Schur functions: tableaux, determinant formulas and lattices paths

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Contents

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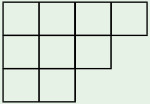
Example

$\lambda = (4, 3, 2)$, $|\lambda| = 9$, $l(\lambda) = 3$

$F^\lambda =$ 

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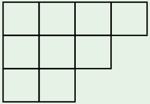
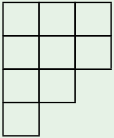
$F^\lambda \quad =$ 

$F^{\lambda'} \quad =$ 

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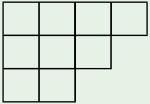
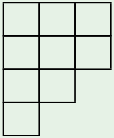
$F^\lambda =$ 

$F^{\lambda'} =$  $\lambda' = (3, 3, 2, 1)$.

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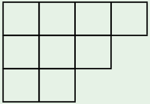
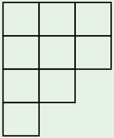
$F^\lambda =$ 

$F^{\lambda'} =$  $\lambda' = (3, 3, 2, 1)$.

- The partition λ' *conjugate* of λ is such that $F^{\lambda'}$ is obtained from F^λ by interchanging rows and columns.

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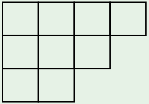
Example

$\lambda = (4, 3, 2)$, $l(\lambda) = 3$, $n = 7$



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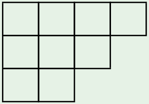

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2	3	4	6
4	4	6	
5	6		

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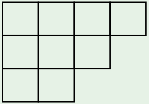

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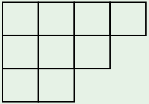

$$T = \begin{array}{|c|c|c|c|} \hline 2 & 3 & 4 & 6 \\ \hline 4 & 4 & 6 \\ \cline{1-3} 5 & 6 \\ \cline{1-2} \end{array}$$

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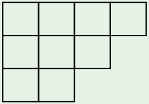

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- Equivalently
  $T : \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq r, 1 \leq j \leq \lambda_i\} \longrightarrow \{1, \ldots, n\}, \quad T(i, j) = T_{ij}.$

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- Let n be a fixed positive integer and $\mathbf{x} = (x_1, \ldots, x_n)$ a sequence of variables.

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- Let n be a fixed positive integer and $\mathbf{x} = (x_1, \ldots, x_n)$ a sequence of variables.
- The *monomial weight* of an n-semistandard tableau T of shape λ is the monomial of degree $|\lambda|$, in the variables $x_1, \ldots, x_n$,

$$X^T = \prod_{T_{ij}} x_{T_{ij}}$$

where T_{ij} runs over all the $|\lambda|$ entries of T.

Example

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$\alpha(T) = (0,1,1,3,1,3,0)$

$$X^T = x_1^0 x_2 x_3 x_4^3 x_5 x_6^3 x_7^0$$

Schur functions continued

- Given a partition λ with $l(\lambda) \leq n$, the Schur function $s_n(\lambda, \mathbf{x})$ associated with the partition λ is the homogeneous polynomial of degree $|\lambda|$ on the variables $x_1 \ldots, x_n$,
$$s_n(\lambda, \mathbf{x}) = \sum_T X^T$$
where T runs over all n-semistandard tableaux of shape λ.

Example

$n = 3$, $\lambda = (2, 1)$, $|\lambda| = 3$

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 \\ \cline{1-1}\end{array}\,.$$

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Then

$$s_3(\lambda, x_1, x_2, x_3) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + 2x_1 x_2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2.$$

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$n = 2$, $\qquad s_2(\lambda, x_1, x_2) = x_1^2 x_2 + x_1 x_2^2.$

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$$s_n(\lambda, \mathbf{x}) = \sum_T X^T$$

$$= \sum_{\alpha \text{ weak composition of } |\lambda| \text{ of length } \leq n} K_{\lambda,\alpha} x^{\alpha},$$

$x^{\alpha} = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$,

$K_{\lambda\alpha}$ is the Kostka number, the number of SSYTs of shape λ and type α.

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- It is enough to check for adjacent transpositions $(i\, i+1) \in S_n$,
$$(i \;\; i+1) s_n(\lambda, \mathbf{x}) = s_n(\lambda, \mathbf{x}).$$

- $K_{\lambda\,\tilde{\alpha}} = K_{\lambda\,\alpha}$, with $\tilde{\alpha} = (i\, i+1)\alpha$?

5. Bender-Knuth involution on semistandard tableaux

- Bender-Knuth involution is a bijection $\xi : T \longrightarrow Q$, on the set of semistandard tableaux of shape λ, such that the numbers of i's and $(i+1)$'s are swapped when passing from T to Q with all other multiplicities staying the same.

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$\xleftrightarrow[\xi_{23}]{}$

  - each column of T contains either an $i, i+1$ pair; exactly one of i, $i+1$; or neither.
  - the numbers in such pairs are called *fixed* and the other occurrences of i's or $i+1$'s are *free*.
  - if in a row one has k free i's followed by ℓ free $i+1$'s then replace them by ℓ free i's followed by k free $i+1$'s.

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Corollary

- $K_{\lambda\beta} = K_{\lambda\alpha}$, with β any permutation of α.
- The Schur function $s_n(\lambda, \mathbf{x})$ is a homogeneous symmetric function in $x_1, \ldots, x_n$.

6. The ring of symmetric functions

- Given λ with $\ell(\lambda) \leq n$,

$$m_\lambda(x_1 \ldots, x_n) = \sum_\alpha x^\alpha,$$

  α a permutation of λ.
  The ring of symmetric functions on variables the $x_1 \ldots, x_n$ is the vector space

$$\Lambda = \mathbb{C} m_\lambda.$$

- For each $k \geq 0$, let Λ^k be the vector space generated by $\{m_\lambda : |\lambda| = k\}$. The Schur functions s_λ with $|\lambda| = k$, on the variables $x_1, \ldots, x_n$, form a basis of the vector space Λ^k.
- The Schur functions s_λ on the variables $x_1, \ldots, x_n$, form an additive basis of the ring Λ.

7. Complete homogeneous symmetric functions and elementary symmetric functions

- row partition $\lambda = (m)$, $l(\lambda) = 1$

Example

$m = 3$, $n = 3$

$T = \boxed{1}\boxed{2}\boxed{2}$ $\quad Q = \boxed{1}\boxed{1}\boxed{3}$

$x^T = x_1 x_2 x_2$ $\quad x^Q = x_1 x_1 x_3.$

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$$h_3(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 + x_1^2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_1 x_3^2 + x_1 x_2 x_3 + x_2^2 x_3 + x_2 x_3^2$$

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$$T = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline\end{array} \qquad Q = \begin{array}{|c|c|c|}\hline 1 & 1 & 3 \\ \hline\end{array}$$

$$x^T = x_1 x_2 x_2 \qquad x^Q = x_1 x_1 x_3.$$

$$h_3(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 + x_1^2 x_3 + x_1^2 x_2 + x_1 x_2^2 + x_1 x_3^2 + x_1 x_2 x_3 + x_2^2 x_3 + x_2 x_3^2$$

- The m^{th} complete homogeneous symmetric function, in the variables $x_1, \ldots, x_n$, is the sum of all monomials of degree m in the variables $x_1, \ldots, x_n$

$$h_m(x_1, \ldots, x_n) := s_n((m), x_1, \ldots, x_n) = \sum_{1 \leq i_1 \leq \cdots \leq i_m \leq n} x_{i_1} x_{i_2} \ldots x_{i_m}$$

continued

- column partition $\lambda = (1, 1, \ldots, 1) = (1^m)$

Example

$m = 3$, $n = 4$

$$T = \begin{array}{|c|}\hline 1 \\ \hline 3 \\ \hline 4 \\ \hline \end{array} \qquad x^T = x_1 x_3 x_4.$$

$$e_3(x_1, x_2, x_3, x_4) = x_1 x_2 x_3 + x_1 x_2 x_4 + x_1 x_3 x_4 + x_2 x_3 x_4$$

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- The m^{th} elementary symmetric function, in the variables $x_1, \ldots, x_n$, is the sum of all monomials $x_{i_1} \ldots x_{i_m}$ for all strictly increasing sequences $1 \leq i_1 < i_2 < \cdots < i_m \leq n$

$$e_m(x_1, \ldots, x_n) := s_n((1^m), x_1, \ldots, x_n) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} x_{i_1} x_{i_2} \ldots x_{i_m}$$

8. The (classical) Jacobi-Trudi determinant formulas

- Let λ be a partition with $l(\lambda) = r \leq n$.
- The original definition of Schur function (and that Schur originally used)

$$s_n(\lambda, \mathbf{x}) = \frac{|x_j^{\lambda_i+n-i}|_{r\times r}}{|x_j^{n-i}|_{r\times r}}$$

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  - $s_n(\lambda, \mathbf{x}) = |h_{\lambda_j-j+i}(\mathbf{x})|_{\mathbf{r}\times\mathbf{r}}$, h-formula

$$= \begin{vmatrix} h_{\lambda_1}(x) & h_{\lambda_2-1}(x) & h_{\lambda_3-2}(x) & \dots & h_{\lambda_{r-1}-r}(x) & h_{\lambda_r-r+1}(x) \\ h_{\lambda_1+1}(x) & h_{\lambda_2}(x) & h_{\lambda_3-1}(x) & \dots & h_{\lambda_{r-1}-r+1}(x) & h_{\lambda_r-r+2}(x) \\ h_{\lambda_1+2}(x) & h_{\lambda_2+1}(x) & h_{\lambda_3}(x) & \dots & h_{\lambda_{r-1}-r+2}(x) & h_{\lambda_r-r+3}(x) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ h_{\lambda_1+r-1}(x) & h_{\lambda_2+r-2}(x) & h_{\lambda_3+r-3}(x) & \dots & h_{\lambda_{r-1}+1}(x) & h_{\lambda_r}(x) \end{vmatrix}$$

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Jacobi-Trudi continued

- $n = 2$, $\mathbf{x} = (x_1, x_2)$, $\lambda = (2, 1)$
  - h-formula

$$|h_{\lambda_j - j + i}(\mathbf{x})| = \begin{vmatrix} h_2 & h_0 \\ h_3 & h_1 \end{vmatrix} = h_2(x)h_1(x) - h_3(x).1$$

$$= (x_1^2 + x_2^2 + x_1x_2)(x_1 + x_2) - (x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3)$$

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Example

$n = 2$, $\lambda = (2, 1)$, $|\lambda| = 3$,

1	1
2	

1	2
2	

Jacobi-Trudi continued

- $n = 2$, $\mathbf{x} = (x_1, x_2)$, $\lambda = (2, 1)$
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Example

$n = 2$, $\lambda = (2, 1)$, $|\lambda| = 3$,

1	1
2	

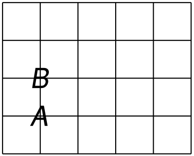
1	2
2	

$$s_2(\lambda, x) = x_1^2x_2 + x_1x_2^2 = \sum_{\text{2-semistandard tableaux } T \text{ of shape } \lambda} x^T.$$

9. The plane integer lattice

- The plane integer lattice is the set of integer points $\mathbb{Z}^2$ equipped with the usual component wise order relation

$$(a, b) \leq (c, d) \Leftrightarrow a \leq c,\ b \leq d.$$



$A \leq B$

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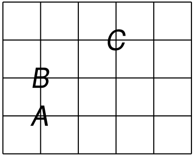

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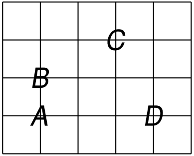

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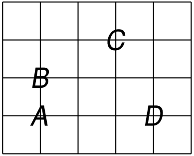

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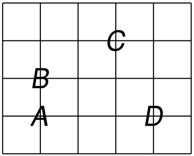

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B, D are not comparable

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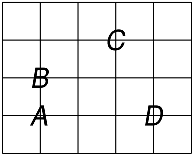

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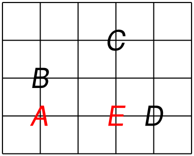

$$B \wedge D = A$$

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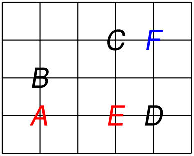

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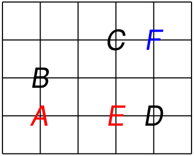

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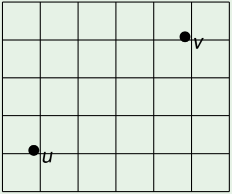
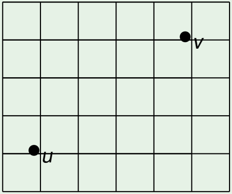
- The plane of integer points $\mathbb{Z}^2$ is a lattice with the relation $\leq$.

10. Lattice paths

- A lattice path from u to v, with $u \leq v$, is a sequence of adjacent points in the integer lattice, *i.e.* a sequence of unit horizontal (East) and vertical (North) steps in the positive direction, starting in u and ending in v.

Example





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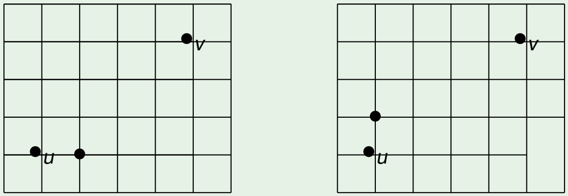

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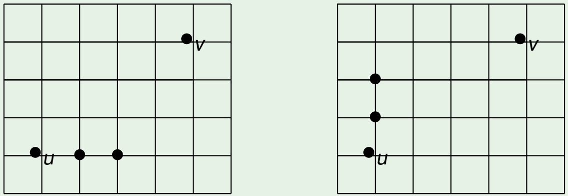

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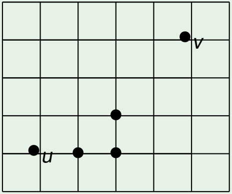

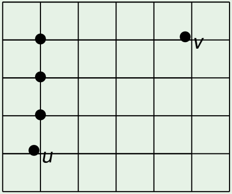

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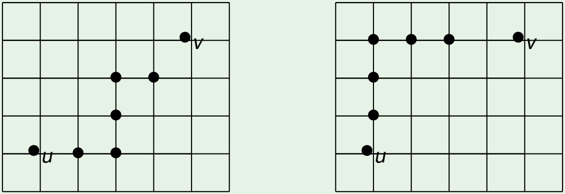

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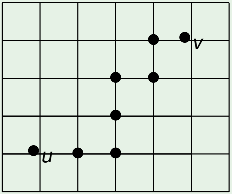

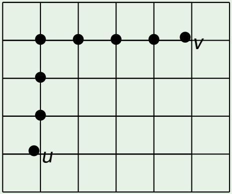

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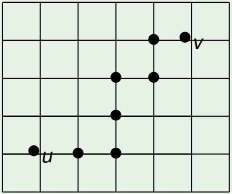

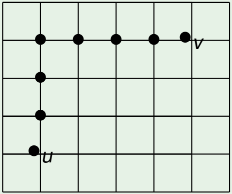

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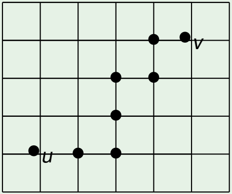

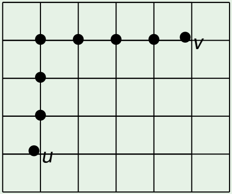

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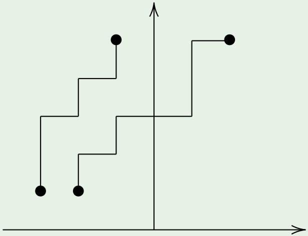
- $$\#\mathcal{P}(u,v) = \#\mathcal{P}(0, v-u) = \binom{v_1 - u_1 + v_2 - u_2}{v_1 - u_1}, \; u=(u_1,u_2) \leq v=(v_1,v_2).$$

$\mathcal{P}(u, v) = \emptyset, \quad u, v$ not comparable.

(continued)

- Two lattice paths are nonintersecting if they do not have any (lattice) point in common.

Example



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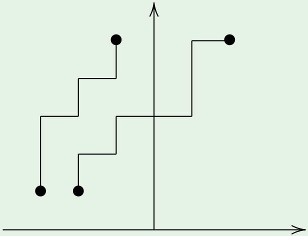

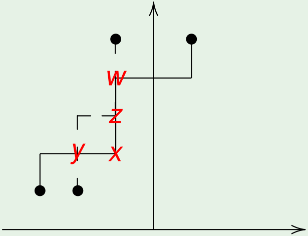

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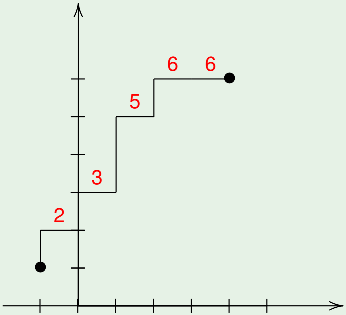

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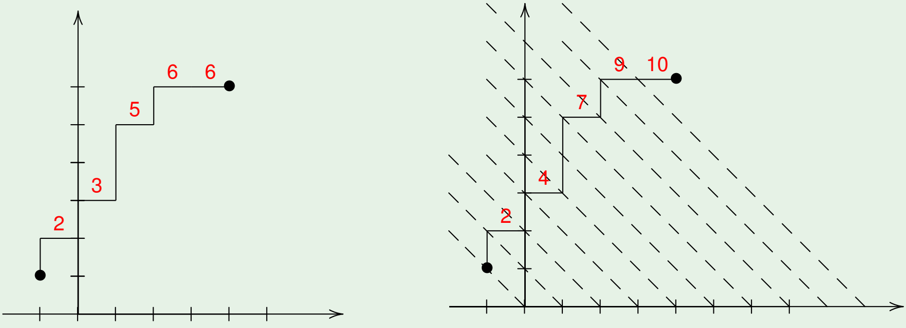

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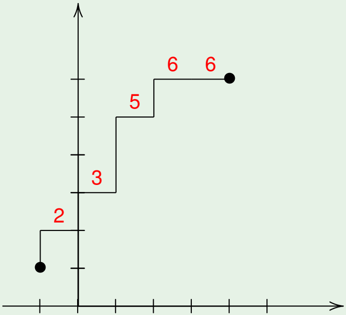
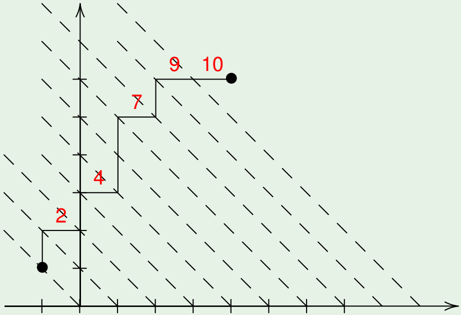


$w_h((i-1, j); (i, j)) = x_j$

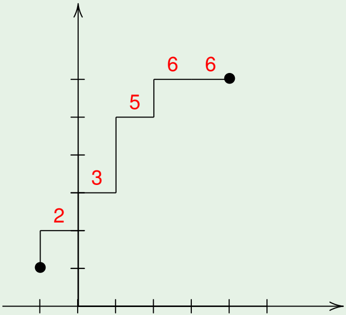


$w_e((i-1, j); (i, j)) = x_{i+j}$

- The weight $w(P)$ of a lattice path P is

$$w(P) = \prod_{a \text{ horizontal step} \in P} w(a).$$

Example



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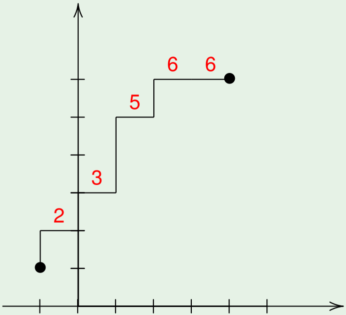
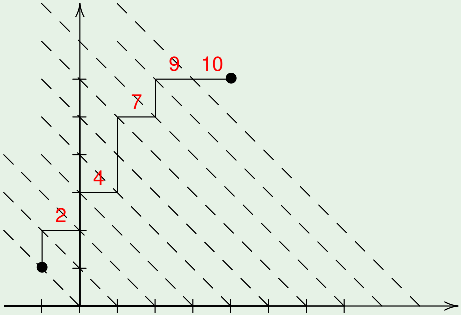


$w_h(P) = x_2 x_3 x_5 x_6^2$



$w_e(P) = x_2 x_4 x_7 x_9 x_{10}$

12. Weight-preserving bijection between tableaux and non intersecting lattice paths

Example

$n = 6$, $\lambda = (5, 3, 2)$

$$T = \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 5 & 6 & 6 \\ \hline 3 & 4 & 6 \\ \cline{1-3} 4 & 5 \\ \cline{1-2} \end{array} \longrightarrow$$

12. Weight-preserving bijection between tableaux and non intersecting lattice paths

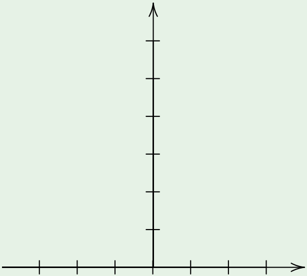
Example

$n = 6$, $\lambda = (5, 3, 2)$

$T =$

2	3	5	6	6
3	4	6		
4	5			

$\longrightarrow$



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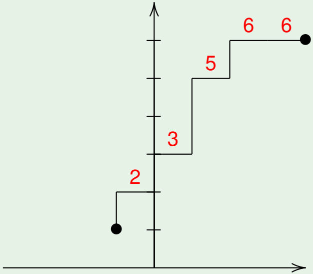

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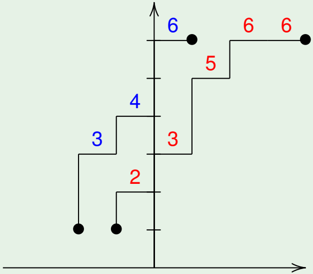

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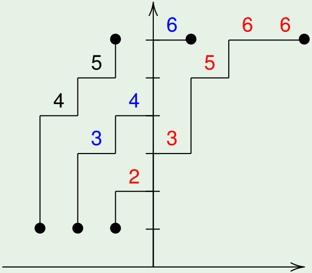

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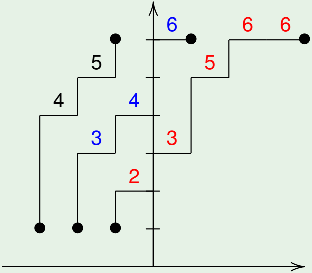

$$X^T = x_2 x_3^2 x_4^2 x_5^2 x_6^3$$

$$\mathbf{P} = (P_1, P_2, P_3)$$
$$w_h(\mathbf{P}) = w_h(P_1) w_h(P_2) w_h(P_3)$$

12. Weight-preserving bijection between tableaux and non intersecting lattice paths

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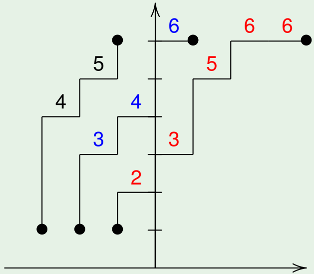

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$$= x_2 x_3 x_5 x_6{}^2 . x_3 x_4 x_6 . x_4 x_5$$

(continued)

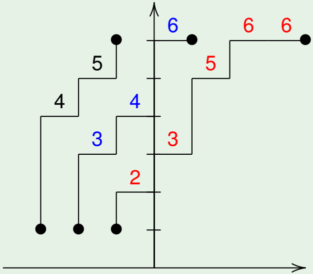
Example

$n = 6$, $\lambda = (5, 3, 2)$

$T =$

2	3	5	6	6
3	4	6		
4	5			

$\longrightarrow$



- Given a n-semistandard tableau of length $r \leq n$, the map

$$T \longrightarrow (P_1, P_2, \ldots, P_r),$$

where $(P_1, P_2, \ldots, P_r)$ is an r-tuple of nonintersecting lattice paths:

  - i-th row of $T \longrightarrow P_i$ from $u_i = (-i, 1)$ to $v_i = (\lambda_i - i, n)$, $i = 1, \ldots, r$, whose heights of horizontal steps are the entries in the i-th row.

(continued)

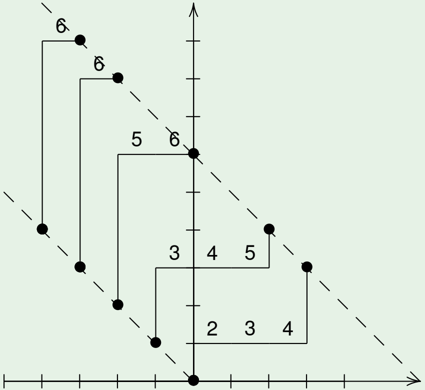
Example

$n = 6$, $\lambda = (5, 3, 2)$

$T =$

2	3	5	6	6
3	4	6		
4	5			

$\longleftrightarrow$



$X^T = x_2 x_3^2 x_4^2 x_5^2 x_6^3$

$\mathbf{Q} = (Q_1, Q_2, Q_3, Q_4, Q_5)$

$w_e(\mathbf{Q}) = w_e(Q_1) w_e(Q_2) w_e(Q_3) w_e(Q_4) w_e(Q_5) = x_2 x_3 x_4 . x_3 x_5 x_6 . x_5 x_6 . x_6 . x_6$

(continued)

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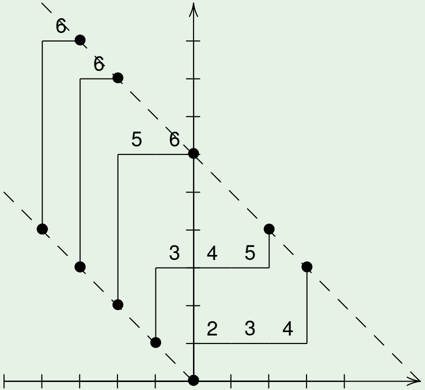

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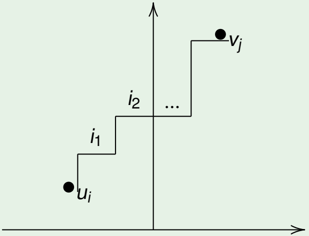
column i of $T \longrightarrow P_i$ from $u_i = (-i + 1, i - 1)$ to $(\lambda_i' - i + 1, n - \lambda_i' + i - 1)$

13. Complete and elementary homogeneous symmetric functions are weight generating functions of lattice paths between two points

Example

$u_i = (-i, 1)$, $v_j = (\lambda_j - j, n)$

$$\underbrace{\boxed{i_1}\boxed{i_2}\boxed{\ldots}\boxed{i_m}}_{\lambda_j - j + i} \longrightarrow$$



$P \in \mathcal{P}(u_i, v_j)$

$$h_{\lambda_j - j + i}(x_1, \ldots, x_n) = \sum_{1 \le i_1 \le \cdots \le i_m \le n} x_{i_1} x_{i_2} \ldots x_{i_m} = GF_h(\mathcal{P}(u_i, v_j))$$

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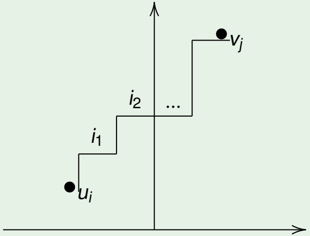

$$P \in \mathcal{P}(u_i, v_j)$$

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$$e_{\lambda'_j - j + i}(x_1, \ldots, x_n) = \sum_{1 \le i_1 < \cdots < i_m \le n} x_{i_1} x_{i_2} \ldots x_{i_m} = GF_e(\mathcal{P}(u_i, v_j))$$

7. The (classical) Jacobi-Trudi determinant formulas

- $|h_{\lambda_j-j+i}(\mathbf{x})|_{\mathbf{r}\times\mathbf{r}}$, h-formula

$$=\begin{vmatrix} h_{\lambda_1}(x) & h_{\lambda_2-1}(x) & h_{\lambda_3-2}(x) & \dots & h_{\lambda_{r-1}-r}(x) & h_{\lambda_r-r+1}(x) \\ h_{\lambda_1+1}(x) & h_{\lambda_2}(x) & h_{\lambda_3-1}(x) & \dots & h_{\lambda_{r-1}-r+1}(x) & h_{\lambda_r-r+2}(x) \\ h_{\lambda_1+2}(x) & h_{\lambda_2+1}(x) & h_{\lambda_3}(x) & \dots & h_{\lambda_{r-1}-r+2}(x) & h_{\lambda_r-r+3}(x) \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ h_{\lambda_1+r-1}(x) & h_{\lambda_2+r-2}(x) & h_{\lambda_3+r-3}(x) & \dots & h_{\lambda_{r-1}+1}(x) & h_{\lambda_r}(x) \end{vmatrix}$$

where we set $h_0 = 1$, $h_k = 0$, $k < 0$.

7. The (classical) Jacobi-Trudi determinant formulas

- $|h_{\lambda_j - j + i}(\mathbf{x})|_{\mathbf{r}\times\mathbf{r}}$, h-formula

$$
= \begin{vmatrix}
h_{\lambda_1}(x) & h_{\lambda_2-1}(x) & h_{\lambda_3-2}(x) & \dots & h_{\lambda_{r-1}-r}(x) & h_{\lambda_r-r+1}(x) \\
h_{\lambda_1+1}(x) & h_{\lambda_2}(x) & h_{\lambda_3-1}(x) & \dots & h_{\lambda_{r-1}-r+1}(x) & h_{\lambda_r-r+2}(x) \\
h_{\lambda_1+2}(x) & h_{\lambda_2+1}(x) & h_{\lambda_3}(x) & \dots & h_{\lambda_{r-1}-r+2}(x) & h_{\lambda_r-r+3}(x) \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
h_{\lambda_1+r-1}(x) & h_{\lambda_2+r-2}(x) & h_{\lambda_3+r-3}(x) & \dots & h_{\lambda_{r-1}+1}(x) & h_{\lambda_r}(x)
\end{vmatrix}
$$

  where we set $h_0 = 1$, $h_k = 0$, $k < 0$.

- $|e_{\lambda'_j - j + i}(\mathbf{x})|_{\lambda_1 \times \lambda_1}$, e-formula,
  where we set $e_0 = 1$, $e_k = 0$, $k > n$.

13. The Lindström-Gessel-Viennot involution

$m = r,\ \lambda_1$

$$|GF\mathcal{P}(u_i, v_j)|_{m\times m} \quad = \quad \sum_{\sigma\in S_m} sgn(\sigma) \prod_{i=1}^{m} GF(\mathcal{P}(u_i, v_{\sigma(i)})$$

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$$\mathcal{P}(u, v_\sigma) = \{P = (P_1, \ldots, P_m) : P_i \in \mathcal{P}(u_i, v_{\sigma(i)}),\ 1 \leq i \leq m\}$$
$$= \mathcal{P}(u_1, v_{\sigma(1)}) \times \cdots \times \mathcal{P}(u_m, v_{\sigma(m)})$$

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$$w(P) = \prod_{i=1}^{m} w(P_i)$$

$$GF(\mathcal{P}(u, v_\sigma)) = \sum_{P\in\mathcal{P}(u,v_\sigma)} w(P) = \prod_{i=1}^{m} GF(\mathcal{P}(u_i, v_{\sigma(i)})).$$

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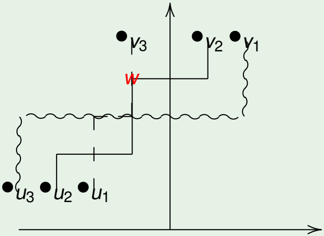
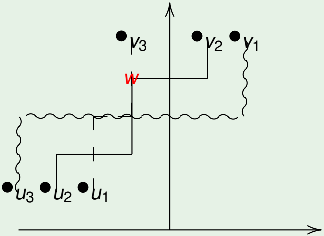
$$= \sum_{\sigma\in S_m,\ P\in\mathcal{P}(u,v_\sigma)} sgn(\sigma) w(P)$$

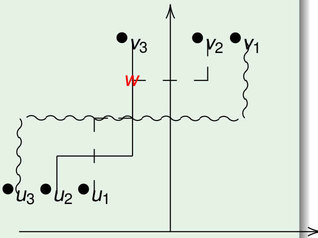
Example



Example



LGV involution



$(P_1, P_2, P_3) \in \mathcal{P}(u, v_\sigma)$

$\sigma = (1\ 3)$

$(P'_1, P'_2, P_3) \in \mathcal{P}(u, v_{\pi\sigma})$

$\pi = (2\,3)(1\,3)$

$$sgn(\pi) = -sign(\sigma)$$

- The LGV-involution is a weight -preserving but sign-reversing involution on the set of all those *m*-tuples that are intersecting.

- By the LGV involution all the intersecting *m*-tuples of paths will cancel and only the nonintersecting will survive. The associated permutation for such *m*-tuples is the identity. These *m* -paths correspond exactly to the *n*-semistandard tableaux of shape λ

$$\begin{aligned} |GF\mathcal{P}(u_i, v_j)|_{m\times m} &= \sum_{\sigma\in S_m} sgn(\sigma) \prod_{i=1}^{m} GF(\mathcal{P}(u_i, v_{\sigma(i)}) \\ &= \sum_{\sigma\in S_m,\ P\in\mathcal{P}(u,v_\sigma)} sgn(\sigma) w(P) \end{aligned}$$

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$$
\begin{aligned}
|GF\mathcal{P}(u_i, v_j)|_{m\times m} &= \sum_{\sigma\in S_m} sgn(\sigma) \prod_{i=1}^{m} GF(\mathcal{P}(u_i, v_{\sigma(i)}) \\
&= \sum_{\sigma\in S_m,\ P\in\mathcal{P}(u,v_\sigma)} sgn(\sigma) w(P) \\
&= \sum_{\text{nonintersecting } m\text{-tuple } P\in\mathcal{P}(u,v)} w(P)
\end{aligned}
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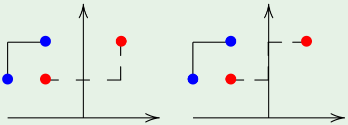
\begin{aligned}
|GF\mathcal{P}(u_i, v_j)|_{m\times m} &= \sum_{\sigma\in S_m} sgn(\sigma) \prod_{i=1}^{m} GF(\mathcal{P}(u_i, v_{\sigma(i)})) \\
&= \sum_{\sigma\in S_m,\ P\in\mathcal{P}(u,v_\sigma)} sgn(\sigma) w(P) \\
&= \sum_{\text{nonintersecting } m\text{-tuple } P \in \mathcal{P}(u,v)} w(P) \\
&= \sum_{n\text{-semistandard tableau } T \text{ of shape } \lambda} x^T \\
&= s_n(\lambda, x)
\end{aligned}
$$

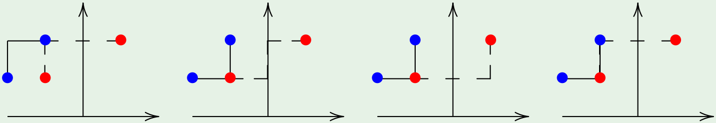
Example

$u_1{=}(-1,1)$, $v_1{=}(1,2)$, $u_2{=}(-2,1)$, $v_2{=}(-1,2)$

- $(P_1,P_2)\in\mathbb{P}(u_1,v_1)\times\mathbb{P}(u_2,v_2)$



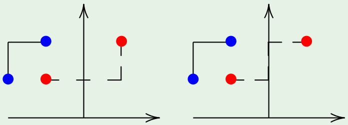
$$x_1^2x_2 + x_1x_2^2$$



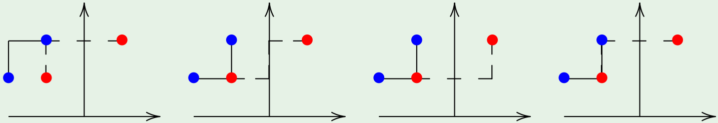
Example

$u_1=(-1,1)$, $v_1=(1,2)$, $u_2=(-2,1)$, $v_2=(-1,2)$

- $(P_1,P_2)\in\mathbb{P}(u_1,v_1)\times\mathbb{P}(u_2,v_2)$



$x_1^2x_2 + x_1x_2^2$



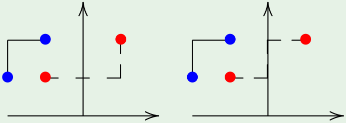
$$(x_1 + x_2)(x_1^2 + x_1x_2 + x_2^2) = x_1^2x_2 + x_1x_2^2 + x_1^3+x_1^2x_2+x_1x_2^2+x_2^3$$

- $(P_1,P_2)\in\mathbb{P}(u_1,v_2)\times\mathbb{P}(u_2,v_1)$

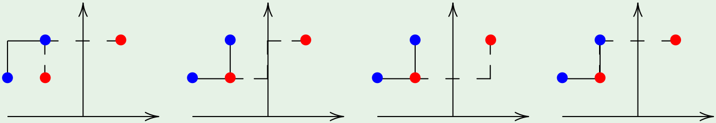
Example

$u_1{=}(-1{,}1)$, $v_1{=}(1{,}2)$, $u_2{=}(-2{,}1)$, $v_2{=}(-1{,}2)$

- $(P_1,P_2)\in\mathbb{P}(u_1,v_1)\times\mathbb{P}(u_2,v_2)$

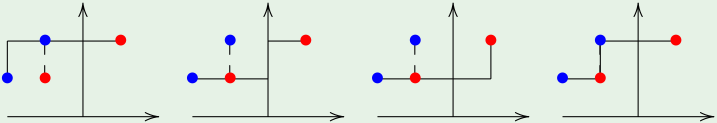


$$x_1^2x_2 + x_1x_2^2$$



$$(x_1 + x_2)(x_1^2 + x_1x_2 + x_2^2) = x_1^2x_2 + x_1x_2^2 + x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3$$

- $(P_1,P_2)\in\mathbb{P}(u_1,v_2)\times\mathbb{P}(u_2,v_1)$



$x_1^3+x_1^2x_2+x_1x_2^2+$

Jacobi-Trudi continued

- $n = 2$, $\mathbf{x} = (x_1, x_2)$, $\lambda = (2, 1)$
  - *h*-formula

$$|h_{\lambda_j - j + i}(\mathbf{x})| = \begin{vmatrix} h_2 & h_0 \\ h_3 & h_1 \end{vmatrix} = h_2(x)h_1(x) - h_3(x).1$$

$$= (x_1^2 + x_2^2 + x_1x_2)(x_1 + x_2) - (x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3)$$

$$= x_1^3 + 2x_1x_2^2 + 2x_1x_2^2 + x_2^3 - (x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3) = x_1^2x_2 + x_1x_2^2.$$

Jacobi-Trudi continued

- $n = 2$, $\mathbf{x} = (x_1, x_2)$, $\lambda = (2, 1)$
  - *h*-formula

$$|h_{\lambda_j - j + i}(\mathbf{x})| = \begin{vmatrix} h_2 & h_0 \\ h_3 & h_1 \end{vmatrix} = h_2(x)h_1(x) - h_3(x).1$$

$$= (x_1^2 + x_2^2 + x_1x_2)(x_1 + x_2) - (x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3)$$

$$= x_1^3 + 2x_1x_2^2 + 2x_1x_2^2 + x_2^3 - (x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3) = x_1^2x_2 + x_1x_2^2.$$

  - *e*-formula

$$|e_{\lambda'_j - j + i}(\mathbf{x})| = \begin{vmatrix} e_2 & e_0 \\ e_3 & e_1 \end{vmatrix} = e_2(x)e_1(x) - 0.1$$

$$= (x_1x_2)(x_1 + x_2) = x_1^2x_2 + x_1x_2^2.$$

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