Littlewood-Richardson fillings and their symmetries

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Abstract

Considering the classical definition of the Littlewood-Richardson rule and its 2-dimensional representation by means of rectangular tableaux, we exhibit 24 symmetries of this rule when considering dualization, conjugation and their composition. Extending the Littlewood-Richardson rule to sequences of nonnegative real numbers, six of these symmetries may be generalized. Our point is to stress the role of different Littlewood-Richardson fillings, opposite (or increasing) [1, 2, 12] and column, [1, 2, 7] in guiding these symmetries. The main result is a bijection in the set of Littlewood-Richardson rectangular tableaux which transforms Littlewood-Richardson fillings of type $[a, b, c]$ into $[b, a, c]$. This bijection is based on the projection of Littlewood-Richardson tableaux of order r into Littlewood-Richardson tableaux of order $r-1$, for each $r \in \mathbb{N}$.

1 Introduction

The Littlewood-Richardson rule (*LR* rule for short) has a lot of symmetries. They do not seem clear from the original definition in terms of tableaux [8, 9]. Our goal is to show up the hidden symmetries of the *LR* fillings in the classical setting.

We consider *LR rectangular tableaux* and *LR rectangular triples.* Given partitions a, b and c (nonnegative integral vectors by weakly decreasing order) with lenght $\leq r$, an *LR* rectangular tableau of type [a,b,c] is an LR tableau of type (a, b, c^*) [9], where $c^* = (m - c_{r-i+1})_{i=1}^r$ for some nonnegative integer $m \geq c_1$, called dual partition of c. We call $[a, b, c]$ an LR rectangular triple. Therefore, $[a, b, c]$ is an LR rectangular triple iff (a, b, c^*) is an LR triple. Let $N_{a\,b}^{c}$ be the Littlewood-Richardson number, *i.e.*, the number of LR tableaux of type (a, b, c). The number of LR rectangular tableaux of type $[a, b, c]$, written $N_{a\,b\,c}$, is precisely $N_{a\,b}^{c^*}$. Let V_a, V_b, V_c be irreducible finite dimensional SL_r-modules with highest weights a, b and c. In [3, 14], $N_{a\,b\,c}$ is the triple multiplicity, that is, the dimension of the space of SL_r-invariants in the triple tensor product $V_a \otimes V_b \otimes V_c$ and $N_{a\,b\,c} = N_{a\,b}^{c^*}$, where c^* is the highest weight of the module V_c^* dual to V_c. In [3], a new combinatorial object is given to calculate $N_{a\,b\,c}$. Using this object many symmetries of $N_{a,b,c}$ become apparent. In [5, 13], several formulations of the classical Littlewood-Richardson rule are given, from which commutativity and other properties also follow.

In [12], increasing LR tableaux (or sequences) are used to point out some symmetries of the LR rule in the classical setting. More precisely, it is shown that a bijection exists between the *LR* tableaux of type (a, b, c) and increasing *LR* tableaux of type (b, a, c), and between LR tableaux of type (a, b, c) and increasing *LR* tableaux of type (a^*, b^*, c^*).

Our approach, as in [12], uses only the classical definition of Littlewood-Richardson rule, that is, the language of tableaux. Using exclusively the language of Littlewood-Richardson fillings, we exhibit bijections between LR fillings of type $[a, b, c]$ and $[a', b', c']$

where (a', b', c') is a permutation of (a, b, c), (a^*, b^*, c^*), $(\tilde{a}, \tilde{b}, \tilde{c})$ ($\sim$ means conjugation) or $(\widetilde{a^*}, \widetilde{b^*}, \widetilde{c^*})$. Combining these transformations we obtain 24 symmetries. The combinatorial algorithms constructed to exhibit bijections between LR fillings of type $[a, b, c]$ and $[a', b', c']$, where (a', b', c') is a permutation of (a, b, c), may be generalized to LR fillings extended to sequences of nonnegative real numbers.

The bijection exhibited between LR fillings of type $[a, b, c]$ and $[b, a, c]$ is based on the projection of LR rectangular tableaux of order r into LR rectangular tableaux of order $r - 1$, for each $r \in \mathbb{N}$. More precisely, there exists a Littlewood-Richardson filling of type $[(a_1, \ldots, a_r); (b_1, \ldots, b_r); (c_1, \ \ldots, c_r)]$ only if, for each $k \in \{2, \ldots, r\}$, there exists a Littlewood-Richardson filling of type $[(a_1^{(k)}, \ldots, a_{k-1}^{(k)}); (b_1, \ldots, b_{k-1}); (c_{r-k+1}, \ \ldots, c_r)]$ where $a_{i+1}^{(k+1)} \leq a_i^{(k)} \leq a_i^{(k+1)}$, for $i = 1, \ldots, k$, is such that

$$b_k + \sum_{s=1}^{j-1}(a_s^{(k)} - a_s^{(k-1)}) \geq b_{k+1} + \sum_{s=1}^{j}(a_s^{(k+1)} - a_s^{(k)}),\ j = 1, ..., k-1. \ (*)$$

(We convention $a^{(0)} := 0$ and $a^{(r+1)} := (a_1, \ldots, a_r)$.)

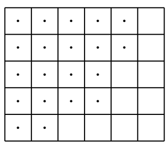
2 LR rectangular tableaux and LR rectangular triples

By a partition a we mean any finite sequence $a = (a_1, \ldots, a_r)$ of nonnegative integers by (weakly) decreasing order. The *weight* of a, written $|a|$, is the sum of of the components. The partition of weight zero is denoted by 0.

Let $m \geq 0$ and $r > 0$ be integers. Let $\mathcal{P}_r = \{a \in \mathbb{Z}^r : 0 \leq a_r \leq \ldots \leq a_1\}$ be the set of all partitions with r components. We write (x^r) to mean the constant partition of $\mathcal{P}_r$ with all components equal to x. We define $\mathcal{P}_{r,m} = \{a \in \mathcal{P}_r : 0 \leq a_r \leq \ldots \leq a_1 \leq m\}$. ($\mathcal{P}_{r,0} = \{0\}$.) Notice that, $\mathcal{P}_r = \bigcup_{m \geq 0} \mathcal{P}_{r,m}$.

Given $a \in \mathcal{P}_{r,m}$, $a^* := (m - a_{r-i+1})_{i=1}^r \in \mathcal{P}_{r,m}$ is called the *dual* partition of a in $\mathcal{P}_{r,m}$.

Consider the rectangular Young diagram of (m^r), *i.e.*, a sequence of r rows of boxes with row lengths m. If $a \in \mathcal{P}_{r,m}$ then $a \subseteq (m^r)$. (We identify a partition with its Young diagram.) Graphically, a^* is the partition defined by the complement of a in the Young diagram of (m^r). For example, if $r = 5$, $m = 6$ and $a = (5, 5, 4, 4, 2)$ we have $a^* = (4, 2, 2, 1, 1)$ (reading from bottom to top) represented by the blank boxes:



Clearly, $(a^*)^* = a$.

Given $a, b \in \mathcal{P}_r$, we say that a and b are *congruent*, written $a \equiv b$, if $b = a + (M^r)$, for some integer $M \geq 0$. Clearly, $a \equiv (a_1 - a_r, \ldots a_{r-1} - a_r, 0) + (a_r^r)$. Therefore, when we write a^* without mentioning an upper bound for the largest component, we mean a partition congruent to $(a_1 - a_{r-i+1})_{i=1}^r$. Moreover, if a and b are congruent, a^* and b^* are congruent. Clearly, $a^* \in \mathcal{P}_{r,k}$, for all $k \geq a_1$.

Given a, b, $c \in \mathcal{P}_r$, we say that (a, b, c) is an *LR triple* if there is an LR tableau of type (a, b, c) [4]. We identify an LR tableau of type (a, b, c) filled with x_{ij} symbols j in row i, for $r \geq i \geq j \geq 1$, with the element $(a, b, c, X) \in \mathbb{Z}^{3r+r^2}$, where $X = [x_{ij}]$ is an, $r \times r$, integral lower triangular matrix, such that the following system of linear inequalities is satisfied [4, 6]:

$$x_{ij} \geq 0, \quad 1 \leq i, j \leq r. \tag{1}$$

$$\sum_{i=1}^{r} x_{ij} = b_j, \quad j = 1, \ldots, r. \tag{2}$$

$$\sum_{j=1}^{r} x_{ij} = c_i - a_i, \quad i = 1, \ldots, r. \tag{3}$$

$$\sum_{i=1}^{k} x_{ij} \geq \sum_{i=1}^{k+1} x_{i,j+1}, \quad 1 \leq k, j \leq r-1. \tag{4}$$

$$a_i + \sum_{j=1}^{k-1} x_{ij} \geq a_{i+1} + \sum_{j=1}^{k} x_{i+1,j}, \quad {\scriptstyle k=1,\ldots,r-1 \text{ and } i=1,\ldots,r-1}. \tag{5}$$

The *Littlewood-Richardson number*, $N_{a\,b}^{c}$, is the number of lower triangular matrices $X \in \mathbb{Z}^{r,r}$ whose entries satisfy this system of linear inequalities for fixed partitions a, b and c.

We may easily extend the LR rule to finite sequences of nonnegative real numbers. Given $\alpha = (\alpha_1, ..., \alpha_r)$, $\beta = (\beta_1, \ldots, \beta_r)$ and $\gamma = (\gamma_1, \ldots, \gamma_r)$ sequences of nonnegative real numbers by weakly decreasing order, we say (α, β, γ) is a *real LR* triple if there is a lower triangular matrix $X = [x_{ij}] \in \mathbb{R}^{r,r}$ such that $(\alpha, \beta, \gamma, X) \in \mathbb{R}^{3r+r^2}$ satisfy the system of linear inequalities above (replacing a by α, b by β and c by γ). We call $(\alpha, \beta, \gamma, X)$ an *LR design* of order r [6]. When α, β, γ and X are integral, we have an integral LR design or, equivalently, an LR tableau of order r.

For $r \geq 1$, let $LRD_r^{\mathbf{R}}$ be the set of elements $(\alpha, \beta, \gamma, X) \in \mathbb{R}_{\geq 0}^{3r+r^2}$ such that the following conditions hold: $\alpha_1 \geq \ldots \geq \alpha_r \geq 0$, $\beta_1 \geq \ldots \geq \beta_r \geq 0$, $\gamma_1 \geq \ldots \geq \gamma_r \geq 0$ and $(\alpha, \beta, \gamma, X)$ satisfy linear inequalities $(1) - (5)$. Let $LRD_r := LRD_r^{\mathbf{R}} \cap \mathbb{Z}^{3r+r^2}$ be the set of integral LR tableaux of order r.

$LRD_r^{\mathbf{R}}$ is a pointed rational polyhedral cone in $\mathbb{R}^{3r+r^2}$. Therefore, $LRD_r^{\mathbf{R}}$ has an integral Hilbert basis [10] and LRD_r is a finitely generated (additive) semigroup. Notice that $(a, b, c, X) + (a', b', c', X') = (a + a', b + b', c + c', X + X')$, with componentwise sum.

Let $LR_r = \{(a, b, c) \in (\mathcal{P}_r)^3 : (a, b, c, X) \in LRD_r, \text{for some, } r \times r, \text{ integral matrix X}\}$ be the set of LR triples of order r. Clearly, LR_r is also a finitely generated (additive) semigroup, called the *Littlewood -Richardson semigroup of order* r [14].

Let $LR_r^{\mathbf{R}}$ be the set of *real LR* triples of order r. $LR_r^{\mathbf{R}}$ is also a pointed rational polyhedral cone, finitely generated by the indecomposable elements of LR_r.

Let $a, b, c \in \mathcal{P}_r$. A *rectangular tableau* of type $[a, b, c]$ is a tableau of type (a, b, c^*).

Notice that rectangular tableaux are in some sense symmetric relatively to a and c. Reading a rectangular tableau from right to left and from bottom to top we obtain an *opposite* (or *increasing*) rectangular tableau of type $[c, b, a]$, replacing each symbol i by $r - i + 1$ (see [2]).

Example 1

						1	1
					1		
		1	2	2			
1	2	2	3				

,

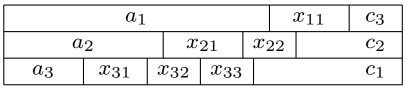
						4	4
					4		
		4	3	3			
4	3	3	2				

are, respectively, a rectangular LR tableau of type $[a, b, c]$ and the corresponding increasing LR rectangular tableau of type $[c, b, a]$.

We define

$$\overline{LRD}_r = \{[a, b, c, X] : (a, b, c^*, X) \in LRD_r\},$$
$$\overline{LR}_r = \{[a, b, c] : (a, b, c^*) \in LR_r\}.$$

Graphically, $[a, b, c, X] \in \overline{LRD}_r$ may be represented as follows:



an *LR rectangular* tableau of type $[a, b, c]$.

Notice that $[a, b, c, X] \in \overline{LRD}_r$ if $X = [x_{ij}]$ satisfy the system of linear inequalities defined by (1), (2), (4), (5) and

$$\sum_{j=1}^{r} x_{ij} = m - a_i - c_{r-i+1}, \; i = 1, \ldots, r,$$

for some nonnegative integer m.

Denoting by $N_{a,b,c}$ the number of matrices X satisfying the conditions $(6) - (10)$ of the system above, it is clear that $N_{a,b,c} = N_{a,b}^{c^*}$. Hence, studying LR_r and LRD_r is the same as studying $\overline{LR}_r$ and $\overline{LRD}_r$, respectively, with the advantage that this triples and these tableaux are more symmetrical (see [3]). As before, we define $\overline{LRD}_r^{\mathbf{R}}$ and $\overline{LR}_r^{\mathbf{R}}$ which are also pointed rational polyhedral cones in $\mathbb{R}^{3r+r^2}$ and $\overline{LR}_r$, $\overline{LRD}_r$ are finitely generated semigroups.

3 Symmetries

We present combinatorial algorithms to define bijections between the set of LR rectangular tableaux of type $[a, b, c]$ and the set of LR rectangular tableaux of type $[a', b', c']$ where (a', b', c') is a permutation of (a, b, c), (a^*, b^*, c^*) or $(\tilde{a}, \tilde{b}, \tilde{c})$. The main tools are: (i) an algorithm exhibiting the commutativity of a and b in LR rectangular tableaux of type $[a, b, c]$, the *opposite* (or *increasing*) LR filling [1, 12] to exhibit the symmetry of a and c in LR rectangular tableaux of type $[a, b, c]$, and the algorithm converting an *opposite* (or *increasing*) LR filling into an LR filling (see [2]); and (ii) the *column* LR filling [7] which is intrinsically related with dualization and conjugation [2, 12]. The algorithms defined in (i) and (ii) can be applied to LR rectangular designs.

We stress that the algoritm mentioned in (i) exhibits a bijection between LR rectangular tableaux of type $[a, b, c]$ and $[b, a, c]$.

3.1 Commutativity

To understand the commutativity of the LR rule it is convenient to formulate conditions (4) as a sequence of partitions $b^{(1)}, ..., b^{(r)} = b$ in $\mathcal{P}_r$, where $b^{(k)}$ has lenght $\leq k$, for $k = 1, ..., r$, satisfying the interlacing inequalities $b_{i+1}^{(k+1)} \leq b_i^{(k)} \leq b_i^{(k+1)}$, for $k = 1, ..., r-1$, $i = 1, ..., k$. Therefore, conditions (5) may also be written

$$a_i + \sum_{j=1}^{k-1}(b_j^{(i)} - b_j^{(i-1)}) \geq a_{i+1} + \sum_{j=1}^{k}(b_j^{(i+1)} - b_j^{(i)}),\ k = 1, \ldots, r-1, \text{ and } i = 1, \ldots, r-1.$$

We identify a rectangular LR tableau $[a, b, c, X]$ with $[a, b, c, b^{(1)}, b^{(2)}, ..., b^{(r)} = b]$, where $b^{(k)} = (\sum_{i=j}^{k} x_{ij}, 0^{r-k})_{j=1}^{k}$, for $k = 1, ..., r$.

Given $\mathcal{T} = [a, b, c, X] \in \overline{LRD}_r$, we define iductively the *deleting matrix* $Z = [z_{ij}] \in \mathbb{Z}^{r,r}$ of $\mathcal{T}$ as follows:

For $r = 1$, we have $\mathcal{T} = ((a_1), (x_{11}), (a_1 + x_{11}); [x_{11}])$ and $Z = [a_1]$.

For $r \geq 2$ let $\mathcal{F} = [(a_i + x_{i1})_{i=2}^r;\ (b_2, \ldots, b_r);\ (c_2, \ldots, c_r);\ X'] \in \overline{LRD}_{r-1}$, where X' is the $(r-1) \times (r-1)$ matrix obtained from X by suppressing the first row and the first column. By induction, let $F = [f_{ij}] \in \mathbb{Z}^{r-1,r-1}$ be the deleting matrix of $\mathcal{F}$, and $Y = [y_{ij}] \in \mathbb{Z}^{r,r}$ such that $y_{ij} = f_{i-1,j-1}$, for $i, j \in \{2, \ldots, r\}$ and $y_{ij} = 0$, otherwise. Then, $Z = [z_{ij}] \in \mathbb{Z}^{r,r}$ with $z_{ij} = 0$, for $j > i$, the deleting matrix of $\mathcal{T}$, is determined in the following way:

Let $x_{i1}^{(r+1)} := x_{i1}$, $i = 1, \ldots, r$.

For $k = r, \ldots, 2$, define

$\theta_i^{(k)} := min\{x_{i1}^{(k+1)}, y_{ki}\}$, $i = 1, \ldots, k$, and $\theta_{k+1}^{(k)} := 0$,

$x_{i,1}^{(k)} := x_{i1}^{(k+1)} - \theta_i^{(k)} + \theta_{i+1}^{(k)}$, $i = 1, \ldots, k-1$,

$z_{k,i} := y_{ki} - \theta_i^{(k)} + \theta_{i+1}^{(k)}$, $i = 1, \ldots, k$, and $z_{11} = a_1 - \sum_{j=2}^{r} \theta_2^{(j)}$.

Let $\mathcal{T}^{(r+1)} := \mathcal{T}$, $a^{(r+1)} := a$ and $X^{(r+1)} := X$. For each $k \in \{1, \ldots, r\}$, let $a_i^{(k)} = a_i^{(k+1)} - z_{k,i}$, $i = 1, \ldots, k-1$. We have, $a_{i+1}^{(k+1)} \leq a_i^{(k)} \leq a_i^{(k+1)}$, for $i = 1, \ldots, k-1$.

We call $(z_{k1}, \ldots, z_{kk})$ the *k-deleting sequence* of $\mathcal{T}^{(k+1)} = [(a_i^{(k+1)})_{i=1}^k, (b_i)_{i=1}^k, (c_i)_{i=r-k+1}^r, X^{(k+1)}] \in \overline{LRD}_k$, for $k = 1, \ldots, r$. (Note that $z_{k,k} = a_k^{(k+1)}$.)

The deleting matrix of a tableau $\mathcal{T}$ is well defined and is unique.

Theorem 1 *Let $n \geq 1$ and $[(a'_1, ..., a'_r), (b_1, ..., b_r), (c_2, ..., c_{r+1}), X'] \in \overline{LRD}_r$ with $\sum_i(a'_i + b_i + c_{r+2-i}) = rm$, and r-deleting sequence $(z_{r1}, ..., z_{r,r-1}, a'_r)$. Let $a = (a_1, ..., a_{r+1})$, $b = (b_1, ..., b_{r+1})$, and $c = (c_1, ..., c_{r+1}) \in \mathcal{P}_{r+1}$ such that $a_{i+1} \leq a'_i \leq a_i$, $i = 1, ..., r$, and $|a| + |b| + |c| = (r+1)m$. Moreover, suppose the following conditions hold*

$$b_r + \sum_{s=1}^{k-1} z_{rs} \geq b_{r+1} + \sum_{s=1}^{k}(a_s - a'_s), k = 1, ..., r.$$

Then, there exists $X \in \mathbb{Z}^{r+1,r+1}$ such that $[a, b, c; X] \in \overline{LRD}_{r+1}$ with $r+1$-deleting sequence $(z_{r+1,1}, ..., z_{r+1,r}, a_{r+1})$ where $z_{r+1,i} = a_i - a'_i$, $i = 1, \ldots, r$.

Theorem 1 defines a bijection between LR rectangular tableaux of type $[a, b, c]$ and $[b, a, c]$. More precisely, a bijection $\pi : \overline{LRD}_r \longleftrightarrow \overline{LRD}_r$ such that $\pi([a, b, c, X])$ is a rectangular tableau of type $[b, a, c]$, with the deleting matrix X. For this theorem says that there exists an LR rectangular tableau $[a, b, c, X]$ only if there exists an LR rectangular tableau of type $[b, a, c]$ with deleting matrix X. Since the number of LR rectangular tableaux of a given type is finite and the deleting matrix of a tableau is unique, we have in fact an injection which transforms an LR rectangular tableau $[a, b, c, X]$ into an LR rectangular tableau of type $[b, a, c]$ with deleting matrix X. So $N_{abc} \leq N_{bac}$. Clearly, we have also a injection which transforms an LR rectangular tableau $[b, a, c, Y]$ into an LR rectangular tableau of type $[a, b, c]$ with deleting matrix Y. So $N_{abc} \geq N_{bac}$. Hence, $N_{abc} = N_{bac}$, ψ is a bijection and in addition we may conclude that distint LR rectangular tableaux of the same type do not have the same deleting matrix. Thus, there exists an LR rectangular tableau of type $[a, b, c]$ with deleting matrix Y iff there exists an LR rectangular tableau $[b, a, c, Y]$, and, therefore, conditions $(*)$ are satisfyied.

We have another bijection $\phi : \overline{LRD}_r \longleftrightarrow \overline{LRD}_r$ such that $\phi([a, b, c, X]) = [b, a, c, Y]$, where Y is the deleting matrix of $[a, b, c, X]$.

It is clear that, if $[a, b, c, X]$ is transformed by π into $[b, a, c, Y]$ with deleting matrix X, then $[b, a, c, Y]$ is transformed by ϕ into an LR tableau $[a, b, c, X]$. This means, $\pi\phi = \phi\pi = id$.

Combinatorialy π may be described by two operations: row-insertion and column sliding.

Row-insertion and column sliding : Consider x symbols $k_1 \leq \ldots \leq k_x$ in $\{0, 1, 2, \ldots\}$. Insert these x symbols $k_1, ..., k_x$ in a row of the tableau by sliding down, to the next row, the left most x symbols $z_1 \leq \ldots \leq z_x$ such that $k_i < z_i$, for $i = 1, ..., k$.

The following example is an illustration of π in $\overline{LRD}_4$. Let

$$[(6,5,2,0);(5,4,1,0);(4,3,2,0); X = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix}] \in \overline{LRD}_4,$$

whose LR rectangular tableau is

						1	1
					1		
		1	2	2			
1	2	2	3				

To calculate the image of this LR rectangular tableau under π the algorithm runs as follows:

We consider the following rectangular numbered diagram, where the boxes of lenght x_{ij} can be thought as being x_{ij} unitary boxes labelled with 0,

<table>
<tr><td colspan="2">x_{11}</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>x_{21}</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td></td><td></td></tr>
<tr><td colspan="2">x_{32}</td><td>x_{31}</td><td>3</td><td>3</td><td></td><td></td><td></td></tr>
<tr><td>x_{43}</td><td colspan="2">x_{42}</td><td>x_{41}</td><td></td><td></td><td></td><td></td></tr>
</table>

Now, the row insertion and column sliding operations are going as follows:

The first row of this numbered diagram defines an LR rectangular tableau of type $[x_{11}, a_1, 0]$.

Insert $x_{21} = 1$ symbol 0 in the first row by sliding down, to the second row, the left most $x_{21} = 1$ symbol 1

<table>
<tr><td colspan="2">x_{11}</td><td>x_{21}</td><td>1</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td>1</td><td>2</td><td>2</td><td>2</td><td>2</td><td>2</td><td></td><td></td></tr>
<tr><td colspan="2">x_{32}</td><td>x_{31}</td><td>3</td><td>3</td><td></td><td></td><td></td></tr>
<tr><td>x_{43}</td><td colspan="2">x_{42}</td><td>x_{41}</td><td></td><td></td><td></td><td></td></tr>
</table>

The two first rows of this numbered diagram define an LR rectangular tableau of type $[(x_{11} + x_{21}, 0); (a_1, a_2); (c_3, 0)]$.

Insert $x_{32} = 2$ symbols 0 in the second row by sliding down, to the third row, the left most $x_{32} = 2$ symbols which are strictly larger than 0 (one symbol 1 and one symbol 2); insert $x_{31} = 1$ symbol 0 in the first row by sliding down, to the second row, the left most $x_{31} = 1$ symbol 1; on its turn, this $x_{31} = 1$ slided symbol 1 is inserted in the second row by sliding down, to the third row, the left most $x_{31} = 1$ symbol which is strictly larger than 1 (one symbol 2),

<table>
<tr><td colspan="2">x_{11}</td><td>x_{21}</td><td>x_{31}</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td colspan="2">x_{32}</td><td>1</td><td>2</td><td>2</td><td>2</td><td></td><td></td></tr>
<tr><td>1</td><td>2</td><td>2</td><td>3</td><td>3</td><td></td><td></td><td></td></tr>
<tr><td>x_{43}</td><td colspan="2">x_{42}</td><td>x_{41}</td><td></td><td></td><td></td><td></td></tr>
</table>

The first three rows of this numbered diagram define an LR rectangular tableau of type $[(\sum_{i=1}^{3} x_{i1}, x_{32}, 0); (a_1, a_2, a_3); (c_2, c_3, 0)]$.

Insert $x_{43} = 1$ symbol 0 in the third row by sliding down, to the 4th row, the left most $x_{43} = 1$ symbol which is strictly larger than 0 (one symbol 1); insert $x_{42} = 2$ symbols 0 in the second row by sliding down, to the third row, the left most $x_{42} = 2$ symbols (one symbol 1 and one symbol 2) which are strictly larger than 0; on their turn, these slided $x_{42} = 2$ symbols are inserted in the third row by sliding down, to the 4th row, the left most $x_{42} = 2$ symbols which are strictly larger respectively than 1 and 2 (one symbol 2 and one symbol 3),

<table>
<tr><td colspan="2">x_{11}</td><td>x_{21}</td><td>x_{31}</td><td>1</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td colspan="2">x_{32}</td><td colspan="2">x_{42}</td><td>2</td><td>2</td><td></td><td></td></tr>
<tr><td>x_{43}</td><td>1</td><td>2</td><td>2</td><td>3</td><td></td><td></td><td></td></tr>
<tr><td>1</td><td>2</td><td>3</td><td>x_{41}</td><td></td><td></td><td></td><td></td></tr>
</table>

Now, insert $x_{41} = 1$ symbols 0 in the first row by sliding down, to the second row, one symbol 1; this symbol 1 will be inserted in the second row by sliding down to the third row, one symbol 2; and this symbol 2 will be inserted in the third row by sliding down to the 4th row one symbol 3; finally, the symbol 3 is inserted in the 4th row,

<table>
<tr><td colspan="2">x_{11}</td><td>x_{21}</td><td>x_{31}</td><td>x_{41}</td><td>1</td><td>1</td><td>1</td></tr>
<tr><td colspan="2">x_{32}</td><td colspan="2">x_{42}</td><td>1</td><td>2</td><td></td><td></td></tr>
<tr><td>x_{43}</td><td>1</td><td>2</td><td>2</td><td>2</td><td></td><td></td><td></td></tr>
<tr><td>1</td><td>2</td><td>3</td><td>3</td><td></td><td></td><td></td><td></td></tr>
</table>

The output is $[b, a, c, Y = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 1 & 2 & 0 \end{bmatrix}]$. Now applying ϕ to the output we obtain $[a, b, c, X]$, which illustrates that $\phi\pi = id$.

3.2 Opposite LR tableaux and symmetries

We exhibit a combinatorial algorithm to define an involution $\psi : \overline{LRD}_r \longleftrightarrow \overline{LRD}_r$ such that $\psi([a, b, c, X]) = [c, b, a, Z]$. First ψ sends $[a, b, c, X]$ to an opposite (or increasing) LR rectangular tableau $[c, b, a, X']$ [1, 12]. The matrix $X' = [x'_{ij}]$ is such that $x'_{ij} = x_{r-i+1,r-j+1}$, for all i, j. Graphically it is precisely the tableau obtained from $[a, b, c, X]$, when reading from bottom to top and from right to left and replacing the symbol j by $r - j + 1$. See, below, the second tableau of our example. Then, using the algorithm defined in [2], we transform the increasing LR rectangular tableau $[c, b, a, X']$ into an LR rectangular tableau $[c, b, a, Z]$,

						1	1
					1		
		1	2	2			
1	2	2	3				

$\rightarrow$

						4	4
					4		
		4	3	3			
4	3	3	2				

$\rightarrow$

						4	4
					4		
		4	3	2			
4	2	2	2				

$\rightarrow$

						3	3
					4		
		3	3	2			
3	2	2	2				

$\rightarrow$

$\rightarrow$

						3	3
					4		
		3	3	2			
2	2	2	2				

Reading the last LR rectangular tableau from left to right and from bottom to top, the output is $[c, b, a, Z = \begin{bmatrix} 4 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{bmatrix}]$. Applying ψ to $[c, b, a, Z]$ we obtain $[a, b, c, X]$.

Now combining ϕ (or π) and ψ we may exhibit six symmetries of the LR fillings and conclude that $N_{a,b,c}$ is invariant under the permutations of (a, b, c). These bijections ϕ (or π) and ψ can be extended to LR rectangular designs.

3.3 Duality and symmetries

A combinatorial algorithm is given to define an involution $\theta : \overline{LRD}_r \longleftrightarrow \overline{LRD}_r$ such that $\theta([a, b, c, X]) = [c^*, b^*, a^*, W]$. The definition of θ is based on a bijection λ between LR rectangular tableaux of type $[a, b, c]$ and *column* LR rectangular tableaux of type $[c, b, a]$ (see [2, 12]). The second rectangular tableau of our example, reading from right to left and from bottom to top, is a column LR rectangular tableau of type $[c, b, a]$. Then, this tableau is sent to a column LR rectangular tableau of type $[a^*, b^*, c^*]$ (we may consider the partitions with the last component equals zero). The column LR filling of type $[a^*, b^*, c^*]$ is as follows: let $\widetilde{b} = (y_1, ..., y_t)$ ($\sim$ means conjugation) and $\widetilde{b^*} = (z_1, ..., z_t)$ (note, $t = b_1$ and $y_i + z_{t-i+1} = r$), then, for each $i = 1, ..., t$, place exactly one symbol i in each of the z_i rows corresponding to the rows, in the previous column rectangular tableau of type $[c, b, a]$, free of symbols $t - i + 1$. See the third rectangular tableau in our example, here $t = 5$. Finally, using the bijection λ we obtain an LR rectangular tableau of type $[c^*, b^*, a^*]$,

						1	1
					1		
		1	2	2			
1	2	2	3				

$\overset{\lambda}{\rightarrow}$

						2	1
					3		
		4	2	1			
5	4	3	1				

$\rightarrow$

								3	2	1
						5	4	2	1	
					3	1				
				4						

$\overset{\lambda}{\rightarrow}$

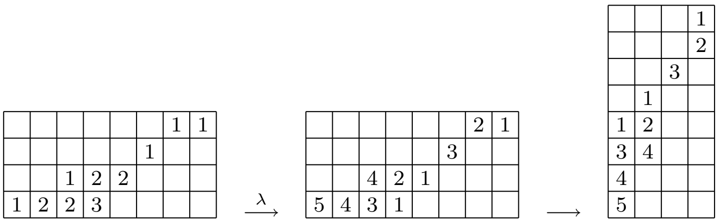
				1	1	1
		1	1	2	2	
	2	3				
2						

Combining ϕ (or π), ψ and θ we may exhibit a bijection $\overline{LRD}_r \longleftrightarrow \overline{LRD}_r$ such that

$\theta([a,b,c,X]) = [a',b',c',X']$, where (a',b',c') is a permutation of (a^*,b^*,c^*). Therefore, $N_{a,b,c}$ is also invariant under the replacement of (a,b,c) by (a^*,b^*,c^*). So far we have twelve symmetries of the LR triples.

3.4 Conjugation and symmetries

For $m \geq 0$, we consider $\overline{LRD}_{r,m} = \{[a,b,c,X] \in \overline{LRD}_r : a,b,c \in \mathcal{P}_{r,m}\}$, and $\overline{LR}_{r,m} = \{[a,b,c] \in \overline{LR}_r : a,b,c \in \mathcal{P}_{r,m}\}$. We construct a bijection $\omega : \overline{LRD}_{r,m} \longleftrightarrow \overline{LRD}_{m,r}$ such that $\omega([a,b,c,X]) = [\widetilde{c},\widetilde{b},\widetilde{a},W]$. The definition of ω is based on the bijection λ defined above. First ω sends $[a,b,c,X]$ to a column LR rectangular tableau of type $[c,b,a]$, via λ, and then, by transposing, to an LR rectangular tableau of type $[\widetilde{c},\widetilde{b},\widetilde{a},W]$.



Combining ω with the previous bijections we obtain 24 symmetries.

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