Non symmetric Cauchy kernels and last passage percolation

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Introduction

We use non-symmetric Cauchy kernel identities to get the law of last passage percolation models in terms of Demazure characters. The construction is based on restrictions of the RSK correspondence compatible with crystal basis theory.

Preliminaries

Cauchy kernel identity

$$\prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_iy_j}=\sum_{\lambda\in\mathcal{P}_{\min(m,n)}}s_\lambda(x)s_\lambda(y)$$

LHS rewritten in the basis of Schur polynomials. $\mathcal{P}_r$ the set of partitions with at most r parts.

Non-symmetric Cauchy kernel identity, Lascoux 2000.

$$\prod_{1\le j\le i\le n}\frac{1}{1-x_iy_j}=\sum_{\mu\in\mathbb{Z}^n_{\ge 0}}\overline{\kappa}^{\mu}(x)\kappa_\mu(y)$$

LHS rewritten in the bases of Demazure and Demazure atom polynomials: $\overline{\kappa}^{\mu}(x_1,\dots,x_n)=\overline{\kappa}_{\sigma_0\mu}(x_n,\dots,x_1)$ *opposite Demazure atoms* and $\kappa_\mu(y)$ *Demazure characters.*

Nonsymmetric q-Cauchy identity: $t=0$ specialization of the Mimachi--Noumi formula, 1996.

$$\prod_{1\le j\le i\le n}\frac{1}{1-x_iy_j}\prod_{1\le i,j\le n}\frac{1}{(qx_iy_j;q)_\infty}=\sum_{\mu\in\mathbb{Z}^n_{\ge 0}}a_\mu(q)E_\mu(x;q,0)E_\mu(y;q^{-1},\infty)$$

$a_\mu(q)$ is the Cherednik norm of E_μ. E_μ stands for type GL_n nonsymmetric Macdonald polynomials.

Bicrystals and RSK correspondence

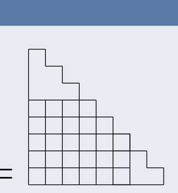
$$\psi:\begin{cases}\mathcal{M}_{m,n}\to\bigsqcup_{\lambda\in\mathcal{P}_{\min(m,n)}}\mathrm{B}_m(\lambda)\times\mathrm{B}_n(\lambda)\\ A\longmapsto(P(A),Q(A))\end{cases}$$

$$\prod_{1\le i\le m,1\le j\le n}\frac{1}{1-x_iy_j}=\sum_{A\in\mathcal{M}_{m,n}}x^{\mathrm{wt}(P(A))}y^{\mathrm{wt}(Q(A))}=\sum_{\lambda\in\mathcal{P}_{\min(m,n)}}s_\lambda(x)s_\lambda(y).$$

Remark: $\mathrm{B}_m(\lambda)$ tableau crystal on the alphabet $[m]$ with highest weight element the key tableau $K(\lambda)$, $\lambda\in\mathcal{P}_{\min(m,n)}$.

Demazure crystals and restriction of RSK to Ferrers shape matrices

Stair shape

$\varrho=$ 

The restriction of the RSK correspondence ψ to $\mathcal{M}^{\varrho}_{n,n}$, $n\times n$ *lower triangular matrices*, gives a one-to-one correspondence

$$\psi:\mathcal{M}^{\varrho}_{n,n}\to\bigsqcup_{\mu\in\mathbb{Z}^n_{\ge0}}\overline{\mathrm{B}}^{\mu}\times\mathrm{B}_\mu \qquad \text{Lascoux, 2000, A.-Emami,15, Choi-Kwon, 18}$$

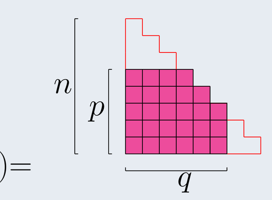
$$A\mapsto(P,Q),\quad K_+(Q)\le K_-(P)=K(\mu)$$

$$\prod_{1\le j\le i\le n}\frac{1}{1-x_iy_j}=\sum_{A\in\mathcal{M}^{\varrho}_{n,n}}x^{\mathrm{wt}(P(A))}y^{\mathrm{wt}(Q(A))}=\sum_{\mu\in\mathbb{Z}^n_{\ge0}}\ \sum_{\substack{(P,Q)\\K_+(Q)\le K^-(P)=K(\mu)}}x^{\mathrm{wt}(P)}y^{\mathrm{wt}(Q)}$$

$$=\sum_{\mu\in\mathbb{Z}^n_{\ge0}}\ \sum_{(P,Q)\in\overline{\mathrm{B}}^{\mu}\times\mathrm{B}_\mu}x^{\mathrm{wt}(P)}y^{\mathrm{wt}(Q)}=\sum_{\mu\in\mathbb{Z}^n_{\ge0}}\overline{\kappa}^{\mu}(x)\kappa_\mu(y).$$

Remark: B_μ Demazure crystal consisting of all tableaux Q with right key $K_+(Q)\le K(\mu)$. $\overline{\mathrm{B}}^{\mu}$ opposite Demazure atom crystal consisting of all tableaux P with left key $K^-(P)=K(\mu)$.

Truncated stair shape, bubble sort operators, and parabolic map

$q\ge p,\quad \Lambda(p,q)=$ 

$$\psi:\mathcal{M}^{\Lambda(p,q)}_{n,n}\to\bigsqcup_{\mu\in\mathbb{Z}^p_{\ge0}}\overline{\mathrm{B}}^{\mu}_p\times\mathrm{B}_{q,\tilde{\mu}}$$

$$A\mapsto(P(A),Q(A))\quad K_-(P)=K(\mu),\ K_+(Q)\le K(\tilde{\mu}).$$

$$\prod_{(i,j)\in\Lambda(p,q)}\frac{1}{1-x_iy_j}=\sum_{\mu\in\mathbb{Z}^p_{\ge0}}\overline{\kappa}^{\mu}(x_{n-p+1},\dots,x_n)\kappa_{\tilde{\mu}}(y_1,\dots,y_q).$$

- The *parabolic map*: for $\sigma\in\mathfrak{S}_n$, the set $\mathfrak{S}^{\le\sigma}_q=\{v\in\mathfrak{S}_q\mid v\le\sigma\}$ has unique maximal element σ^{I_q} for the (strong) Bruhat order $\le$; A. -Gobet-Lecouvey, 22.
- For $\mu=(\mu_1,\dots,\mu_p)\in\mathbb{Z}^p_{\ge0}$, let $\lambda\in\mathcal{P}_p$ and $\tau\in\mathfrak{S}^{\lambda}_p$ such that $\mu=\tau\lambda$: $\tilde{\mu}=(\sigma_0\tau)^{I_q}(\lambda,0^{q-p},0^{n-q})$, $\sigma_0\in\mathfrak{S}_n$. A.-Emami, 14, A.-Gobet-Lecouvey, 22.

$n=5$, $p=3$, $q=4$, $\lambda=(4,3,1)$, $\mu=(1,3,4)$

$$A_{\Lambda(3,4)}=\begin{pmatrix}\mathbf{0}&0&0&0&0\\0&\mathbf{0}&0&0&0\\\mathbf{0}&\mathbf{1}&\mathbf{0}&0&0\\\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{0}&0\\\mathbf{1}&\mathbf{0}&\mathbf{2}&\mathbf{1}&\mathbf{0}\end{pmatrix}\quad 45\otimes34\otimes455\otimes5\otimes\emptyset\quad 5\to4\to5\to5\to3\to4\to45$$

$$P=\begin{array}{|c|c|c|c|}\hline 3&4&4&5\\\hline 4&5&5\\\cline{1-3} 5\\\cline{1-1}\end{array}=K^-(P)=K(0,0,1,3,4)=K(0,0,\mu),\qquad Q=\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 3&3&3\\\cline{1-3} 4\\\cline{1-1}\end{array}$$

$$K_+(Q)=\begin{array}{|c|c|c|c|}\hline 2&2&2&2\\\hline 3&3&3\\\cline{1-3} 4\\\cline{1-1}\end{array}=K(0,4,3,1,0)\ \le K(\tilde{\mu})=\begin{array}{|c|c|c|c|}\hline 2&3&3&3\\\hline 3&4&4\\\cline{1-3} 4\\\cline{1-1}\end{array}=K(0,1,4,3,0)$$

$\sigma_0\in\mathfrak{S}_5,\quad \sigma_0(\mu,0,0)=(00431)=s_2s_3s_4s_1s_2s_3(43100)\ \ (\mathbf{s}_2\mathbf{s}_3\mathbf{s}_4\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3)^{I_4}:=\mathbf{s}_2\mathbf{s}_3\hat{\mathbf{s}}_4\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3=\mathbf{s}_2\mathbf{s}_3\mathbf{s}_1\mathbf{s}_2\mathbf{s}_3=\mathbf{s}_2\mathbf{s}_1\mathbf{s}_3\mathbf{s}_2\mathbf{s}_3$

$\tilde{\mu}=s_2s_1s_3s_2s_3(43100)=(01430)=\pi_2\pi_1\pi_3\pi_2\pi_3(43100)\ \ \mathrm{B}_{q,\tilde{\mu}}=\mathrm{B}_{\pi_2\pi_1\pi_3\pi_2\pi_3(43100)}$, π_i bubble sort operator (♣).

Last passage percolation in a Young diagram

For $A=[a_{i,j}]\in\mathcal{M}_{n,n}$, the **last passage percolation** associated to A:

$$perc(A)=\max_{\pi\text{ in }A}\{\textstyle\sum\text{ entries along a path }\pi\text{ in }A\text{ with steps }\leftarrow,\downarrow\text{ starting in }(1,n)\text{ and ending in }(n,1)\}$$
$$=\text{ maximal row lenght of }P(A)\text{ (or }Q(A)).$$

$$perc(A_{\Lambda(3,4)})=4,\quad perc(A_{(7,4,2,2,2)})=5$$

The random matrix $\mathcal{W}$ in $\mathcal{M}_{m,n}$ whose entry $w_{i,j}$ follows a geometric distribution of parameter u_iv_j

$\mathbb{P}(w_{i,j}=k)=(1-u_iv_j)(u_iv_j)^k$ for any $k\in\mathbb{Z}_{\ge0}$ gives $\mathbb{P}(\mathcal{W}=A)=\prod_{1\le i\le n,1\le j\le n}(1-u_iv_j)\prod_{1\le i\le n,1\le j\le n}(u_iv_j)^{a_{i,j}}$.

The law of the random variable $G=perc(\mathcal{W})$:

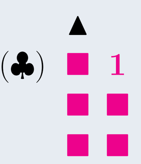
$$\mathcal{W}\in\mathcal{M}_{n,n},\qquad \mathbb{P}(G=k)=\prod_{1\le i\le n,1\le j\le n}(1-u_iv_j)\sum_{\lambda\in\mathcal{P}_n|\lambda_1=k}s_\lambda(u)s_\lambda(v).$$

$$\mathcal{W}\in\mathcal{M}^{\Lambda(n,n)}_{n,n},\qquad \mathbb{P}(G=k)=\prod_{1\le j\le i\le n}(1-u_iv_j)\sum_{\mu\in\mathbb{Z}^n_{\ge0}|\max(\mu)=k}\overline{\kappa}^{\mu}(u)\kappa_\mu(v).$$

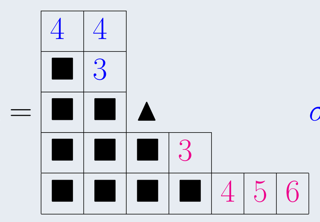
$$\mathcal{W}\in\mathcal{M}^{\Lambda(p,q)}_{m,n},\quad \mathbb{P}(G=k)=\prod_{(i,j)\in\Lambda(p,q)}(1-u_iv_j)\sum_{(\mu_1,\dots,\mu_p)\in\mathbb{Z}^p_{\ge0}|\max(\mu)=k}\overline{\kappa}_{(\mu_p,\dots,\mu_1)}(u_n,\dots,u_{n-p+1})\kappa_{\tilde{\mu}}(v_1,\dots,v_q).$$

Ferrers shape

$n=5$, $\varrho_\Lambda=(3,2,1)$, $\Lambda=(3,4,4)$; $\quad n=8$, $\Lambda=(7,4,2,2,2)$, $m=4$, $\varrho_\Lambda=(4,3,2,1)$

(♣) 

$\sigma(\Lambda,NW)=1$, $\sigma(\Lambda,SE)=s_2s_1s_3s_2s_3$

$\Lambda=$ 

$\sigma(\Lambda,NW)=s_4s_3s_4$, $\sigma(\Lambda,SE)=s_3s_6s_5s_4$

$$\psi:\mathcal{M}^{D_\Lambda}_{n,n}\to\bigsqcup_{(\mu_1,\dots,\mu_m)\in\mathbb{Z}^m_{\ge0}}\iota\left(\dot{\Delta}_{\sigma(\Lambda,NW)}\overline{\mathrm{B}}_{(\mu_m,\dots,\mu_1)}\right)\times\Delta_{\sigma(\Lambda,SE)}\mathrm{B}_{(\mu_1,\dots,\mu_m)}$$

where $\Delta_{\sigma(\Lambda,SE)}=\Delta_{j_1}\cdots\Delta_{j_b}$, $\dot{\Delta}_{\sigma(\Lambda,NW)}=\dot{\Delta}_{i_1}\cdots\dot{\Delta}_{i_a}$, ι Schützenberger involution,

$$\prod_{(i,j)\in\Lambda}\frac{1}{1-x_iy_j}=$$
$$\sum_{(\mu_1,\dots,\mu_m)\in\mathbb{Z}^m}D_{\sigma(\Lambda,NW)}\overline{\kappa}_{(\mu_m,\dots,\mu_1)}(x_n,\dots,x_{n-m+1})D_{\sigma(\Lambda,SE)}\kappa_{(\mu_1,\dots,\mu_m)}(y_1,\dots,y_m)$$

where $D_{\sigma(\Lambda,NW)}=D_{i_1}\cdots D_{i_a}$ and $D_{\sigma(\Lambda,SE)}=D_{j_1}\cdots D_{j_b}$ are Demazure operators.

$$A_{(7,4,2,2,2)}=\begin{pmatrix}0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&0\\\mathbf{0}&\mathbf{1}&0&0&0&0&0&0\\\mathbf{1}&\mathbf{1}&0&0&0&0&0&0\\\mathbf{0}&\mathbf{0}&0&0&0&0&0&0\\\mathbf{2}&\mathbf{0}&\mathbf{1}&\mathbf{1}&0&0&0&0\\\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{1}&\mathbf{0}&\mathbf{2}&0\end{pmatrix}\qquad 577\otimes45\otimes7\otimes7\otimes8\otimes\emptyset\otimes88\otimes\emptyset$$

$$P=\begin{array}{|c|c|c|c|c|}\hline 4&5&5&7&7\\\hline 7&7&8\\\cline{1-3} 8&8\\\cline{1-2}\end{array}\qquad K_-(P)=\begin{array}{|c|c|c|c|c|}\hline 4&4&4&4&4\\\hline 7&7&8\\\cline{1-3} 8&8\\\cline{1-2}\end{array}=K(0^3,5,0^2,2,3)$$

$$Q=\begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&2\\\hline 3&4&7\\\cline{1-3} 5&7\\\cline{1-2}\end{array}\qquad K_+(Q)=\begin{array}{|c|c|c|c|c|}\hline 2&2&2&2&2\\\hline 4&4&7\\\cline{1-3} 7&7\\\cline{1-2}\end{array}=K(0,5,0,2,0^2,3,0)$$

$\iota\dot{\Delta}_4\dot{\Delta}_3\dot{\Delta}_4\overline{\mathrm{B}}_{(\mu_4,\dots,\mu_1,0^4)}=$

$$=\begin{cases}\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,0,0^4)}\sqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_2,0^4)}\sqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0^2,\mu_2,0^3)}\ \text{if }\mu_2>\mu_1=0\\ \iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,0,0^4)}\ \text{if }\mu_1=\mu_2=0\\ \iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,\mu_1,0^4)}\sqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,0,\mu_1,0^3)}\sqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_2,\mu_1,0^3)}\text{if scriptstyle}\mu_1=\mu_2>0\\ \emptyset,\ \text{if }\mu_1>\mu_2\ge0\\ \iota\overline{\mathrm{B}}_{(\mu_4,\dots,\mu_1,0^4)}\sqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_1,\mu_2,0^4)}\sqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_2,0,\mu_1,0^3)}\sqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_2,\mu_1,0^3)}\sqcup\\ \sqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,0,\mu_1,\mu_2,0^3)}\sqcup\iota\overline{\mathrm{B}}_{(\mu_4,\mu_3,\mu_1,0,\mu_2,0^3)},\ \text{if }\mu_2>\mu_1>0.\end{cases}$$

$K^-(P)=K(0^3,5,0^2,2,3)\Leftrightarrow P\in\iota\overline{\mathrm{B}}_{(3,2,0^2,5,0^3)}=\overline{\mathrm{B}}^{(0^3,5,0^2,2,3)}\Rightarrow\mu=(0,5,2,3)$

$K_+(Q)=K(0,5,0,2,0^2,3,0)\le K(\pi_3\pi_6\pi_5\pi_4(\mu,0^4))=K(0,5,0,2,0^2,3,0).$

$(P,Q)\in\iota\mathrm{B}_{(3,2,0^2,5,0^3)}\times\mathrm{B}_{(0,5,0,2,0^2,3,0)}.$

$$\mathcal{W}\in\mathcal{M}^{\Lambda}_{m,n},\quad \mathbb{P}(A_\Lambda=k)=$$
$$\prod_{(i,j)\in D_\Lambda}(1-u_iv_j)\sum_{(\mu_1,\dots,\mu_m)\in\mathbb{Z}^m|\max(\mu)=k}D_{\sigma(\Lambda,NW)}\overline{\kappa}_{(\mu_m,\dots,\mu_1)}(u_n,\dots,u_{n-m+1})D_{\sigma(\Lambda,SE)}\kappa_{(\mu_1,\dots,\mu_m)}(v_1,\dots,v_m).$$