Symplectic cacti, virtualization and Berenstein–Kirillov groups

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The cactus group $J_{\mathfrak{g}}$

[Henriques-Kamnitzer, 2006; Halacheva, 2016]

- Let $\mathfrak{g}$ be a finite dimensional, complex, semisimple Lie algebra and
  - I its Dynkin diagram, $\Delta = \{\alpha_i\}_{i \in I}$ the simple roots.
  - $W_{\mathfrak{g}}$ the Weyl group, $w_0 \in W_{\mathfrak{g}}$ the longest element.
  - $\theta : I \to I$ the Dynkin diagram automorphism of I defined by

$$\alpha_{\theta(i)} = -w_0.\alpha_i, \ i \in I.$$

Example: For $\mathfrak{g} = \mathfrak{gl}_n$, Cartan type A_{n-1}: $I = [n-1]$, $\Delta = \{\alpha_i = \mathbf{e_i} - \mathbf{e_{i+1}}\}_{\mathbf{i \in [n-1]}}$, $W_{\mathfrak{g}} = \mathfrak{S}_n$, $\theta(i) = n - i$,

$n = 5$ $\quad A_4$

$\alpha_1 = (1, -1, 0, 0, 0) \to \alpha_4 = (0, 0, 0, 1, -1) = -w_0\alpha_1$

  - $\theta_J : J \to J$ the Dynkin diagram automorphism of a connected subdiagram $J \subseteq I$, defined by

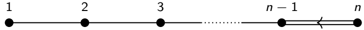
$$\alpha_{\theta_J(j)} = -w_0^J.\alpha_j, \ j \in J,$$

w_0^J the long element of the parabolic subgroup $W_{\mathfrak{g}}^J \subseteq W_{\mathfrak{g}}$.

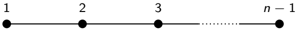
Example: $\mathfrak{gl}_5$, $J = [1, 2]$, $J = \{2\}$, $J = \{3\}$: A_2, A_1, A_1,

$\mathfrak{g} = \mathfrak{sp}_{2n}$: Dynkin diagram automorphism and restrictions

C_n



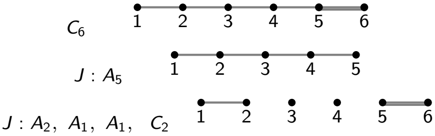
A_{n-1}



- Cartan type C_n: $I = [n]$, $\Delta = \{\alpha_i = \mathbf{e_i} - \mathbf{e_{i+1}}\}_{\mathbf{i \in [n-1]}} \cup \{\alpha_\mathbf{n} = \mathbf{2e_n}\}$,
  $W = B_n = < r_1, \ldots, r_{n-1}, r_n : R1, R2, R3, R4 >$

$$\begin{array}{lll} R1: & r_i^2 = 1, & 1 \leq i \leq n, \\ R2: & (r_i r_j)^2 = 1, & |i-j| > 1, \\ R3: & (r_i r_{i+1})^3 = 1, & 1 \leq i \leq n-2, \\ R4: & (r_{n-1} r_n)^4 = 1 & \end{array}$$

$$\alpha_{\theta(i)} = -w_0.\alpha_i = -(-\alpha_i) = \alpha_i, \quad \theta(i) = i.$$



$$\theta_{[1,5]}(i) = 5 - i, \quad \theta_{[5,6]} = 1$$

- [Henriques-Kamnitzer 2006] The *cactus group* $J_{\mathfrak{gl}_n} = J_n$.
- [Halacheva 2016]. The *cactus group* $J_{\mathfrak{g}}$ corresponding to $\mathfrak{g}$ is the group defined by:
  - **Generators**: s_J, $J \subseteq I$ running over all connected subdiagrams of the Dynkin diagram I of $\mathfrak{g}$, and
  - **Relations**:
    1 $\mathfrak{g}$. $s_J^2 = 1$, for all $J \subseteq I$,

    2 $\mathfrak{g}$. $s_J s_{J'} = s_{J'} s_J$, for all $J, J' \subseteq I$ such that $J \sqcup J'$ is not connected,

    $J = [1,2],\ J' = \{4\},\ J \sqcup J'$ is not connected diagram

    3 $\mathfrak{g}$. $s_J s_{J'} = s_{\theta_J(J')} s_J$, for all $J' \subseteq J \subseteq I$.

- Related groups: Like the braid group, the cactus group $J_{\mathfrak{g}}$ surjects onto the Weyl group $W_{\mathfrak{g}}$

$$s_J \mapsto w_0^J.$$

The kernel of the latter contains the elements $(s_{\{i\}} i s_{\{j\}})^{m_{ij}}$ such that $(r_i r_j)^{m_{ij}} = 1$ in $W_{\mathfrak{g}}$ as a Coxeter group.

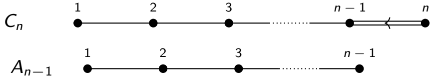
The cacti $J_n := J_{\mathfrak{gl}_n}$ and $J_{\mathfrak{sp}_{2n}}$

- The cactus group $J_{\mathfrak{sp}_{2n}}$ is the group defined by
  - Generators: s_J, J connected subdiagrams of the C_n Dynkin diagram,
  - Relations:

1C. $s_J^2 = 1$, $J \subseteq [n]$,

2C. $s_J s_{J'} = s_{J'} s_J$, $\quad J, J' \subseteq [n]$ such that $J \sqcup J'$ is not connected,

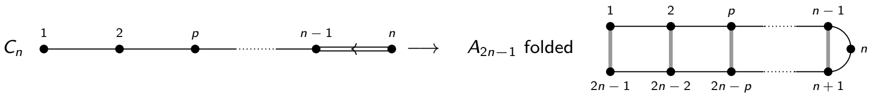
3C. 1. $s_{[p,q]} s_{[k,l]} = s_{[p+q-l,p+q-k]} s_{[p,q]}$, $[k,l] \subset [p,q] \subseteq [n-1]$.
    2. $s_{[p,n]} s_{[q,l]} = s_{[q,l]} s_{[p,n]}$, $[q,l] \subset [p,n] \subseteq [n]$,



- $J_n = J_{\mathfrak{gl}_n} \subseteq J_{\mathfrak{sp}_{2n}}$.
- Alternative $n-1$ generators for J_n,
  $s_{[1,p]}$, $1 \leq p \leq n-1$,
  $s_{[p,n-1]}$, $1 \leq p \leq n-1$,
- Alternative $2n-1$ generators for $J_{\mathfrak{sp}_{2n}}$: $s_{[1,p]}$, $1 \leq p \leq n-1$, $\quad s_{[p,n]}$, $1 \leq p \leq n$.

Embedding of $J_{\mathfrak{sp}_{2n}}$ into J_{2n}

- Dynkin diagram folding $C_n \hookrightarrow A_{2n-1}$



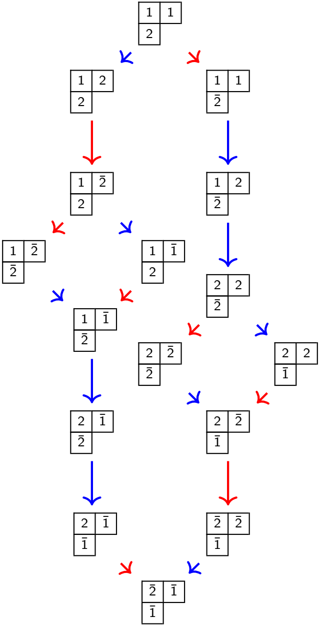
- $J_n \subseteq J_{\mathfrak{sp}_{2n}} \hookrightarrow J_{2n}$ [A–Tarighat–Torres, 22].

$$\begin{array}{rcll} \tilde{\iota} : J_{\mathfrak{sp}_{2n}} & \hookrightarrow & J_{2n} & \\ s_{[p,q]} & \mapsto & \tilde{s}_{[p,q]\sqcup[2n-q,2n-p]} = s_{[p,q]}s_{[2n-q,2n-p]}, & 1 \leq p \leq q < n, \\ s_{[p,n]} & \mapsto & s_{[p,2n-p]}, & 1 \leq p \leq n. \end{array}$$

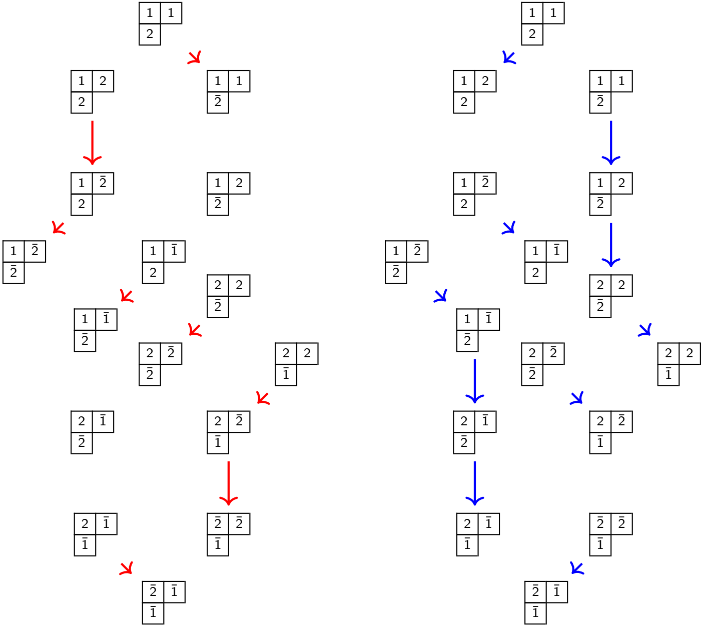
- $J_n \subseteq J_{\mathfrak{sp}_{2n}} \cong \tilde{J}_n := \tilde{\iota}(J_{\mathfrak{sp}_{2n}}) \subseteq J_{2n}$.

- $\tilde{J}_n$ is the virtual symplectic cactus group
  - generators: $\tilde{s}_{[p,q]\sqcup[2n-q,2n-p]}$, $1 \leq p \leq q < n$, and $s_{[p,2n-p]}$, $1 \leq p \leq n$,
  - $J_{\mathfrak{sp}_{2n}}$ symplectic cactus relations

Normal crystals: C_2 crystal $\mathrm{KN}(\lambda, 2)$, Kashiwara-Nakashima tableaux



Levi restrictions for $J \subseteq I$: $\mathrm{KN}_{\{2\}}(\lambda, 2)$ and $\mathrm{KN}_{\{1\}}(\lambda, 2)$



Schützenberger–Lusztig involution on crystals

- $B(\lambda)$ $\mathfrak{g}$-normal crystal with h.w. λ and u_λ^{high} and u_λ^{low}.
- The *Schützenberger–Lusztig involution* $\xi : B(\lambda) \to B(\lambda)$ is the unique set involution such that, for all $b \in B(\lambda)$, and $i \in I$,
  - $e_i \xi(b) = \xi f_{\theta(i)}(b)$
  - $f_i \xi(b) = \xi e_{\theta(i)}(b)$
  - $\text{wt}(\xi(b)) = w_0 \text{wt(b)}$

  where w_0 is the long element of the Weyl group W.

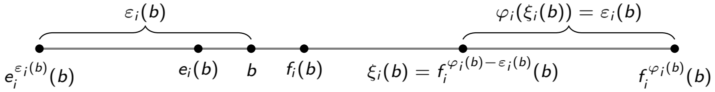
- Let $b = f_{j_r} \cdots f_{j_1}(u_\lambda^{high})$, for $j_r, \ldots, j_1 \in I$. Then
  - type A_{n-1}, $\xi(b) = e_{n-j_r} \cdots e_{n-j_1}(u_\lambda^{\text{low}})$, and $\text{wt}(\xi(b)) = w_0 \text{wt}(b)$, $w_0 \in \mathfrak{S}_n$.

    On $SSYT(\lambda, n)$, ξ coincides with *Schützenberger evacuation*.
  - type C_n, $\xi(b) = e_{j_r} \cdots e_{j_1}(u_\lambda^{\text{low}})$, and $\text{wt}(\xi(b)) = -\text{wt}(b)$.

    On $KN(\lambda, n)$, ξ coincides with *Santos symplectic evacuation*, 2021.

$J_{\mathfrak{g}}$-cactus action on a normal $\mathfrak{g}$-crystal

- The *partial Schützenberger–Lusztig involution* ξ_J is the Schützenberger–Lusztig involution ξ on the normal crystal B_J, for J any sub-diagram of I.
- When $J = \{i\}$, ξ_i is the Schützenberger–Lusztig involution on the i-strings $B_{\{i\}}$ and coincides the Weyl group $W_{\mathfrak{g}}$ action on the i-strings $B_{\{i\}}$: ξ_i, $i \in I$, satisfy the Weyl group relations.



Theorem

Halacheva, 2016 (Henriques–Kamnitzer $\mathfrak{g} = \mathfrak{gl}_n$, 2006) The map $s_J \mapsto \xi_J$, for all $J \subseteq I$ connected Dynkin sub-diagrams of I, defines an action of the cactus group $J_{\mathfrak{g}}$ on the set $B(\lambda)$; that is, the ξ_J satisfy the $J_{\mathfrak{g}}$ cactus relations, and the following is a group homomorphism

$$\begin{array}{rccc} \Phi_{\mathfrak{g}}: & J_{\mathfrak{g}} & \to & \mathfrak{S}_B \\ & s_J & \mapsto & \xi_J. \end{array}$$

- On $SSYT(\lambda, n)$, ξ_J is realized by J-partial Benkart-Sottile-Stroomer-reversal.
- On $KN(\lambda, n)$, ξ_J, $J = [p, n]$, is realized by the colourful J-partial symplectic reversal, A.-Tarighat-Torres, 2022.

Colourful partial symplectic reversal

$$P = \begin{array}{|c|c|c|c|}\hline 1 & 2 & 2 & \bar{1} \\ \hline 4 & 4 & \bar{3} \\ \cline{1-3} \bar{4} & \bar{2} & \bar{1} \\ \cline{1-3} \bar{3} \\ \cline{1-1}\end{array} \quad \in \mathrm{KN}((4,3,3,1),4).\ \mathrm{wt}(P) = (-1,1,-2,1).$$ How to compute $\xi_{[2,4]}$?

1. Symplectic rectification of $P_{[\pm 2,4]}$: apply symplectic jeu de taquin SJDT.

$$(U_0, P_{[\pm 2,4]}) = \begin{array}{|c|c|c|c|}\hline g & 2 & 2 & \\ \hline 4 & 4 & \bar{3} \\ \cline{1-3} \bar{4} & \bar{2} & \\ \cline{1-3} \bar{3} \\ \cline{1-1}\end{array} \overset{\mathrm{SJDT}}{\to} \begin{array}{|c|c|c|c|}\hline g & p & 2 & \\ \hline 4 & 4 & \bar{3} \\ \cline{1-3} \bar{4} & p' & \\ \cline{1-3} \bar{3} \\ \cline{1-1}\end{array} \overset{\mathrm{SJDT}}{\to} \begin{array}{|c|c|c|c|}\hline g & 2 & \bar{3} & \\ \hline 4 & 4 & p \\ \cline{1-3} \bar{4} & p' & \\ \cline{1-3} \bar{3} \\ \cline{1-1}\end{array} \overset{\mathrm{SJDT}}{\to} \begin{array}{|c|c|c|c|}\hline 2 & 4 & \bar{3} & \\ \hline 4 & g & p \\ \cline{1-3} \bar{4} & p' & \\ \cline{1-3} \bar{3} \\ \cline{1-1}\end{array}$$

$$\overset{\mathrm{SJDT}}{\to} \begin{array}{|c|c|c|c|}\hline r & 4 & \bar{3} & \\ \hline 2 & g & p \\ \cline{1-3} \bar{3} & p' & \\ \cline{1-3} r' \\ \cline{1-1}\end{array} \overset{\mathrm{SJDT}}{\to} \begin{array}{|c|c|c|c|}\hline 2 & 4 & \bar{3} & \\ \hline \bar{3} & g & p \\ \cline{1-3} r & p' & \\ \cline{1-3} r' \\ \cline{1-1}\end{array} = (\mathrm{rect} P_{[\pm 2,4]}, V) \Rightarrow \mathrm{rect} P_{[\pm 2,4]} = \begin{array}{|c|c|c|}\hline 2 & 4 & \bar{3} \\ \hline \bar{3} \\ \cline{1-1}\end{array},$$

$$V = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & g & p \\ \cline{1-3} r & p' & \\ \cline{1-3} r' \\ \cline{1-1}\end{array}, \; r < r' < g < p < p'.$$

2. Santos $\mathrm{evac}^{C_3}\, \mathrm{rect} P_{[\pm 2,4]}$. $\begin{array}{|c|c|c|}\hline 2 & 4 & \bar{3} \\ \hline \bar{3} \\ \cline{1-1}\end{array} \to \begin{array}{|c|c|c|}\hline & & 3 \\ \hline 3 & \bar{4} & \bar{2} \\ \hline\end{array} \overset{\mathrm{SJDT}}{\to} \begin{array}{|c|c|c|}\hline 3 & 3 & \bar{2} \\ \hline \bar{4} \\ \cline{1-1}\end{array} = \mathrm{evac}^{C_3}\, \mathrm{rect} P_{[\pm 2,4]}.$

3. Reversal of $P_{[\pm 2,4]}$. Replace $\mathrm{rect}P_{[\pm 2,4]}$ with $\mathrm{evac}^{C_3}\mathrm{rect}P_{[\pm 2,4]}$ in $(\mathrm{rect}P_{[\pm 2,4]}, V)$ and apply *RSJDT*.

$$(\mathrm{evac}^{C_3}(\mathrm{rect}P_{[\pm 2,4]}), V) = \begin{array}{cccc} 3 & 3 & \bar{2} & \square \\ \bar{4} & g & p & \\ r & p' & \square & \\ r' & & & \end{array} \xrightarrow{\text{RSJDT}} \begin{array}{cccc} r & 3 & \bar{2} & \square \\ 3 & g & p & \\ \bar{4} & p' & \square & \\ r' & & & \end{array} \xrightarrow{\text{RSJDT}} \begin{array}{cccc} 2 & 3 & \bar{2} & \square \\ 3 & g & p & \\ \bar{4} & p' & \square & \\ \bar{2} & & & \end{array}$$

$$\xrightarrow{\text{RSJDT}} \begin{array}{cccc} g & 2 & \bar{2} & \square \\ 3 & 3 & p & \\ \bar{4} & p' & \square & \\ \bar{2} & & & \end{array} \xrightarrow{\text{RSJDT}} \begin{array}{cccc} g & p & 3 & \square \\ 3 & 3 & \bar{3} & \square \\ \bar{4} & p' & \square & \\ \bar{2} & & & \end{array} \xrightarrow{\text{RSJDT}} \begin{array}{cccc} g & 2 & 3 & \square \\ 3 & 3 & \bar{3} & \square \\ \bar{4} & \bar{2} & \square & \\ \bar{2} & & & \end{array} = (U_0, \mathrm{reversal}^{C_3} P_{[\pm 2,4]})$$

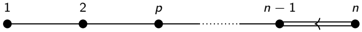
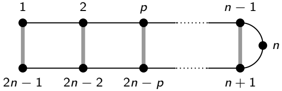
$$\Rightarrow \mathrm{reversal}^{C_3} P_{[\pm 2,4]} = \begin{array}{cccc} \square & 2 & 3 & \square \\ 3 & 3 & \bar{3} & \square \\ \bar{4} & \bar{2} & \square & \\ \bar{2} & & & \end{array}.$$

4. Replace $P_{[\pm 2,4]}$ with $\mathrm{reversal}^{C_3} P_{[\pm 2,4]}$ in P

$$\mathrm{reversal}^{C_4}_{[2,4]} P = \begin{array}{cccc} 1 & 2 & 3 & \bar{1} \\ 3 & 3 & \bar{3} & \\ \bar{4} & \bar{2} & \bar{1} & \\ \bar{2} & & & \end{array}$$

Baker, 2006, virtualization of KN tableau crystals

- Dynkin diagram folding $C_n \hookrightarrow A_{2n-1}$

C_n  $\longrightarrow$ A_{2n-1} folded 

- Baker virtualization is an injective map

$$E : \mathsf{KN}(\lambda, n) \hookrightarrow \mathsf{SSYT}(\lambda^A, n, \bar{n})$$

$$T = \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \bar5 \\ \hline \bar4 & \bar3 \\ \hline \bar3 & \\ \hline \end{array} \mapsto \quad E(T) = [\emptyset \leftarrow w(\psi(C_2)) \leftarrow w(\psi(C_1))] = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 4 & \bar5 \\ \hline 3 & \bar5 & \bar4 & \bar3 \\ \hline 5 & \bar4 & \bar2 & \\ \hline \bar5 & \bar3 & & \\ \hline \bar4 & \bar2 & & \\ \hline \bar3 & & & \\ \hline \end{array}.$$

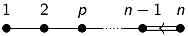
such that $E(K(\lambda, n))$ has crystal structure with $f_i^E = f_i^A f_{2n-i}^A$, $i < n$, and $f_n^E = (f_n^A)^2$, isomorphic to $\mathsf{KN}(\lambda, n)$ such that

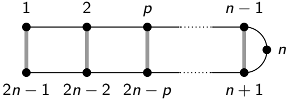
$$E f_i(T) = f_i^E E(T), \text{ for } T \in \mathsf{KN}(\lambda, n), 1 \leq i \leq n.$$

- $(E(T), Q_\lambda) = \mathrm{RSK} \circ \Psi(T) = (P(w_T), Q_\lambda)$ and

$$E^{-1} = \Psi^{-1} \mathrm{RSK}^{-1}_{|E(\mathsf{KN}(\lambda,n)) \times \{Q_\lambda\}}$$

where $\mathrm{RSK}^{-1}_{|\mathsf{KN}(\lambda,n) \times \{Q_\lambda\}}$ denotes the inverse of RSK restricted to $E(\mathsf{KN}(\lambda, n)) \times \{Q_\lambda\}$.

- C_n 

$\longrightarrow$ A_{2n-1} folded 

  The virtualization map E behaves very nicely with respect to Levi restriction!

$$\begin{array}{lcll} \mathrm{KN}_{[1,p]}(\lambda, n) & \underset{E}{\hookrightarrow} & \mathrm{SSYT}_{[1,p]\cup[2n-p,2n-1]}(\lambda^A, n, \bar{n}), & p < n, \\ \mathrm{KN}_{[p,n]}(\lambda, n) & \underset{E}{\hookrightarrow} & \mathrm{SSYT}_{[p,2n-p]}(\lambda^A, n, \bar{n}), & p \leq n \end{array}$$

$$\begin{array}{ccc} \mathrm{KN}(\lambda, n) & \xrightarrow{\;E\;} & \mathrm{SSYT}(\lambda^A, n, \bar{n}) \\ \xi^{C_n}_{[p,n]} \Big\downarrow \xi^{C_n}_{[p,q]} & & \xi^{A_{2n-1}}_{[p,2n-p]} \Big\downarrow \xi^{A_{2n-1}}_{[p,q]} \xi^{A_{2n-1}}_{[2n-q,2n-p]} \\ \mathrm{KN}(\lambda, n) & \xrightarrow[\;E\;]{} & \mathrm{SSYT}(\lambda^A, n, \bar{n}) \end{array}$$

- Virtualization of the symplectic cactus action of $J_{\mathfrak{sp}(2n,\mathbb{C})}$ on the crystal $\mathrm{KN}(\lambda, n)$

$$\begin{array}{ccc} J_{\mathfrak{sp}(2n,\mathbb{C})} & \xrightarrow{\;\Phi_{\mathfrak{sp}(2n,\mathbb{C})}\;} & \mathfrak{S}_{\mathrm{KN}(\lambda,n)} \\ \tilde{\imath} \Big\downarrow & & \Big\downarrow \imath \\ \widetilde{J}_{2n} & \xrightarrow[\;\widetilde{\Phi}^E_{\mathfrak{gl}(2n,\mathbb{C})}\;]{} & \mathfrak{S}_{E(\mathrm{KN}(\lambda,n))} \end{array} \qquad \widetilde{\Phi}^E_{\mathfrak{gl}(2n,\mathbb{C})}\tilde{\imath} = \imath\Phi_{\mathfrak{sp}(2n,\mathbb{C})}$$

The Berenstein–Kirillov group

The *Berenstein–Kirillov group* $\mathcal{BK}$ (*Gelfand-Tsetlin group*) [Berenstein, Kirillov, 1995], is the free group generated by the Bender-Knuth involutions t_i, for $i > 0$, modulo the relations they satisfy on `straight shaped semistandard Young tableaux`.

$$t_1 \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 2 & 2 & 3 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array} = \begin{array}{|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & 1 & 2 & 3 \\ \hline 2 & 2 & 2 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

Proposition

[Berenstein–Kirillov, 1995] *Let $\mathcal{BK}_n$ be the subgroup of $\mathcal{BK}$ generated by $t_1, \ldots, t_{n-1}$.*

- *The elements $q_{[1,1]}, \ldots, q_{[1,n-1]}$ are generators of $\mathcal{BK}_n$, $q_{[1,i]} = \xi_{[1,i]}$, $i \geq 1$.*
- $t_1 = q_{[1,1]}$, $\quad t_i = q_{[1,i-1]} q_{[1,i]} q_{[1,i-1]} q_{[1,i-2]}$, *for $i \geq 2$,* $\quad q_{[1,0]} := 1$.

- The following are group epimorphisms from J_n to $\mathcal{BK}_n$.
  1. $s_{[i,j]} \mapsto q_{[i,j]}$ [Chmutov–Glick–Pylyavskii 2016,2020].
  2. $s_{[1,j]} \mapsto q_{[1,j]}$ [Halacheva 2016, 2020].

The group $\mathcal{BK}_n$ is isomorphic to a quotient of J_n.

The known relations for the $\mathcal{BK}_n$ group

$$
\begin{aligned}
t_i^2 &= 1, \text{ for } i \geq 1 \\
t_i t_j &= t_j t_i, \text{ for } |i-j| > 1, \\
(t_1 q_{[1,i]})^4 &= 1, \text{ for } i > 2, \\
(t_1 t_2)^6 &= 1, \\
(t_i q_{[j,k-1]})^2 &= 1, \text{ for } \quad i+1 < j < k,
\end{aligned}
$$

where

$$
\begin{aligned}
q_{[1,i]} &:= t_1(t_2 t_1)\cdots(t_i t_{i-1}\cdots t_1), \text{ for } i \geq 1, \\
q_{[j,k-1]} &:= q_{[1,k-1]} q_{[1,k-j]} q_{[1,k-1]}, \quad \text{for } j < k.
\end{aligned}
$$

The type C_n Berenstein–Kirillov group $\mathcal{BK}^{C_n}$

Definition (A–Tarighat–Torres 2022)

The *symplectic Berenstein–Kirillov group* $\mathcal{BK}^{C_n}$, $n \geq 1$, is the free group generated by the $2n-1$ symplectic partial Schützenberger-Lusztig involutions

$$q^{C}_{[1,i]} =: \xi^{C_n}_{[1,i]},\ 1 \leq i < n, \quad \text{and} \quad q^{C}_{[i,n]} =: \xi^{C_n}_{[i,n]},\ 1 \leq i \leq n,$$

on straight shaped KN tableaux on the alphabet $[\pm n]$ modulo the relations they satisfy on those tableaux.

- [A–Tarighat–Torres 2022] The following is a group epimorphism from $J_{\mathfrak{sp}_{2n}}$ to $\mathcal{BK}^{C_n}$:

  $$s_{[1,j]} \mapsto q^{C_n}_{[1,j]},\ 1 \leq j < n, \qquad s_{[j,n]} \mapsto q^{C}_{[j,n]},\ 1 \leq j \leq n.$$

  $\mathcal{BK}^{C_n}$ is isomorphic to a quotient of $J_{\mathfrak{sp}_{2n}}$.

- [A–Tarighat–Torres 2022] For $n \geq 1$, the *symplectic Bender–Knuth involutions* $t^{C_n}_i$, $1 \leq i \leq 2n-1$, on straight shaped KN tableaux on the alphabet $[\pm n]$, are defined as

  $$t^{C_n}_i := q^{C_n}_{[1,i-1]} q^{C_n}_{[1,i]} q^{C_n}_{[1,i-1]} q^{C_n}_{[1,i-2]} = E^{-1} t^{A_{2n-1}}_i \tilde{t}^{A_{2n-1}}_{2n-i} E,\ \ 1 \leq i \leq n-1,$$

  $$\tilde{t}^{A_{2n-1}}_{2n-i} := q^{A_{2n-1}}_{[1,2n-1]} t^{A_{2n-1}}_i q^{A_{2n-1}}_{[1,2n-1]} = evac\ t^{A_{2n-1}}_i\ evac,\ \ 1 \leq i \leq n-1,$$

  $$t^{C_n}_{n-1+i} := q^{C_n}_{[n-i+1,n]} q^{C_n}_{[n-i+2,n]} = E^{-1} q^{A_{2n-1}}_{[n-(i-1),n+(i-1)]} q^{A_{2n-1}}_{[n-(i-2),n+(i-2)]} E,\ \ 1 \leq i \leq n.$$

  The symplectic Bender-Knuth involutions $t^{C_n}_i$, $1 \leq i \leq 2n-1$ also generate $\mathcal{BK}^{C_n}$.

Proposition (A–Tarighat–Torres 2022)

The symplectic Bender–Knuth involutions $t_i^{C_n} = 1$, $i = 1, \ldots, 2n-1$, *satisfy the following relations:*

1. $(t_i^{C_n})^2 = 1$, $i = 1, \ldots, 2n-1$.
2. $(t_{n+i-1}^{C_n} t_{n+j-1}^{C_n})^2 = 1$, $1 \leq i, j \leq n$.
3. $(t_i^{C_n} t_j^{C_n})^2 = 1$, $|i-j| > 1$, $1 \leq i, j < n$.
4. $(t_i^{C_n} t_{n+j-1}^{C_n})^2 = 1$, $i < n-j$.
5. $(t_i^{C_n} q_{[j,k-1]}^{C_n})^2 = 1$, $i+1 < j < k \leq n$.
6. $(t_i^{C_n} q_{[j,n]}^{C_n})^2 = 1$, $i+1 < j \leq n$.
7. $(t_{n+i-1}^{C_n} q_{[j,n]}^{C_n})^2 = 1$, $1 \leq i, j \leq n$.
8. $(t_{n+i-1}^{C_n} q_{[j,k-1]}^{C_n})^2 = 1$, $n-i+1 < j < k \leq n$.
9. $(t_1^{C_n} t_2^{C_n})^6 = 1$, $n \geq 3$.
10. $(t_{n-1}^{C_n} \cdots t_2^{C_n} t_1^{C_n} t_2^{C_n} \cdots t_{n-1}^{C_n} t_n^{C_n})^4 = 1$.