Skew-shapes with interval support in the dominance lattice

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Outline

Introduction

I. Classification of a direct sum of a class ribbon of Schur functions with interval support

II. Bad configurations. Full configurations. Classification of multiplicity free skew Schur functions with interval support

Skew Schur functions

- Schur functions are considered to be the most important basis for the ring of symmetric functions.

Skew Schur functions

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- Let $x = (x_1, x_2, \ldots)$. Given partitions $\mu \subseteq \lambda$, $A := \lambda/\mu$.

  The skew-Schur function s_A is the generating function for SSYT T of shape A

  $$s_A(x) = \sum_T x^T,$$

  where the sum is over all SSYT T of shape A. Thus it is a symmetric function.
- $$s_A = \sum_\nu c_A^\nu s_\nu,$$

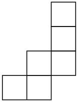
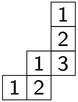
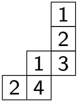
  where $c_A^\nu := c_{\mu,\lambda}^\nu \geq 0$ is the number of SSYT of shape A and content ν, satisfying the Littlewood-Richardson rule.

Skew Schur functions

- $c^{\nu}_{A} = c^{\nu'}_{A'}$

$$s_A = \sum_{c(A) \preceq \nu' \preceq r(A)'} c^{\nu}_{\mu,\lambda} s_{\nu} = s_{r(A)} + \cdots + c^{\nu}_{A} s_{\nu} + \cdots + s_{c(A)'} = s_{A^{\pi}}$$

$$s_{A'} = \sum_{c(A) \preceq \nu' \preceq r(A)'} c^{\nu}_{\mu,\lambda} s_{\nu'} = s_{c(A)} + \cdots + c^{\nu}_{A} s_{\nu'} + \cdots + s_{r(A)'}$$

$A =$   $c(A) = 321,$  $r(A)' = 42$

$$[321, 42] = \{321, 33, 411, 42\}.$$

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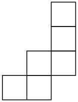
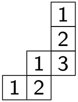
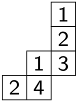
$A =$  

$c(A) = 321,$ 

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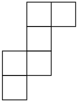
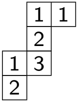
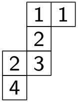
$A =$  



$$[321, 42] = \{321, 33, 411, 42\}.$$

Skew Schur function support

- The support of a skew shape A, $\mathrm{supp}A$, considered as a subposet of the *dominance lattice*, has a top element and a bottom element uniquely defined by the shape A,

$$r(A), c(A)' \in \mathrm{supp}A = \{\nu : c_A^\nu > 0\} \subseteq [r(A), c(A)']$$

$$c(A), r(A)' \in \mathrm{supp}A' = \{\nu' : c_A^\nu > 0\} \subseteq [c(A), r(A)']$$

$$\mathrm{supp}A = \mathrm{supp}A^\pi$$

$$c_A^{c(A)} = c_A^{r(A)'} = 1$$

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- The support of s_A is the support of A.

Problems

Given the skew shape A and $\nu' \in [c(A), r(A)']$

1. How does the shape of A govern the positivity of c^{ν}_{A}?
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Problems

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   How does the shape of A govern the support of A?

2. Under what conditions do we have $c_A^\nu > 0$ whenever $\nu' \in [c(A), r(A)']$?

   Which skew shapes have interval support?

   A., *The admissible interval for the invariant factors of a product of matrices*, Linear and Multilinear Algebra (1999).

If A is a skew shape with two or more components and A has interval support, then the components of A are ribbon shapes.

3 Which are the ribbon shapes with interval support?

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Which are the disconnected shapes with interval support?

Ribbon shapes and direct sums of ribbon shapes

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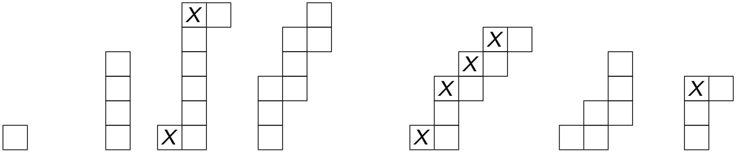
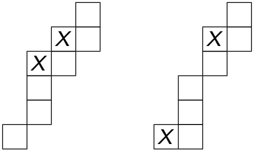
- **Answer.** *Ribbons whose column (row) lengths are at least two except possibly the top and bottom columns (rows).*



Direct sums of similar ribbons



Example. *(Direct sum) Ribbons such that all columns and row lenghts differ by at most one.*

Dominance order on partitions

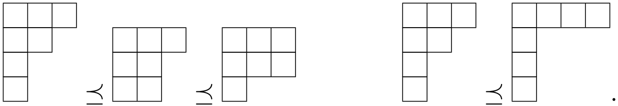
- The *dominance order* $\preceq$ on partitions of N, $\lambda = (\lambda_1, \ldots, \lambda_l)$, $\mu = (\mu_1, \ldots, \mu_s)$ is defined by setting $\lambda \preceq \mu$ if

$$\lambda_1 + \cdots + \lambda_i \leq \mu_1 + \cdots + \mu_i,$$

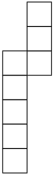
  for $i = 1, \ldots, l$, where we set $\mu_i = 0$ if $i > l$.

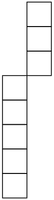
  The set of partitions of size N equipped with the dominance order is a lattice with maximum element (N) and minimum element (1^N).

- $\lambda \preceq \mu$ if and only if the Young diagram of μ is obtained by *"lifting"* at least one box in the Young diagram of λ.



Skew Schur functions and support

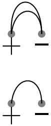
$A =$  $s_{A'} = s_{53} + s_{62} + s_{71}$

$B =$  $s_{B'} = s_{53} + s_{62} + s_{71} + s_{8}$

$r(A)' = 71$

62

$c(A) = 53$



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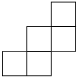
- $A =$  $s_{A'} = s_{221} + s_{311} + s_{32}$

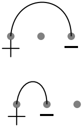
$r(A)' = 32$

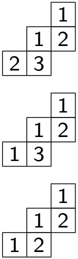
|

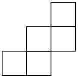
311

|

$c(A) = 221$





- $A =$  $\qquad s_{A'} = s_{221} + s_{311} + s_{32}$

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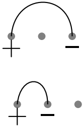
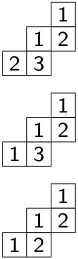
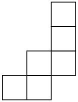
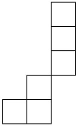




$0 \leq 1 - 1, \quad 1 \leq 2 + 1 - 2$

Skew Schur functions and support

- $A =$  $B =$ 

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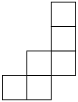
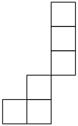
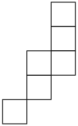
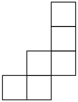
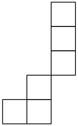
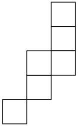
- $A =$  $B =$  $C =$ 

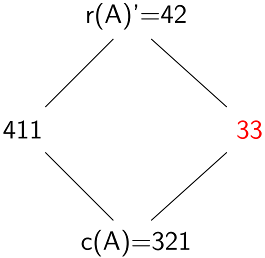
$$s_{A'} = s_{321} + s_{411} + \color{red}{0}\color{black}s_{33} + s_{42}$$
$$s_{B'} = s_{321} + s_{411} + \color{red}{0}\color{black}s_{33} + s_{42} + s_{51}$$

Skew Schur functions and support

- $A =$  $B =$  $C =$ 

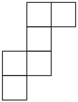
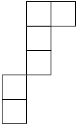
$$s_{A'} = s_{321} + s_{411} + 0s_{33} + s_{42}$$

$$s_{B'} = s_{321} + s_{411} + 0s_{33} + s_{42} + s_{51} \qquad s_{C'} = s_{321} + s_{411} + s_{33} + 2s_{42} + s_{51}$$

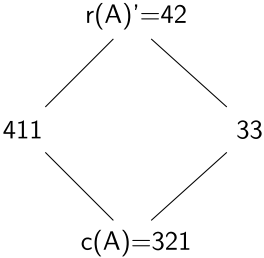


$$1 \not< 1 - 1, \qquad 1 \leq 2 + 1 - 2$$

Skew Schur functions and support

- $A =$  $B =$ 

  $s_{A'} = s_{321} + s_{411} + s_{33} + s_{42}$ $s_{B'} = s_{321} + s_{411} + s_{33} + s_{42} + s_{51}$



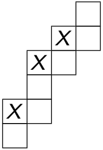
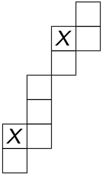
Ribbon shapes and direct sums of ribbon shapes

Definition

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$ be a composition with $\alpha_i \geq 2$, $i \neq 1, s$.
R_α denotes a skew-shape consisting of s column strips (1^{α_i}), $i = 1, \ldots, s$, right to left, where any two of them overlap at most in one row.
Let $0 \leq p < s$ be the number rows of size two. When $p = s - 1$, R_α is a ribbon and one writes $R_\alpha =< \alpha >$. Otherwise, it is a direct sum of $s - p$ ribbons.

$\mathrm{supp}\, R'_\alpha \subseteq [\alpha^+; (|\alpha| - p, p)]$, $\alpha^+ = (\alpha_1^+, \ldots, \alpha_s^+)$ the decreasing rearrangement of α.

$R_{(2,2,3,2)}$

$< 2232 >=$  $p = 3$, $< 22 > \oplus < 32 >=$  $p = 2$

$[\alpha^+ = 32^3; 63]$ $[\alpha^+ = 3\,2^3; 72]$

Ribbon shapes and direct sums of ribbon shapes

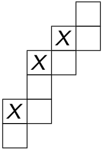
Definition

Given the composition $\alpha = (\alpha_1, \ldots, \alpha_s)$, $\alpha_i \geq 2$, $i \neq 1, s$, and a skew shape R_α, let

$$R_\alpha^1 := R_\alpha, \text{ and } R_\alpha^{i+1} := R_\alpha^i \setminus < \alpha_i^+ >, \; i = 1, \ldots, s-2,$$

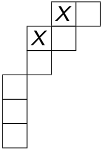
giving priority to the rightmost column strip $< \alpha_i^+ >$ of R_α, in case of equal size.

The *overlapping sequence* of R_α is the non increasing sequence of nonnegative integers $p_1 = p, p_2, \ldots, p_{s-1}$, where p_i is the number of rows with size two of R_α^i, $1 \leq i \leq s-1$. Note that $0 \leq p_{i+1} \leq p_i \leq s - i$, for $i = 1, \ldots, s-2$.

$< 2232 >=$ 

$p_1 = 3, p_2 = 1, p_3 = 0$

$\mathrm{supp} R' \subseteq [32^3; 63]$

$< 1, 2, 2 > \oplus < 3 >=$ 

$p_1 = 2 = p_2, p_3 = 0$

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Support criterion for a direct sum of a class of ribbons

Theorem

Given the composition $\alpha = (\alpha_1, \ldots, \alpha_s)$ with $\alpha_i > 1$, $i \neq 1, s$, consider R_α with overlapping sequence $(p_1, \ldots, p_{s-1})$, and $\nu' \in [\alpha^+; (|\alpha| - p, p)]$.

$$c^{\nu}_{R_\alpha} > 0 \quad \text{if and only if} \quad \nu'_i \leq \sum_{j=i}^{s} \alpha^+_j - p_i, \quad i = 1, \ldots, s-1.$$

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Equivalently, $c^{\nu}_{R_\alpha} > 0$ if and only if, for all $i \in \{1, \ldots, s-1\}$,

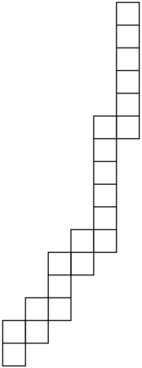
$$0 \leq \epsilon_i \leq \sum_{j=i+1}^{s} \alpha^+_j - p_i,$$

where ϵ_i is the number of lifted boxes from the last $s - i$ rows of α^+ to the ith row α^+_i.

- $R = < 662322 >$ $\quad (6^2\,3\,2^3) \preceq (7761) \preceq (777) \preceq (876) \preceq (21, 21-5)$

$$\epsilon_3 = 3 = \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - 3 = 2+2+2-3 \quad p_3 = 3$$
$$\epsilon_2 = 1 < \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - 3 = 3+2+2+2-3 \quad p_2 = 3$$
$$\epsilon_1 = 2 < \alpha_2^+ + \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - 5 = 6+3+2+2+2-5, p_1 = 5$$
$$\Rightarrow (876)' \in \mathrm{supp} R$$

$$4 > \alpha_4^+ + \alpha_5^+ + \alpha_6^+ = 2+2+2-3 \Rightarrow (777)' \notin \mathrm{supp} R$$



- $R = <662322> \quad (6^2\,3\,2^3) \preceq (7761) \preceq (777) \preceq (876) \preceq (21, 21-5)$

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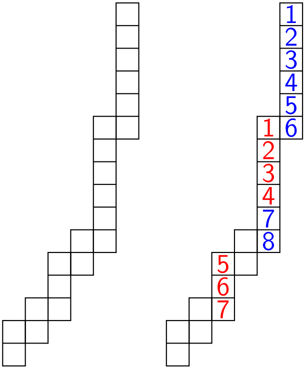
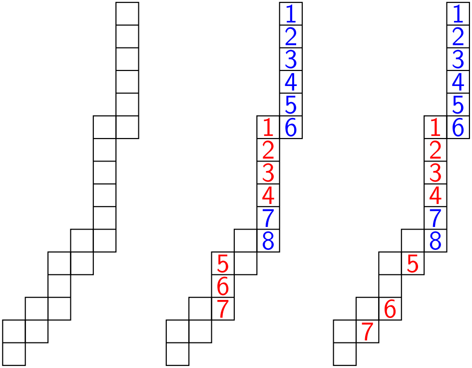


- $R =< 662322 >$ $(6^2\,3\,2^3) \preceq (7761) \preceq (777) \preceq (876) \preceq (21, 21-5)$

$$\epsilon_3 = 3 = \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - 3 = 2+2+2-3 \quad p_3 = 3$$
$$\epsilon_2 = 1 < \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - 3 = 3+2+2+2-3 \quad p_2 = 3$$
$$\epsilon_1 = 2 < \alpha_2^+ + \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - 5 = 6+3+2+2+2-5, p_1 = 5$$
$$\Rightarrow (876)' \in \mathrm{supp} R$$

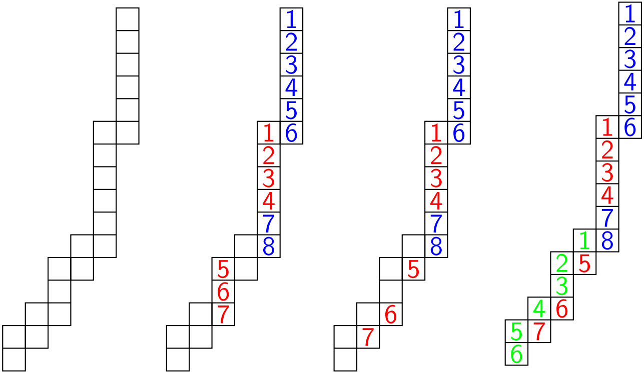
$$4 > \alpha_4^+ + \alpha_5^+ + \alpha_6^+ = 2+2+2-3 \Rightarrow (777)' \notin \mathrm{supp} R$$



- $R = <662322> \quad (6^2\,3\,2^3) \preceq (7761) \preceq (777) \preceq (876) \preceq (21, 21-5)$

  $\epsilon_3 = 3 = \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - 3 = 2+2+2-3 \quad p_3 = 3$
  $\epsilon_2 = 1 < \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - 3 = 3+2+2+2-3 \quad p_2 = 3$
  $\epsilon_1 = 2 < \alpha_2^+ + \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - 5 = 6+3+2+2+2-5, p_1 = 5$
  $\Rightarrow (876)' \in \mathrm{supp} R$

  $4 > \alpha_4^+ + \alpha_5^+ + \alpha_6^+ = 2+2+2-3 \Rightarrow (777)' \notin \mathrm{supp} R$



- $R =< 662322 >$ $(6^2\,3\,2^3) \preceq (7761) \preceq (777) \preceq (876) \preceq (21, 21-5)$

$$\epsilon_3 = 3 = \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - 3 = 2+2+2-3 \quad p_3 = 3$$
$$\epsilon_2 = 1 < \alpha_3^+ + \alpha_4^+ + \alpha_5^+ + \alpha_6^+ - 3 = 3+2+2+2-3 \quad p_2 = 3$$
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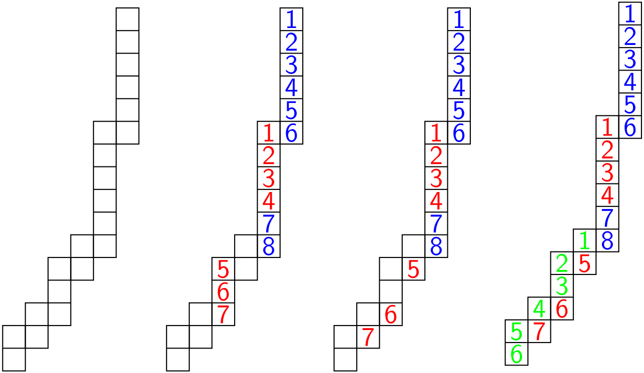

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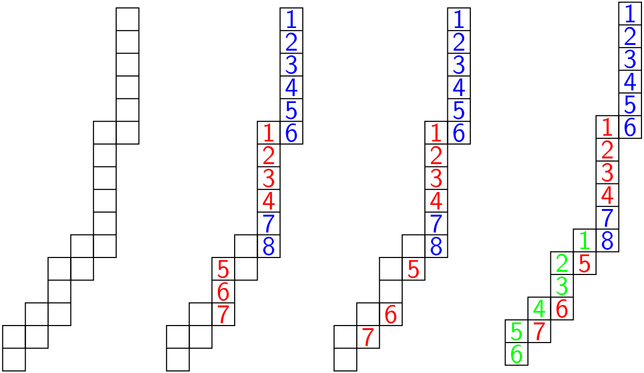


Classification of a direct sum of a class of ribbon Schur functions with interval support

Theorem

Given the composition $\alpha = (\alpha_1, \ldots, \alpha_s)$ with $\alpha_i > 1$, $i \neq 1, s$, consider R_α with overlapping sequence $(p_1, \ldots, p_{s-1})$.
$\mathrm{supp} R'_\alpha \subsetneq [\alpha^+; (|\alpha| - p, p)]$ if and only if for some $1 \leq i \leq s-2$ with $p_{i+1} \geq 1$, there exist integers $g_1, \ldots, g_i \geq 0$ with $\sum_{j=1}^{i} g_j \leq p_{i+1} - 1$, such that

$$\alpha_j^+ + g_j \geq \sum_{q=i+1}^{s} \alpha_q^+ - p_{i+1} + 1, \quad j = 1, \ldots, i.$$

In this case,
$(\alpha_1^+ + g_1, \ldots, \alpha_i^+ + g_i, \sum_{q=i+1}^{s} \alpha_q^+ - p_{i+1} + 1, p_{i+1} - \sum_{j=1}^{i} g_j - 1)^+$ is not in the $\mathrm{supp} R'_\alpha$.

Theorem

Given the composition $\alpha = (\alpha_1, \ldots, \alpha_s)$ with $\alpha_i > 1$, $i \neq 1, s$, consider R_α with overlapping sequence $(p_1, \ldots, p_{s-1})$.
$c^{\nu}_{R_\alpha} > 0$ whenever $\nu' \in [\alpha^+; (|\alpha| - p, p)]$ if and only if for all $1 \leq i \leq s-2$ with $p_{i+1} \geq 1$, and for all integers $g_1, \ldots, g_i \geq 0$ with $\sum_{j=1}^{i} g_j \leq p_{i+1} - 1$, one has always, for some $f \in \{1, \ldots, i\}$,

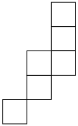
$$\alpha_f^+ + g_f \leq \sum_{q=i+1}^{s} \alpha_q^+ - p_{i+1}.$$

Corollary

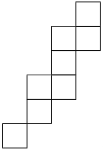
- If $p_1 = 0$ or $p_2 = 0$, $\mathrm{supp} R_\alpha = [\alpha^+; (|\alpha| - p, p)]$.

$$< \alpha_1 > \oplus \cdots \oplus < \alpha_s >$$

$$< {\alpha_1}^+, \alpha_2^+ > \oplus \cdots \oplus < \alpha_s >$$



$$< {\alpha_1}^+, \alpha_2^+, \alpha_3^+ > \oplus \cdots \oplus < \alpha_s >$$



.

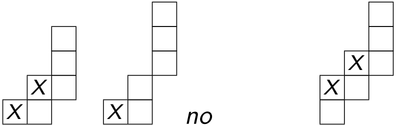
- If $p_2 = 1, p_3 = 0$, R_α has interval support except when

$$\alpha_1^+ \geq \sum_{q=2}^{s} \alpha_q^+$$

- If $p_2 = 1 = p_3$, $p_4 = 0$, R_α has interval support except when

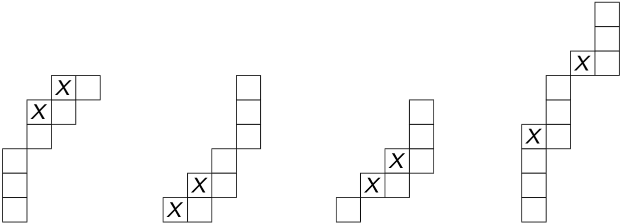
$$\alpha_1^+ \geq \sum_{q=2}^{s} \alpha_q^+ \quad \textit{or} \quad \alpha_1^+, \alpha_2^+ \geq \sum_{q=3}^{s} \alpha_q^+$$

$R_{(\alpha_1,\alpha_2,\alpha_3)}$ *has interval support* *except* *when* $\alpha_1 \geq \alpha_2 + \alpha_3$ *or* $\alpha_3 \geq \alpha_1 + \alpha_2$.



no *yes*

$s = 4$

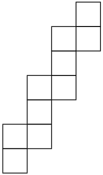
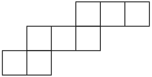


$p_1 = p_2 = 2$	$p_1 = p_2 = 2$	$p_1 = 2, p_2 = 1$	$p_1 = 2, p_2 = 1$
$p_3 = 0$	$p_3 = 1$	$p_3 = 0$	$p_3 = 0$
no	*no*	*yes*	*yes*

Corollary

McNamara, van Willigenburg, 2011

Ribbon shapes whose column and row lengths differ at most one have full support.

$R = \langle 2332 \rangle =$  $R' =$ 

More examples

- $\alpha = (6,2,2,2,2,7,6), \quad \alpha^+ = (7,6,6,2,2,2,2), \ i = 3, \quad p_4 = 3$

$$\begin{array}{l} 7,6 \geq 2+2+2+2-2 \\ g_1 + g_2 + g_3 = 0 \end{array} .$$

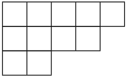
$\nu = (7,6,6,6,2) \notin \mathrm{supp} R'_\alpha$

$\nu = (7, 6+1, 6+1, 2+2+2+2-2) = (7,7,7,6) \notin \mathrm{supp} R'_\alpha,$
$g_1 = 0, g_2 = g_3 = 1$
$\nu = (6+2, 7, 6, 2+2+2+2-2) = (8,7,6,6) \notin \mathrm{supp}(R'_\alpha),$
$g_1 = 0 = g_2, g_3 = 2$
$\nu = (7+2, 6, 6, 2+2+2+2-2) = (9,6,6,6) \notin \mathrm{supp}(R'_\alpha),$
$g_1 + g_2 + g_3 = 2.$

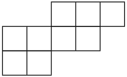
Note that $p_4 = 3 \Rightarrow 3 \leq p_2, p_3 \leq 4.$
If $p_3 = 4$, $7+3, 6+3 \not\geq 6+2+2+2+2-3$

II. Bad configurations. Full configurations.
Classification of multiplicity free skew Schur functions with interval support.

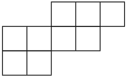
Definitions

Example: $\lambda = (5, 4, 2) =$ 

Definitions

Example: $\lambda/\mu = (5,4,2)/(2) =$ 

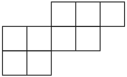
Definitions

Example: $\lambda/\mu = (5,4,2)/(2) =$ 

$$s_{in} = (2,2,1,3), \quad m^n - \text{shortness of } \mu \text{ is } 1$$

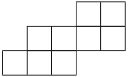
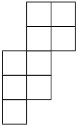
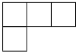
$$s_{out} = (2,1,2,1,1,1), \quad m^n - \text{shortness of } \lambda^* \text{ is } 1$$

Definitions

Example: $\lambda/\mu = (5,4,2)/(2) =$ 

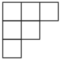
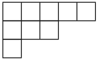
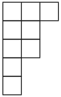
$$s_{in} = (2,2,1,3), \quad m^n - \text{shortness of } \mu \text{ is } 1$$

$$s_{out} = (2,1,2,1,1,1), \quad m^n - \text{shortness of } \lambda^* \text{ is } 1$$

$(\lambda/\mu)^\pi =$ , $(\lambda/\mu)' =$ , $(\lambda^*)^\pi =$ 

The *sum* $\lambda + \mu$ of two partitions λ and μ, is the partition whose parts are equal to $\lambda_i + \mu_i$, with $i = 1, \ldots, \max\{\ell(\lambda), \ell(\mu)\}$. Using conjugation, we define the *union* $\lambda \cup \mu := (\lambda' + \mu')'$. Equivalently, $\lambda \cup \mu$ is obtained by taking all parts of λ jointly with those of μ and rearranging all these parts in descending order.

Example

Let $\lambda =$  and $\mu =$ . Then, $\lambda + \mu =$ , $\lambda \cup \mu =$  .

Fix a positive integer n, and let λ and μ be two partitions with length $\leq n$. The *product* $\lambda^\pi \bullet_n \mu$ of two partitions λ and μ is defined as

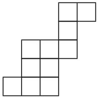
$$(\lambda_1 + \mu_1, \ldots, \lambda_1 + \mu_n)/(\lambda^*)^\pi,$$

where $\lambda^* = \lambda_1^n/\lambda$.

Example

If $\lambda = (3, 2^2, 0^2)$ and $\mu = (2, 1^2, 0^2)$, we obtain $\lambda^\pi \bullet_5 \mu = (5, 4^3, 3^2)/(3^2, 1^2)$,

$$\lambda^\pi \bullet \mu =$$



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Problems

If A is a skew shape with two or more components and A has interval support, then the components of A are ribbon shapes.

Problems

If A is a skew shape with two or more components and A has interval support, then the components of A are ribbon shapes.

When do the Schur function products have interval support?

Under what conditions do we have $c_A^{\nu} = 1$ whenever $\nu' \in [c(A), r(A)']$? (Equivalently, when is it the case that s_A can be expressed as $\sum_{\nu'} s_{\nu}$ where the sum runs over the interval $[c(A), r(A)']$ in dominance order?)

When are the Schur function products multiplicity free and with interval support?

Is there any representation-theoretic or geometric explanation?

Bad configurations

Lemma

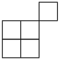
Let $\xi \in [c(A), r(A)' = (n_1, \ldots, n_s)]$ where $\xi = (n_1) \cup \xi^1$. Then

$$\xi^1 \notin \mathrm{supp}(A \setminus V_1)' \Rightarrow \xi \notin \mathrm{supp} A'.$$

Proposition

Let A be a skew diagram with two or more connected components. If there is a component containing a two by two block of boxes, then A has no interval support.

Example

The support of $A =$  is not an interval.
$[c(A), r(A)'] = \{c(A) = 221, \xi = 311, r(A)' = 32\}$ with
$\xi \notin \mathrm{supp} A' = \{c(A), r(A)'\}$.

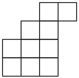
Corollary

If A is a skew diagram with two or more components and the support of A is an interval, then the connected components of A are ribbon shapes.

Corollary

Let A be a skew diagram such that $\ell(c(A)) > \ell(r(A)') = s$ (equivalently, it has no block of maximal width), and the vertical strip V_s is a column of A of length greater than, or equal to 2. Then, the support of A is not an interval.

Example

$A =$ 

We have $c(A) = (4, 3, 2, 1) \preceq \xi = (4, 4, 1, 1) \preceq r(A)' = (4, 4, 2)$, but $\xi \notin \mathrm{supp} A'$.

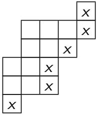
Proposition

Let A be a connected skew diagram such that

$$\sigma^1 = (n_1, \overline{w}_2, \ldots, \overline{w}_l, w_{l+1}, \ldots, w_r) \in [c(A) = (w_1, \ldots, w_r); r(A)' = (n_1, \ldots, n_s)]$$

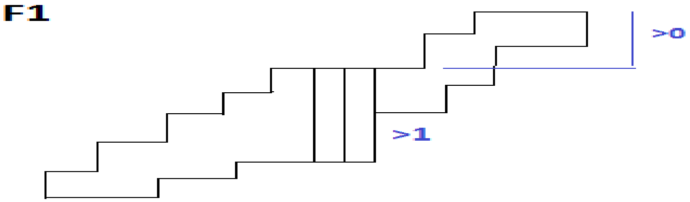
for some $3 \leq l \leq r$ such that $\overline{w}_k \leq w_k$ for $k = 1, \ldots, l$ and $\overline{w}_l < w_l$. Moreover, assume the existence of two integers $2 \leq i < j \leq l$ such that $\overline{w}_i \geq \overline{w}_j + 2$ and $w_j > \overline{w}_j$. Then the support of A is not an interval.

Example

$A =$ 

$$c(A) = (4, 4, 3, 2, 2)$$
$$r(A)' = (6, 4, 4, 1)$$
$$\sigma^1 = (6, \underline{4}, 2, 2, \underline{1})$$

$\mathrm{supp} A' \subsetneq [c(A), r(A)']$, $\xi = (5, 3, 3, 2, 2) \notin \mathrm{supp} A'$ but $c(A) \preceq \xi \preceq r(A)'$



Theorem

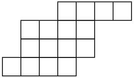
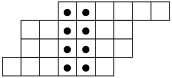
If, up to a π-rotation and/or conjugation, A is an $F1$ configuration then $\mathrm{supp}(A') \subsetneq [c(A), r(A)']$.

Full configurations

Lemma

Let v, w be partitions of length ℓ.
(*a*) If $A = v^{\pi} \bullet w$ and $B = [v + (x^n)]^{\pi} \bullet w$, then $b \in \mathrm{supp}A$ if and only if $b \cup (n^x) \in \mathrm{supp}B$.
(*b*) (A. 99) If $A = (v^{\pi} \bullet w)'$ and $B = ([v + (x^n)]^{\pi} \bullet w)'$, then $b \in \mathrm{supp}A$ if and only if $b + (x^n) \in \mathrm{supp}B$.

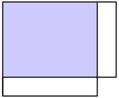
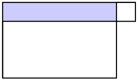
$v = (3, 2, 2, 0), w = (4, 2, 2, 1)$ with $\ell = 4$

$v^{\pi} \bullet w =$  $, [v + (2^4)]^{\pi} \bullet w =$ 

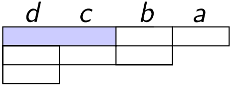
Proposition

- (A. 99; C. Bessenrodt and A. Kleshchev 99) $c(\lambda/\mu) = r(\lambda/\mu)'$ if and only if $\lambda/\mu = \nu$ or $\lambda/\mu = \nu^{\pi}$.
- (van Willigenburg 04) $s_{\lambda/\mu} = s_{\nu}$ if and only if $\lambda/\mu = \nu$ or $\lambda/\mu = \nu^{\pi}$.
- (A. 99) Let A be a skew diagram with $c(A) \precneqq r(A)'$. Then, $\mathrm{supp} A' = \{c(A), r(A)'\}$ if and only if, up to a π–rotation/ or conjugation and up to a block of maximal width or maximal depth, A either is an $F1$ or an $\widetilde{F}1$ configuration.

$F1 = ((a+1)^x, a)/(a^x)$, and $\tilde{F}1 = (a+1, a^x)/(a)$, a, $x \geq 1$:

$F1$  $\widetilde{F}1$  . (1)

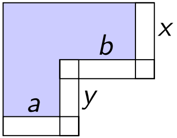
Consider skew diagram λ/μ with three rows, where $\mu = (d + c)$ is one row rectangle and $\lambda^* = (a + b + c, a)$ is a fat hook, or vice versa, for some integers $a, d \geq 1$ and $b, c \geq 0$:

$F2$ 

Proposition

Let λ/μ be the skew diagram $F2$. Then, the support of λ/μ is an interval if and only if it is $a \leq c + 1$ and $d \leq b + 1$. In this case we say that λ/μ is an $A2$ configuration.

Consider the ribbon skew diagram λ/μ, with $\mu = ((a+b+1)^x, a^y)$ and $\lambda^* = ((b+1)^{y+1})$, for some integers $a, b, x, y \geq 1$, as illustrated by:

$F3$ 

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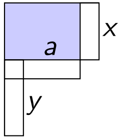
Proposition

Let λ/μ be a $F3$ configuration. Then its support is an interval if and only if $a = x = 1$, or $a = 1$ and $x \leq y+1$, or $a \leq b+1$ and $x = 1$, in which case it is called an $A3$ configuration.

Consider the skew diagram λ/μ of type $F4$ defined by the partitions $\lambda = ((a+2)^x, a+1, 1^y)$ and $\mu = ((a+1)^x)$ for some a, x, $y \geq 1$ such that not both x and y are equal to 1:

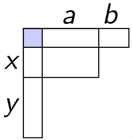


$F4$

Proposition

If λ/μ is the skew diagram $F4$, then its support is an interval if and only if $a = 1$ and $x \leq y + 1$, or $a \geq 2$ and $x = 1$, in which case it is called an $A4$ configuration.

The next family of skew diagrams λ/μ is designated by $F6$ and it is defined by partitions $\lambda = (a+b+1, (a+1)^x, 1^y)$ and $\mu = (1)$, for some integers $a, x > 0$ and $b, y \geq 1$:

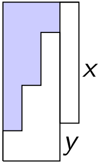
$F6$ 

.

Proposition

The support of the skew diagram $F6$ is an interval if and only if $b = y = 1$. In this case it is called an $A6$ configuration.

Finally, consider the skew diagram A, with $f + 1 \geq 4$ columns and $x + y$ rows, $x, y \geq 1$, having the form

F7 

,

where

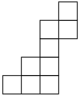
- the first f columns end in the same row and have pairwise distinct lengths,
- x is the length of the $(f + 1)$th column,
- the fth column starts at least one box below the starting row of the $(f + 1)$th column, and
- the first column has length $\leq y$.

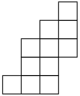
Proposition

Let A be an $F7$ configuration. Denote by $(w_1, \ldots, w_f)$ the partition formed by the first f columns of A, and let $k \geq 0$ be the number of rows shared by the fth and $(f+1)$th columns of A. The support of A is an interval if and only if $f = 3$, $w_1 = 1$, $x = w_2$ and $k = w_2 - 1$. In this case the diagram is called an $A7$ configuration.

Example

Some $A7$ configurations are shown below:

(i)  ,

(ii)  .

Theorem (Gutschwager '06, Thomas and Yong '05, King et al '09)

The basic skew Schur function $s_{\lambda/\mu}$ is multiplicity-free if and only if one or more of the following is true:

- *R0* μ or λ^* is the zero partition 0;
- *R1* μ or λ^* is a rectangle of m^n-shortness 1;
- *R2* μ is a rectangle of m^n-shortness 2 and λ^* is a fat hook;
- *R3* μ is a rectangle and λ^* is a fat hook of m^n-shortness 1;
- *R4* μ and λ^* are rectangles.

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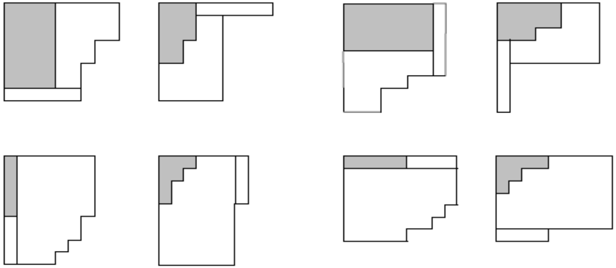
- *R3* μ is a rectangle and λ^* is a fat hook of m^n-shortness 1;
- *R4* μ and λ^* are rectangles.

Corollary (Stembridge '00)

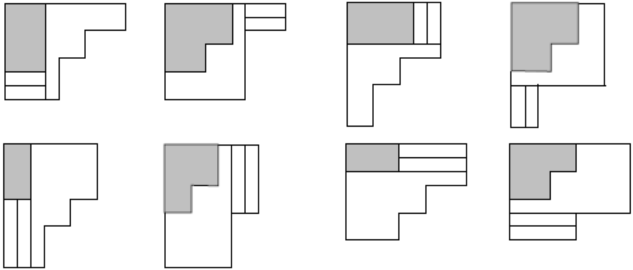
The product $s_\mu s_\nu$ is multiplicity-free if and only if

- (i) μ or ν is a one-line rectangle, or
- (ii) μ is a two line rectangle and ν is a fat hook, or
- (iii) μ is a rectangle an ν is a near rectangle, or
- (iv) μ and ν are rectangles.

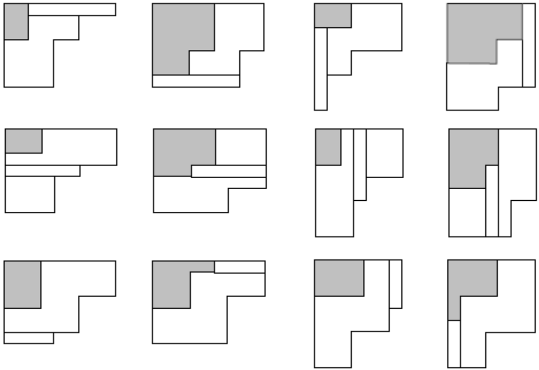
R1



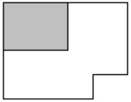
R2



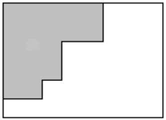
R3



R4



R0



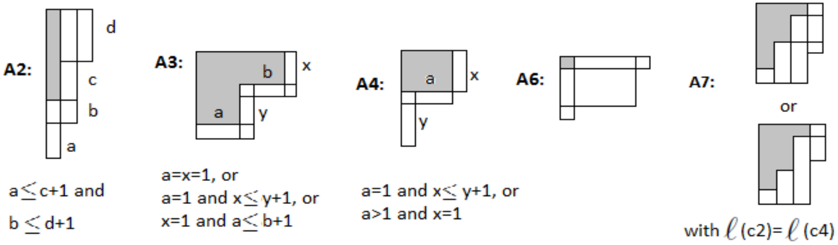
Theorem

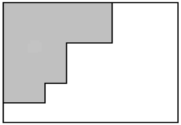
The basic skew Schur function $s_{\lambda/\mu}$ is multiplicity-free and has interval support if and only if, up to a block of maximal width or maximal length, and up to a π-rotation and/or conjugation, one or more of the following is true:

(*i*) μ or λ^* is the zero partition 0.

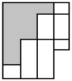
(*ii*) λ/μ is a two column or a two row diagram.

(*iii*) λ/μ is an $A2, A3, A4, A6$ or $A7$ configuration.

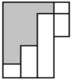




A7:

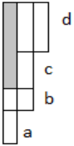


or



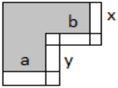
with $\ell(c2)=\ell(c4)$

A2:



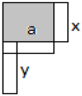
$a \leq c+1$ and $b \leq d+1$

A3:



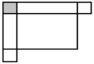
a=x=1, or
a=1 and $x \leq y+1$, or
x=1 and $a \leq b+1$

A4:



a=1 and $x \leq y+1$, or
a>1 and x=1

A6:



Corollary

The Schur function product $s_\mu s_\nu$ is multiplicity–free and has interval support if and only if one or more of the following is true:

(a) μ or ν is the zero partition.

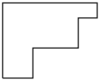
(b) μ and ν are both rows or both columns.

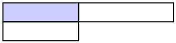
(c) $\mu = (1^x)$ is a one–column rectangle and $\nu = (a, 1^y)$ is a hook such that either $a = 2$ and $1 \leq x \leq y + 1$, or $a \geq 3$ and $x = 1$ (or vice versa).

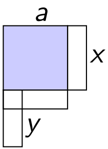
(c') $\mu = (x)$ is a one–row rectangle and $\nu = (z, 1^a)$ is a hook such that either $a = 1$ and $1 \leq x \leq z$, or $a \geq 2$ and $x = 1$ (or vice versa).

Corollary

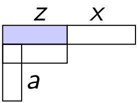
The Schur function product $s_\mu s_\nu$ has interval support if and only if one of the conditions above or one of the following is true: $\mu = (r_1, 1^{r_2})$ and $\nu = (s_1, 1^{s_2})$ are hooks such that $s_2 = r_2 = 1$, and either $r_1 = s_1 \geq 2$ or $r_1 = 2, \;\; s_1 = r_1 + 1$ (or vice versa).



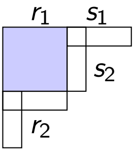




$a = 2$ and $1 \leq x \leq y + 1$,
or $a \geq 3$ and $x = 1$;



$a = 1$ and $1 \leq x \leq z$,
or $a \geq 2$ and $x = 1$;



$s_2 = r_2 = 1$, and either
$r_1 = s_1 \geq 2$ or $r_1 = 2$
and $s_1 = r_1 + 1$.