We remark that in this situation the terminal object is also the unit element 0 and that $\overline{\mathbb{R}}_+$ is $>$-atomic, hence we may apply our results. For simplicity, we use the same notation for the given (lax) monad and its extension.

6.1. **The identity monad.** Barr's extension of the identity monad $(\mathrm{Id}, 1, 1)$ in **Set** into **Rel** gives the identity monad. The same occurs in the next step: its extension into $\mathrm{Mat}(\mathbf{V})$ as defined here is the identity monad. (We remark that this monad may have other lax extensions, as it is shown in [7].)

6.2. **The powerset monad.** The powerset monad $(P, \eta, \mu)$ in **Set** is defined by:

- $P$ is the *powerset functor*, assigning to each set $X$ its powerset $PX$ and to each map its direct image,
- $\eta_X(x) = \{x\}$ for every $x \in X \in$ **Set**, and
- $\mu_X(\mathcal{A}) = \bigcup \mathcal{A}$ for every set $\mathcal{A}$ of subsets of $X$.

It is easy to check that the functor $P$ satisfies (BC), hence this monad can be extended to **Rel**, with

$$A(Pr)B \;\Leftrightarrow\; \forall x \in A\; \exists y \in B \;:\; xry \text{ and } \forall y \in B\; \exists x \in A \;:\; xry.$$

For $\mathbf{V} = \overline{\mathbb{R}}_+$, $d : X \times Y \to \overline{\mathbb{R}}_+$, $A \subseteq X$ and $B \subseteq Y$, the extension $Pd(A, B)$ is defined by

$$\inf\{v \in \overline{\mathbb{R}}_+ \,|\, \forall x \in A\; \exists y \in B \;:\; d(x, y) \leq v \text{ and } \forall y \in B\; \exists x \in A \;:\; d(x, y) \leq v\}.$$

In case $d$ is a premetric in $X$, $\widetilde{P}d$ is the *usual premetric* in $PX$.

6.3. **The lax powerset monad.** If we consider now $H : \mathbf{Rel} \to \mathbf{Rel}$ with $HX := PX$ the powerset of $X$ and

$$A(Hr)B \text{ if for each } b \in B \text{ there exists } a \in A \text{ such that } a\, r\, b,$$

it is easy to check that $1_{HX} \leq H1_X$ and $Hr \cdot Hs \leq H(r \cdot s)$, hence $H$ is a lax functor. We may equip $H$ with the structure of a lax monad, considering the (strict) natural transformations $\eta : \mathrm{Id}_{\mathbf{Rel}} \to H$ and $\mu : H^2 \to H$, defined by

$$x(\eta_X)A \text{ if } x \in A \quad \text{and} \quad \mathcal{A}(\mu_X)A \text{ if } \bigcup \mathcal{A} \subseteq A,$$

for $x \in X$, $A \subseteq X$ and $\mathcal{A} \subseteq HX$. It is easy to check that, for every set $X$, $A, A' \subseteq X$ and $\mathfrak{A} \subseteq HHX$,

$$A(\mu_X \eta_{HX})A' \Leftrightarrow A(\mu_X H\eta_X)A' \Leftrightarrow A(H1_X)A' \Leftrightarrow A \subseteq A', \text{ and}$$

$$\mathfrak{A}(\mu_X H\mu_X)A \Leftrightarrow \mathfrak{A}(\mu_X \mu_{HX})A \Leftrightarrow \bigcup\bigcup \mathfrak{A} \subseteq A.$$

Hence, $1_{HX} \leq \mu_X \eta_{HX} = \mu_X H\eta_X = H1_X$ and $\mu_X H\mu_X = \mu_X \mu_{HX}$, and then $(H, \eta, \mu)$ is a lax monad in **Rel**. (We remark that it is not a lax monad in the sense of Bunge [2], since $\mu_X H\eta_X \not\leq 1_{HX}$.)

It has an interesting lax extension to $\mathrm{Mat}(\overline{\mathbb{R}}_+)$: given $d : X \nrightarrow Y$ in $\mathrm{Mat}(\overline{\mathbb{R}}_+)$, for each $A \subseteq X$ and $B \subseteq Y$,

$$Hd(A, B) = \inf\{v \geq 0 \,|\, A(Hd_v)B\} = \inf\{v \geq 0 \,|\, \forall x \in A \;\exists y \in B \,:\, d(x, y) \leq v\}.$$

For a premetric $d : X \nrightarrow X$, $Hd$ assigns to each pair of subsets $A, B$ of $X$, its *Hausdorff (non-symmetric) premetric*

$$d_H(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y).$$

This identification holds in the general case of a $\mathbf{V}$-matrix $d : X \nrightarrow Y$, considering $d_H$ defined as above. Indeed, for $v \in \overline{\mathbb{R}}_+$,

$$\begin{aligned} \forall x \in A \;\exists y \in B \,:\, d(x, y) \leq v \;&\Rightarrow\; \forall x \in A \;\inf_{y \in B} d(x, y) \leq v \;\Rightarrow\; d_H(A, B) \leq v \\ &\Rightarrow\; d_H(A, B) \leq Hd(A, B). \end{aligned}$$

On the other hand,

$$\begin{aligned} d_H(A, B) < v \;&\Rightarrow\; \forall x \in A \;\inf_{y \in B} d(x, y) < v \;\Rightarrow\; \forall x \in A \;\exists y \in B \,:\, d(x, y) \leq v \\ &\Rightarrow\; Hd(A, B) \leq v. \end{aligned}$$

Hence, $d_H = Hd$ as claimed.

6.4. **The ultrafilter monad.** We consider now the ultrafilter monad $(U, \eta, \mu)$ in **Set**, with:

- the functor $U : \mathbf{Set} \to \mathbf{Set}$ such that $UX$ is the set of ultrafilters of $X$ for every set $X$, and $Uf(\mathfrak{x})$ the ultrafilter generated by $f(\mathfrak{x})$, for every map $f : X \to Y$ and every ultrafilter $\mathfrak{x}$ in $X$.
- $\eta_X : X \to UX$ assigns to each point $x$ the principal ultrafilter

$$\dot{x} = \{A \subseteq X \,|\, x \in A\};$$

- $\mu_X : U^2X \to UX$ is the *Kowalsky multiplication*, i.e.

$$\mu_X(\mathfrak{X}) = \bigcup_{\mathcal{X} \in \mathfrak{X}} \bigcap_{\mathfrak{x} \in \mathcal{X}} \mathfrak{x}.$$

The functor $U$ satisfies (BC), hence it has an extension in **Rel**, given by

$$\mathfrak{x}(Ur)\mathfrak{y} \iff r[\mathfrak{x}] \subseteq \mathfrak{y} \iff r^\circ[\mathfrak{y}] \subseteq \mathfrak{x},$$

for every relation $r : X \nrightarrow Y$, $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$. This can be equivalently described by

$$\mathfrak{x}(Ur)\mathfrak{y} \iff \forall A \in \mathfrak{x}\ \forall B \in \mathfrak{y}\ \exists x \in A\ \exists y \in B\ :\ xry.$$

Its lax extension $U$ to $\mathrm{Mat}(\mathbf{V})$ coincides with Clementino-Tholen lax extension [9] (which we will denote below by $U'$), as we show next.

For each $d : X \nrightarrow Y$ in $\mathrm{Mat}(\mathbf{V})$,

$$\begin{aligned} Ud(\mathfrak{x},\mathfrak{y}) &= \bigvee\{v \in V \mid \mathfrak{x}(Ud_v)\mathfrak{y}\} \\ &= \bigvee\{v \in V \mid \forall A \in \mathfrak{x}\ \forall B \in \mathfrak{y}\ \exists x \in A\ \exists y \in B\ :\ d(x,y) \geq v\}, \end{aligned}$$

while

$$U'd(\mathfrak{x},\mathfrak{y}) = \bigwedge_{A\in\mathfrak{x},B\in\mathfrak{y}}\ \bigvee_{x\in A,y\in B} d(x,y).$$

For each $v \in V$ such that $\mathfrak{x}(Ud_v)\mathfrak{y}$, $v \leq \bigvee\limits_{x\in A,y\in B} d(x,y)$, hence $v \leq U'd(\mathfrak{x},\mathfrak{y})$, and therefore $Ud(\mathfrak{x},\mathfrak{y}) \leq U'd(\mathfrak{x},\mathfrak{y})$.

If $w$ is $\sqsubset$-atomic and $w \sqsubset U'd(\mathfrak{x},\mathfrak{y})$, then $w \sqsubset \bigvee\limits_{x\in A,y\in B} d(x,y)$ for each $A \in \mathfrak{x}$ and $B \in \mathfrak{y}$. Hence there exists $x \in A$ and $y \in B$ such that $w \leq d(x,y)$, and therefore $w \leq Ud(\mathfrak{x},\mathfrak{y})$. Hence, $U'd(\mathfrak{x},\mathfrak{y}) \leq Ud(\mathfrak{x},\mathfrak{y})$ and the equality follows.

We point out that, although $U : \mathbf{Rel} \to \mathbf{Rel}$ is a (strict) functor, its extension $U : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ is not always op-lax. It is the case when $\mathbf{V} = ([-\infty,+\infty],\geq)$, with tensor product $\otimes = +$ (where $-\infty + (+\infty) = +\infty$), as we show next.

Consider $X = \{n \mid n \in \mathbb{N}$, non-zero and even$\}$, $Z = \{-m \mid m \in \mathbb{N}$, non-zero and odd$\}$ and $Y = X \cup Z$. For

$$\begin{aligned} d_1 : X \times Y &\to [-\infty,+\infty] \\ (x,y) &\mapsto xy \end{aligned} \quad\text{and}\quad \begin{aligned} d_2 : Y \times Z &\to [-\infty,+\infty], \\ (y,z) &\mapsto yz \end{aligned}$$

and free ultrafilters $\mathfrak{x} \in UX$ and $\mathfrak{z} \in UZ$, we have

$$\begin{aligned} U(d_2\cdot d_1)(\mathfrak{x},\mathfrak{z}) &= \inf\{v \in V \mid \forall A \in \mathfrak{x}\ \forall C \in \mathfrak{z}\ \exists x \in A\ \exists z \in C : (d_2\cdot d_1)(x,z) \geq v\} \\ &= -\infty, \end{aligned}$$

since $(d_2\cdot d_1)(x,z) = \inf\limits_{y\in Y} y(x+z) = -\infty$. To calculate $(Ud_2\cdot Ud_1)(\mathfrak{x},\mathfrak{z})$, let $\mathfrak{y} \in UY$. If $X \in \mathfrak{y}$, then $Ud_1(\mathfrak{x},\mathfrak{y}) = +\infty$ since every $A \in \mathfrak{x}$ is unlimited and every $B \in \mathfrak{y}$ has a positive element. If $X \notin \mathfrak{y}$, then $Z \in \mathfrak{y}$; hence $Ud_2(\mathfrak{y},\mathfrak{z}) = +\infty$ since every $C \in \mathfrak{z}$ is unlimited and every $B \in \mathfrak{y}$ contains a negative element. Now

$$(Ud_2\cdot Ud_1)(\mathfrak{x},\mathfrak{z}) = \inf_{\mathfrak{y}\in UY} Ud_1(\mathfrak{x},\mathfrak{y}) + Ud_2(\mathfrak{y},\mathfrak{z}) = +\infty,$$

and therefore $U(d_2 \cdot d_1) \nleq Ud_2 \cdot Ud_1$; that is $U$ is not op-lax.

6.5. **The filter monad.** The filter monad $(F, \eta, \mu)$ in **Set**, with $FX$ the set of filters of $X$, $Ff(\mathfrak{x}) = \{B \subseteq Y \mid f^{-1}(B) \in \mathfrak{x}\}$ for every $f : X \to Y$ and $\mathfrak{x} \in FX$, and $\eta$ and $\mu$ defined as in the example above, satisfies (BC). Hence $F$ can be extended into an endofunctor of **Rel**, that may be described by

$$\mathfrak{x}(Fr)\mathfrak{y} \;\Leftrightarrow\; r[\mathfrak{x}] \subseteq \mathfrak{y} \text{ and } r^\circ[\mathfrak{y}] \subseteq \mathfrak{x},$$

for every relation $r : X \nrightarrow Y$, $\mathfrak{x} \in FX$ and $\mathfrak{y} \in FY$. We observe that, contrarily to the case of the ultrafilter monad, in this situation we have to impose both conditions, $r[\mathfrak{x}] \subseteq \mathfrak{y}$ and $r^\circ[\mathfrak{y}] \subseteq \mathfrak{x}$, since each of them does not follow from the other. This was the reason why Pisani in [15] had to restrict the codomain in order to get a functor extension with the "non-symmetric" definition. Indeed, if we define $G : \mathbf{Rel} \to \mathbf{Rel}$, $\varepsilon : \mathrm{Id}_{\mathbf{Rel}} \to G$ and $\nu : GG \to G$ by $GX = FX$,

$$\begin{array}{rcl} \mathfrak{x}(Gr)\mathfrak{y} & \Leftrightarrow & r^\circ[\mathfrak{y}] \subseteq \mathfrak{x}, \\ x\,\varepsilon_X\,\mathfrak{x} & \Leftrightarrow & \forall A \in \mathfrak{x}\;\; x \in A, \text{ and} \\ \mathfrak{X}\,\nu_X\,\mathfrak{x} & \Leftrightarrow & \mathfrak{x} \subseteq \nu_X(\mathfrak{X}), \end{array}$$

we obtain a lax monad $(G, \varepsilon, \nu)$ in **Rel**.

6.6. **The double powerset monad.** The double powerset functor $P^\circ P^\circ$ : **Set** $\to$ **Set** is obtained composing the contravariant powerset functor $P^\circ$ with itself; that is, $P^\circ P^\circ X = PPX$ and $P^\circ P^\circ f(\mathcal{A}) = \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{A}\}$, for every $f : X \to Y$ and $\mathcal{A} \subseteq PX$. This functor does not satisfy (BC), although it is part of a **Set**-monad $(P^\circ P^\circ, \eta, \mu)$ [11]. Indeed, it is easy to check that the $P^\circ P^\circ$-image of the following pullback

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \{0,1\} \arrow[d, "g"] \\ \{0,1\} \arrow[r, "f"] & \{0,1\}, \end{tikzcd}$$

where $f(0) = f(1) = 0$ and $g(0) = g(1) = 1$, does not satisfy (BC):

$$P^\circ P^\circ f(\{\emptyset, \{0,1\}\}) = P^\circ P^\circ g(\{\emptyset, \{0,1\}\}) = P(\{0,1\}),$$

although there is no element on $P^\circ P^\circ \emptyset$ mapped into $\{\emptyset, \{0,1\}\}$ by the pullback projections.

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