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An analogue of the Robinson-Schensted-Knuth correspondence, growth diagrams, and non-symmetric Cauchy kernels

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In the name of God

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0. ABSTRACT

We prove a restriction of an analogue of the Robinson-Schensted-Knuth correspondence for semi-skyline augmented fillings, due to Mason, to multisets of cells of a staircase, in French convention, possibly truncated by a smaller staircase at the upper left end corner, or at the bottom right end corner. The condition to be imposed on the pairs of semi-skyline augmented fillings is that the pair of shapes, rearrangements of each other, satisfies an inequality in the Bruhat order, w.r.t. the symmetric group, where one shape is bounded by the reverse of the other. For semi-standard Young tableaux the inequality means that the pair of their right keys is such that one key is bounded by the Schützenberger's evacuation of the other. This bijection is then used to obtain an expansion formula of the non-symmetric Cauchy kernel, over staircases or truncated staircases, in the basis of Demazure characters of type A, and the basis of Demazure atoms. The expansion implies a Lascoux's expansion formula over arbitrary Ferrers shapes, when specialised to staircases or truncated staircases, and make explicit, in the latter, the Young tableaux in the Demazure crystal by interpreting Demazure operators via elementary bubble sorting operators acting on weak compositions.

Fomin has introduced a growth diagram presentation for the Robinson-Schensted correspondence and van Leeuwen and Roby developed it further. Krattenthaler uses the same approach to treat fillings of diagrams under certain restrictions. The growth diagram approach avoids insertion operations and makes symmetries transparent. We also introduce a growth diagram presentation for the analogue of the Robinson-Schensted-Knuth correspondence. This is done in a natural way via reverse semi-standard Young tableaux which are in bijection with semi-skyline augmented fillings. While our strategy, in the first part of our work, is to consider the smallest staircase containing the truncated shape, it is however observed by Lascoux that the Cauchy kernel expansion, over an arbitrary Ferrers shape, can be recovered from the expansion on staircases, by considering the biggest staircase inside the Ferrers shape. The strategy, in the last chapter, is then to use a convenient growth diagram to reduce the expansion, over arbitrary Ferrers shapes, to staircases by moving the cells outside the biggest staircase into inside. This is achieved by interpreting the action of crystal operators, on words, as matchings and slides of cells marked with 1 in the 01-filling of our growth diagram, and, then, convert the resulting information to semi-skyline augmented fillings. With these tools we give a combinatorial proof for the Lascoux's non-symmetric Cauchy kernel expansion over an arbitrary Ferrers shapes, in the cases, of one and two non-consecutive boxes above the staircase, and an idea in the general case. In particular, this affords another combinatorial proof for strict truncated staircases, considered in the first part of our work.

A previous expansion of the non-symmetric Cauchy kernel, over staircases, was given by Lascoux, based on the structure of double crystal graphs, and, by Fu and Lascoux, relying on Demazure operators properties. The expansion, over an arbitrary Ferrers shape, was derived by Lascoux algebraically.

0. RESUMO

Prova-se uma restrição de uma análoga da correspondência de Robinson-Schensted-Knuth, para preenchimentos semistandard de linhas de horizonte aumentadas, da autoria de Mason, a multiconjuntos de células em diagramas em escada, segundo a convenção Francesa, possivelmente truncados por escadas mais pequenas nos cantos superior esquerdo ou inferior direito. A condição imposta nos pares de linhas de horizonte aumentadas com preenchimento semistandard é que o par de formatos, rearranjos um do outro, satisfaça uma desigualdade na ordem de Bruhat, com respeito ao grupo simétrico, onde um formato é limitado pelo outro em ordem reversa. Para tableaux de Young semistandard, a desigualdade significa que o par de chaves à direita é tal que uma chave é limitada pela evacuação de Schützenberger da outra. Esta bijecção é seguidamente usada para obter uma expansão não simétrica do *kernel* de Cauchy, sobre diagramas em escada ou diagramas em escada truncados, na base dos caracteres de Demazure do tipo A, e na base dos átomos de Demazure. Esta expansão implica uma outra, da autoria de Lascoux, sobre diagramas de Ferrers arbitrários, quando especializada a diagramas em escada ou diagramas em escada truncados, e explicita, no último, os tableaux de Young no cristal de Demazure via operadores elementares de bolha actuando em vectores de entradas inteiras não negativas.

Fomin introduziu uma apresentação em diagramas de crescimento para a correspondência de Robinson-Schensted, e van Leeuwen e Roby desenvolveram-na. Krattenthaler usou a mesma abordagem para tratar preenchimentos de diagramas sujeitos a certas restrições. Os diagramas de crescimento evitam operações de inserção e tornam as simetrias transparentes. Nós também introduzimos nesta tese uma apresentação de diagramas de crescimento para a análoga da correspondência de Robinson-Schensted Knuth referida acima. Isto é feito naturalmente via tableaux de Young reversos os quais estão em bijecção com as linhas de horizonte aumentadas com preenchimentos semistandard. Enquanto, na primeira parte do nosso trabalho, a estratégia foi considerar o diagrama em escada mais pequeno contendo o diagrama truncado, é, no entanto, observado por Lascoux que a expansão não simétrica do *kernel* de Cauchy sobre diagramas de Ferrers arbitrários, pode ser reduzida ao caso da expansão dos diagramas em escada, considerando para esse efeito o maior diagrama em escada contido no diagrama de Ferrers. De acordo com esta observação, usamos agora diagramas de crescimento convenientes para reduzir a expansão sobre diagramas de Ferrers arbitrários a diagramas em escada, deslocando, para esse efeito, as células que estão fora da maior escada para o seu interior. Isto é alcançado interpretando a acção dos operadores de cristal em palavras como emparelhamentos e deslocamentos de células marcadas com 1 no preenchimento 0-1 do nosso diagrama de crescimento, e converter, em seguida, a informação resultante para linhas de horizonte aumentadas com preenchimento semistandard. Com estas ferramentas obtemos uma prova combinatória para a expansão não simétrica do *kernel* de Cauchy sobre diagramas de Ferrers, nos casos, de uma ou duas caixas não consecutivas em cima do diagrama em escada, e uma ideia no caso geral. Em particular, este método fornece uma outra prova combinatória para a expansão do *kernel* Cauchy sobre diagramas em escada truncados, considerados na primeira parte do trabalho.

A expansão não simétrica do *kernel* de Cauchy previamente obtida por Lascoux, sobre diagramas em escada, foi baseada na estrutura de grafos de cristal duplos, e a obtida por Fu e Lascoux apoiou-se em propriedades dos operadores de Demazure. No caso dos diagramas de Ferrers arbitrários, esta foi deduzida algebricamente por Lascoux.

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1. INTRODUCTION

The main definitions and notations that appear throughout the thesis are discussed in this chapter. Although the contents of this chapter is well known, we reproduce it here for the convenience of the reader. For more information about these objects, including proofs of elementary facts, see [4, 12, 30, 38, 49]. We start with objective and background of our research.

1.1 Background and objective

Let λ be a partition. Kashiwara [19, 20] has associated with λ a crystal $\mathfrak{B}^\lambda$, which can be realised in type A as a coloured directed graph whose vertices are all semi-standard Young tableaux (SSYTs) of shape λ with entries $\leq n$, and the edges are coloured with a colour i, for each pair of crystal operators f_i, e_i, such that there exists a coloured i-arrow from the vertex P to P' if and only if $f_i(P) = P'$, equivalently, $e_i(P') = P$, for $i = 1, \ldots, n-1$. For a given permutation w in the symmetric group $\mathfrak{S}_n$, the shortest in its class modulo the stabiliser of λ, the subset $\mathfrak{B}_{w\lambda} \subseteq \mathfrak{B}^\lambda$ is a certain subgraph called Demazure crystal, [21, 34] and the Demazure character corresponding to λ and w, is the sum of the weight monomials of the SSYTs in the Demazure crystal $\mathfrak{B}_{w\lambda}$.

Demazure characters (or key polynomials) are also defined through Demazure operators (or isobaric divided differences). They were introduced by Demazure [8] for all Weyl groups and were studied combinatorially, in the case of $\mathfrak{S}_n$, by Lascoux and Schützenberger [28, 32] who produce a crystal structure. We assume throughout $\mathbb{N}$ as the set of nonnegative integers. The action of the simple transpositions $s_i \in \mathfrak{S}_n$ on weak compositions in $\mathbb{N}^n$, by permuting the entries i and $i+1$, induces an action of $\mathfrak{S}_n$ on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ by considering weak compositions $\alpha \in \mathbb{N}^n$ as exponents of monomials $x^\alpha := x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$ [29], and defining $s_i x^\alpha := x^{s_i \alpha}$ as the transposition of x_i and x_{i+1} in the monomial x^α. If $f \in \mathbb{Z}[x_1, \ldots, x_n]$, $s_i f$ indicates the result of the action of s_i in each monomial of f. For $i = 1, \ldots, n-1$, one defines the linear operators π_i, $\widehat{\pi}_i$ on $\mathbb{Z}[x_1, \ldots, x_n]$ by

$$\pi_i f = \frac{x_i f - s_i(x_i f)}{x_i - x_{i+1}}, \qquad \widehat{\pi}_i f = (\pi_i - 1)f = \pi_i f - f, \tag{1.1}$$

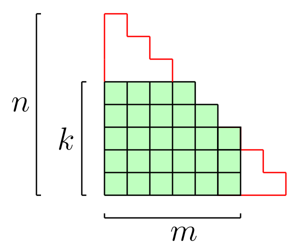
where 1 is the identity operator on $\mathbb{Z}[x_1, \ldots, x_n]$. These operators are called isobaric divided differences [29], and the first is the Demazure operator [8] for the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$.

The 0-Hecke algebra $H_n(0)$ of $\mathfrak{S}_n$, a deformation of the group algebra of $\mathfrak{S}_n$, can be faithfully realized either by its action on $\mathbb{Z}[x_1, \ldots, x_n]$ via isobaric divided difference operators $\{\pi_i : 1 \leq i < n\}$ or $\{\hat{\pi}_i : 1 \leq i < n\}$, or by the action on weak compositions in $\mathbb{N}^n$ via the elementary bubble sort operators, *i.e.* π_i is viewed as the operator which sorts the entries in positions i and $i+1$ in weakly increasing order. They are used to generate two kinds of key polynomials [32, 45], the Demazure characters [8, 18], and the Demazure atoms [40]. For $\alpha \in \mathbb{N}^n$, the key polynomial κ_α (resp. $\hat{\kappa}_\alpha$) is $\kappa_\alpha = \hat{\kappa}_\alpha = x^\alpha$, if α is a partition. Otherwise, $\kappa_\alpha = \pi_i \kappa_{s_i\alpha}$ (resp. $\hat{\kappa}_\alpha = \hat{\pi}_i \hat{\kappa}_{s_i\alpha}$), if $\alpha_{i+1} > \alpha_i$. The key polynomial κ_α lifts the Schur polynomial $s_{(\alpha_n, \ldots, \alpha_1)}$ when $\alpha_1 \leq \ldots \leq \alpha_n$, and then $\kappa_\alpha = s_{(\alpha_n, \ldots, \alpha_1)}$. It should be noticed that the action of Demazure operators on key polynomials is described by the action of the elementary bubble sort operators on weak compositions: $\pi_i \kappa_\alpha = \kappa_{s_i\alpha}$, if $\alpha_i > \alpha_{i+1}$, otherwise, $\pi_i \kappa_\alpha = \kappa_\alpha$ [45]. Both families of key polynomials $\{\kappa_\alpha : \alpha \in \mathbb{N}^n\}$, and $\{\hat{\kappa}_\alpha : \alpha \in \mathbb{N}^n\}$ form linear $\mathbb{Z}$-bases for $\mathbb{Z}[x_1, \ldots, x_n]$.

If $w' < w$ in the Bruhat order on the classes modulo the stabiliser of λ, $\mathfrak{B}_{w'\lambda} \subset \mathfrak{B}_{w\lambda}$. Setting $\widehat{\mathfrak{B}}_{w\lambda} := \mathfrak{B}_{w\lambda} \setminus \bigcup_{w'<w} \mathfrak{B}_{w'\lambda}$, one has the decomposition $\mathfrak{B}_{w\lambda} = \bigcup_{w' \leq w} \widehat{\mathfrak{B}}_{w'\lambda}$ [34]. A key tableau is a SSYT whose content is a rearrangement of the shape. Each component $\widehat{\mathfrak{B}}_{w\lambda}$ has exactly one key tableau, $key(w\lambda)$, with shape λ and content $w\lambda$. Lascoux and Schützenberger [32] have characterised $\widehat{\mathfrak{B}}_{w\lambda}$ as the set of those SSYTs whose right key is the unique key tableau in $\widehat{\mathfrak{B}}_{w\lambda}$, and defined the Demazure atom (or standard basis) $\widehat{\kappa}_{w\lambda}$ to be the sum of the weight monomials over $\widehat{\mathfrak{B}}_{w\lambda}$. As the sum of the weight monomials over all the crystal $\mathfrak{B}^\lambda$ gives the Schur polynomial s_λ, the Demazure atoms decompose the Schur polynomials. Specialising the combinatorial formula for the nonsymmetric Macdonald polynomial $\widehat{E}_\alpha(x; q; t)$ given in [15], by setting $q = t = 0$, implies that $\widehat{E}_\alpha(x; 0; 0)$ is the sum of the

weight monomials of all semi-skyline augmented fillings (SSAF) of shape α which are fillings of diagrams of weak compositions with positive integers, weakly decreasing upwards along columns, and the rows satisfy inversion conditions. These polynomials are also a decomposition of the Schur polynomial s_λ, with $\alpha^+ = \lambda$ the decreasing rearrangement of α. Semi-skyline augmented fillings are in bijection with semi-standard Young tableaux such that the content is preserved and the right key of the SSYT is the unique key with content the shape of the SSAF [39]. Hence, the Demazure atom $\widehat{\kappa}_\alpha$ and $\widehat{E}_\alpha(x;0;0)$ are equal [15, 40].

Mason shows [39] that semi-skyline augmented fillings also satisfy a variation of the Robinson-Schensted-Knuth algorithm which commutes with RSK and retains its symmetry. Semi-standard Young tableaux of shape λ and entries $\leq n$ decompose into subsets according to the right key. We see this RSK analogue as a refinement of the ordinary RSK where the right keys are provided. In chapters 2 and 4, we consider the following Ferrers diagram, in the French convention, $\lambda = (m^{n-m+1}, m-1, \dots, n-k+1)$, $\;1 \leq m \leq n$, $1 \leq k \leq n$, $\;n+1 \leq m+k$, shown in green colour below,



Theorem 10, in Chapter 2, exhibits a bijection between multisets of cells of λ and pairs of SSAFs whose shapes satisfy an inequality in the Bruhat order, in the symmetric group $\mathfrak{S}_\mathfrak{n}$ such that one shape is bounded by the reverse of the other. In particular, if $m+k = n+1$ then λ is a rectangle and it reduces to the ordinary RSK correspondence in the sense that the inequality on the right keys is relaxed. We then use this bijection, in Chapter 4, to give an expansion of the non-symmetric Cauchy kernel $\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}$ in the basis of Demazure characters, and the basis of Demazure atoms, where the product is over all the cells (i,j) of λ in French convention. The expansion is obtained in two steps: firstly, the bijection provides an expansion as a sum of products of Demazure atoms and the generating functions of SSYTs in the intersection of two Demazure crystals; secondly, interpreting the action of Demazure operators on key polynomials via the action of sorting operators on weak compositions, we compute the Demazure crystal resulting from that intersection, and, thereby, the key polynomial with that generating function. More precisely, one obtains the general formula

$$\prod_{\substack{(i,j)\in\lambda\\ k\leq m}}(1-x_iy_j)^{-1}=\sum_{\mu\in\mathbb{N}^k}\widehat{\kappa}_\mu(x)\kappa_{(0^{m-k},\alpha)}(y), \tag{1.2}$$

where $\alpha=(\alpha_1,\dots,\alpha_k)\in\mathbb{N}^k$ is such that, for each $i=k,\dots,1$, the entry α_i is the maximum element among the last $\min\{i,n-m+1\}$ entries of μ in reverse order, after deleting α_j, for $i<j\leq k$. The Demazure crystal $\mathfrak{B}_{(0^{m-k},\alpha)}$ consists of all SSYTs with entries $\leq m$, shape $(\mu^+,0^{m-k})$, and right key bounded by $key(0^{m-k},\alpha)$. If $m<k$, the formula is symmetrical, swapping in (1.2) x with y, and k with m.

If λ is a rectangle, $\alpha=\omega\mu^+$ and the classical Cauchy identity in the basis of Schur polynomials is recovered; and if λ is the staircases of length n, α is the reverse of μ, and the Lascoux's expansion, in Theorem 6 of [30], and in [11], of the non symmetric Cauchy kernel in the basis of Demazure characters and the basis of Demazure atoms, is also recovered. The proofs, given by Lascoux, for the latter expansion, use double crystal graphs in [30], and, in [11], with Fu, is based on algebraic properties of isobaric divided differences. For truncated staircases, the expansion (1.2) implies Lascoux's formula in Theorem 7 of [30], and makes explicit the SSYTs of the Demazure crystal.

Fomin has introduced a growth diagram presentation to the Robinson-Schensted correspondence, and van Leeuwen and Roby developed it further. Krattenthaler uses the same approach to treat fillings of diagrams under certain restrictions. The growth diagram approach avoids insertion operations and makes symmetries transparent. In this chapter and in the next, we introduce a growth diagram presentation for the analogue of the Robinson-Schensted-Knuth correspondence, based on the reverse Robinson-Schensted-Knuth correspondence for reverse semi-standard Young tableaux. Reverse semi-standard Young tableaux are in bijection with semi-skyline augmented fillings. While, in Chapter 4, the strategy is to consider the smallest staircase containing the truncated staircase, it is, however, observed by Lascoux that the Cauchy kernel expansion, over an arbitrary Ferrers shape, can be recovered from the staircases by considering the biggest staircase inside the Ferrers shape. The strategy, in the last chapter, is to use our growth diagram approach to reduce the expansion over arbitrary Ferrers shapes to the staircase, by moving the cells outside the biggest staircase into inside. This is achieved by interpreting crystal operators on words as matching and slides of cells marked with 1 in the 01-filling of the growth diagram, and then convert the resulting information to semi-skyline augmented fillings. With these tools we give a combinatorial proof for the Lascoux's non-symmetric Cauchy kernel expansion over an arbitrary Ferrers shapes, in the cases, of one and two non-consecutive boxes above the staircase, and an idea in the general case. The expansion over an arbitrary Ferrers shape was derived by Lascoux algebraically.

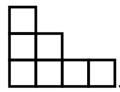
The objective of our research is to give a combinatorial proof for the Lascoux's non-symmetric Cauchy kernel expansion for some Ferrers shapes, (Theorem 7 in [30]), and, in particular, to give explicitly the Young tableaux in the expansion formula over truncated staircases.

The original results of this thesis are included in two articles: one published [3], and the other submitted although available electronically [1]. The results of the last chapter is the content of an article in preparation [2].

1.2 Semi-standard and reverse semi-standard Young tableaux

Let $\mathbb{N}$ denote the set of non-negative integers. Fix a positive integer n, and define $[n]$ as the set $\{1,\dots,n\}$. A weak composition $\gamma=(\gamma_1\dots,\gamma_n)$ is a vector in $\mathbb{N}^n$. We call $\sum_{i=1}^n \gamma_i$ the weight of γ, denoted $|\gamma|$. If $\gamma_i=\cdots=\gamma_{i+k-1}$ then we also write $\gamma=(\gamma_1\dots,\gamma_{i-1},\ \gamma_i^k,\gamma_{i+k},\ \dots,\gamma_n)$. We often concatenate weak compositions $\alpha\in\mathbb{N}^r$ and $\beta\in\mathbb{N}^s$, with $r+s=n$, to form the weak composition $(\alpha,\beta)=(\alpha_1,\dots,\alpha_r,\ \beta_1,\dots,\beta_s)\in\mathbb{N}^n$. When there is no danger of confusion we also use the notation $\gamma_1\dots\gamma_n$ for the weak composition $\gamma=(\gamma_1,\dots,\gamma_n)$. A weak composition γ whose entries are in weakly decreasing order, that is, $\gamma_1\geq\cdots\geq\gamma_n$, is said to be a partition. Every weak composition γ determines a unique partition γ^+ obtained by arranging the entries of γ in weakly decreasing order. A partition $\lambda=(\lambda_1,\dots,\lambda_n)$ is identified with its Young diagram (or Ferrers shape) $dg(\lambda)$ in French convention, an array of left-justified cells with λ_i cells (or boxes) in row i from the bottom, for $1\leq i\leq n$. The cells are located in the diagram $dg(\lambda)$ by their row and column indices (i,j), where $1\leq i\leq n$ and $1\leq j\leq\lambda_i$. The number $\ell(\lambda)$ of positive entries of λ is said to be the length of the partition λ. If the context does not require the number of the entries of λ, we do not distinguish λ and $(\lambda_1,\dots,\lambda_{\ell(\lambda)})$, and write $\lambda=(4,2,1,0)=(4,2,1)$. In this sense we identify the null partition $(0,\dots,0)$ with the empty partition

() or empty tableau, denoted $\emptyset$. The conjugate of the partition λ is the partition $\bar{\lambda} = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1})$ where λ'_j is the length of the j-th column in the Ferrers diagram $dg(\lambda)$. For instance the partition $\lambda = (4, 2, 1, 0)$ has weight $|\lambda| = 4 + 2 + 1 = 7$ and $\ell(\lambda) = 3$, and its Young diagram is



The conjugate of the partition $\lambda = (4, 2, 1, 0)$ is $\bar{\lambda} = (3, 2, 1, 1)$.

We define a partial order $\subseteq$ on partitions by containment of their Ferrers diagrams. If $\mu = (\mu_1, \mu_2, \ldots, \mu_n) \in \mathbb{N}^n$ and $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{N}^n$, then $\mu \cup \nu$ is the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $\lambda_i = \max\{\mu_i, \nu_i\}$ for $i = 1, 2, \ldots, n$. The intersection $\mu \cap \nu$ of two partitions μ and ν, $\mu \cap \nu$, is the partition $\rho = (\rho_1, \rho_2, \ldots, \rho_n)$, where $\rho_i = \min\{\mu_i, \nu_i\}$ for $i = 1, 2, \ldots, n$.

Let $\mathcal{A}$ be a finite completely ordered alphabet. A word in this alphabet is a finite sequence of letters in $\mathcal{A}$, and its content is the weak composition $(\alpha_1, \ldots, \alpha_n)$ where α_i is the multiplicity of the letter a_i in the word. A filling of shape λ in the alphabet $\mathcal{A}$ is a map $T : dg(\lambda) \to \mathcal{A} = \{a_1 < \cdots < a_n\}$. A *semi-standard Young tableau* (SSYT) T of shape $sh(T) = \lambda$, in the alphabet $\mathcal{A}$, is a filling $T : dg(\lambda) \to \mathcal{A}$, which is weakly increasing in each row from left to right and strictly increasing up in each column. Let SSYT_n denote the set of all semi-standard Young tableaux in the alphabet $[n]$. The column word of $T \in \mathrm{SSYT}_n$, $col(T)$, is the word, over the alphabet $[n]$, which consists of the entries of each column, read top to bottom and left to right. The content or weight of $T \in \mathrm{SSYT}_n$ is the content or weight of its column word in the alphabet $[n]$, denoted $c(T)$. Below, T is a SSYT in the alphabet $[7]$, $sh(T) = (4, 2, 1)$, $c(T) = (2, 2, 1, 1, 0, 1, 0)$ and $col(T) = 4216123$.

$$T = \begin{array}{|c|c|c|c|} \cline{1-1} 4 \\ \cline{1-2} 2 & 6 \\ \hline 1 & 1 & 2 & 3 \\ \hline \end{array}\,.$$

Sometimes we forget the cells and write only the filling of the tableau T.

$$T = \begin{array}{cccc} 4 & & & \\ 2 & 6 & & \\ 1 & 1 & 2 & 3. \end{array}$$

A skew shape (or a skew-diagram) is a pair of partitions (λ, μ) such that the Young diagram of λ contains the Young diagram of μ; it is denoted by λ/μ. A semi-standard Young tableau of skew shape λ/μ (or skew semi-standard Young tableau) is the filling of the cells in the Young diagram of λ not in the Young diagram of μ which is weakly increasing in each row from left to right and strictly increasing up in each column.

Example 1. *A SSYT T of skew shape $\lambda/\mu = (4,3,1)/(2,1)$*

$$\begin{array}{cccc} \boxed{3} & & & \\ & \boxed{3} & \boxed{4} & \\ & & \boxed{2} & \boxed{3} \end{array}$$

A reverse semi-standard Young tableau (RSSYT) of shape λ is a filling of a Ferrers diagram of shape λ such that the entries in each row are weakly decreasing from left to right, and strictly decreasing from bottom to top.

Example 2. *The reverse semi-standard Young tableau $\widetilde{T}$ with shape $sh(\widetilde{T}) = (4,2,1)$ and content $c(\widetilde{T}) = (1,3,1,1,1)$.*

$$\widetilde{T} = \begin{array}{cccc} 1 & & & \\ 3 & 2 & & \\ 5 & 4 & 2 & 2 \end{array}$$

1.3 Key tableau

A key tableau is a semi-standard Young tableau such that the set of entries in the $(j+1)^{th}$ column is a subset of the set of entries in the j^{th} column, for all j. There is a bijection in [45] between weak compositions in $\mathbb{N}^n$ and key tableaux in the alphabet $[n]$, given by $\gamma \to key(\gamma)$, where $key(\gamma)$ is the key such that for all $j \in [n]$, the first γ_j columns contain the letter j. The inverse map is defined by sending the key tableau to its content. Any key tableau is of the form $key(\gamma)$ with γ its content and γ^+ the shape. When $\gamma = \gamma^+$ one obtains the key tableau of shape and content γ, called Yamanouchi tableau of shape γ, that is the SSYT with all entries equal to i in row i, for all $1 \leq i \leq \ell(\gamma)$.

Example 3. *Let* $\gamma = (3,0,1,2,1)$, *then* $\gamma^+ = (3,2,1,1,0)$

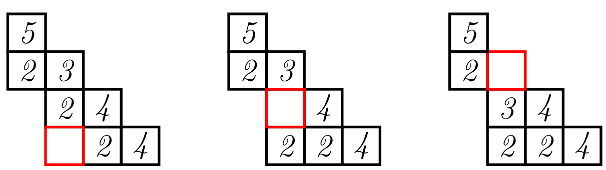
$$key(\gamma) = \begin{array}{lll} 5 & & \\ 4 & & \\ 3 & 4 & \\ 1 & 1 & 1 \end{array} \qquad and \qquad key(\gamma^+) = \begin{array}{lll} 4 & & \\ 3 & & \\ 2 & 2 & \\ 1 & 1 & 1. \end{array}$$

1.4 Schützenberger's evacuation

In this section we follow close [12]. To define the evacuation of a tableau T, we need the definition of *jeu de taquin*.

jeu de taquin.
Given a skew tableau T of skew shape λ/μ, pick an empty cell, denoted c, that is in μ and can be added to λ/μ; what this means is that c is in μ and must share at least one edge with some cell in T, and the result of adding it to T must also be a skew shape. Slide the number from its neighbouring cell into c; if c has neighbours both to its right and above, then pick the smallest of these two numbers, and in the case of equality choose the above number. (This rule preserves the property of increasing rows and columns in the SSYT.) If the cell that just has been emptied has no neighbour to its right or above, then the slide is completed. Otherwise, slide a number into that cell according to the same rule as before, and continue this manner until the slide is completed. After this transformation, the resulting tableau is still a skew (or possibly straight, that is, the inner shape $\mu = \emptyset$) tableau.

Example 4.



For any word $w = x_1x_2\dots x_r$ in the alphabet $[n]$, let $w^* := x_r^* \dots x_2^* x_1^* = n+1-x_r \dots n+1-x_1$, where $x_i^* = n+1-x_i$. We consider the anti-isomorphism on words over the alphabet $[n]$, $w \longrightarrow w^*$.

Schützenberger's evacuation.
Given a tableau T in the alphabet $[n]$, we construct a dual tableau, denoted $evac(T)$, on the alphabet $[n]$ as follows. Remove the entry, say x, from the

lower left corner of T, and play *jeu de taquin* on the skew tableau that is left. We obtain a tableau, denoted ΔT, whose diagram is the diagram of T with one box removed. Put the letter x^* in this removed box from T. Repeat this algorithm on ΔT, getting the smaller tableau $\Delta^2 T$, and putting y^* in the box removed from ΔT, where y is the letter in the lower corner of ΔT. Continue until all the entries have been removed. This happens after $|\lambda|$ steps that is $\Delta^{|\lambda|}(T) = \emptyset$. The Young diagram of T has been filled with the duals of the letters in T, and the result is the $evac(T)$. This procedure of constructing $evac(T)$ is called the evacuation of T.

Example 5. *The evacuation of the tableau T:*

$$\underset{T}{\begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 2 & 2 \\ \cline{1-2} 1 & 1 & 2 \\ \hline\end{array}} \quad \underset{\Delta T}{\begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 2 & 1^* \\ \cline{1-2} 1 & 2 & 2 \\ \hline\end{array}} \quad \underset{\Delta^2 T}{\begin{array}{|c|c|c|}\hline 1^* \\ \cline{1-1} 3 & 1^* \\ \cline{1-2} 2 & 2 & 2 \\ \hline\end{array}} \quad \underset{\Delta^3 T}{\begin{array}{|c|c|c|}\hline 1^* \\ \cline{1-1} 3 & 1^* \\ \cline{1-2} 2 & 2 & 2^* \\ \hline\end{array}} \quad \underset{\Delta^4 T}{\begin{array}{|c|c|c|}\hline 1^* \\ \cline{1-1} 3 & 1^* \\ \cline{1-2} 2 & 2^* & 2^* \\ \hline\end{array}} \quad \underset{\Delta^5 T}{\begin{array}{|c|c|c|}\hline 1^* \\ \cline{1-1} 2^* & 1^* \\ \cline{1-2} 3 & 2^* & 2^* \\ \hline\end{array}} \quad \underset{\Delta^6 T = \emptyset}{\begin{array}{|c|c|c|}\hline 1^* \\ \cline{1-1} 2^* & 1^* \\ \cline{1-2} 3^* & 2^* & 2^* \\ \hline\end{array}}$$

$$\underset{evac(T)}{\begin{array}{|c|c|c|}\hline 3 \\ \cline{1-1} 2 & 3 \\ \cline{1-2} 1 & 2 & 2 \\ \hline\end{array}}$$

Proposition 1. *Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ then $key(\alpha_n, \ldots, \alpha_1) = evac(key(\alpha))$.*

Proof. We prove by induction on $|\alpha|$. As usual let $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in \mathbb{N}^n$, where 1 is in the position i, for some $1 \le i \le n$. If $|\alpha| = 1$, $\alpha = e_i$, for some $1 \le i \le n$. Since $evac(key(e_i)) = n+1-i = key(e_{n+1-i})$, the statement holds where $|\alpha| = 1$.

Let $m \ge 1$ and suppose that the statement holds for $|\alpha| < m$. Now consider a weak composition α with $|\alpha| = m$. Let α_i be the first non-zero entry of α, that is $\alpha = (0^{i-1}, \alpha_i, \ldots, \alpha_n)$. So i appears in the bottom left corner of $key(\alpha)$. We play *jeu de taquin* on the bottom left corner of $key(\alpha)$. The chosen neighbours will be the right neighbours with entry i, up to the column α_i. Since each column is contained in the previous one, the rest of the chosen neighbours are in the column α_i. Hence the removed box will appear in the top of the column α_i and $\Delta(key(\alpha))$ is still a key tableau, $\Delta(key(\alpha)) = key(\alpha')$, where $\alpha' = (0^{i-1}, \alpha_i - 1, \ldots, \alpha_n)$, with $|\alpha'| = m-1$. By induction hypothesis $evac(key(\alpha')) = key(\alpha_n, \ldots, \alpha_i - 1, 0^{i-1})$. The $evac(key(\alpha))$ is equal to the $evac(key(\alpha')) = key(\alpha_n, \ldots, \alpha_i - 1, 0^{i-1})$ with one box on the top of the column α_i with the entry $n+1-i$ and this is exactly $key(\alpha_n, \ldots, \alpha_i, 0^{i-1})$.

$\square$

For example in the Example 5, $T = key(2,3,1)$, $\Delta(T) = key(1,3,1)$, $\Delta^2(T) = key(0,3,1)$, $\Delta^3(T) = key(0,2,1)$, $\Delta^4(T) = key(0,1,1)$, $\Delta^5(T) = key(0,0,1)$, $\Delta^6(T) = \emptyset$ and $evac(T) = key(1,3,2)$.

1.5 Bruhat orders on $\mathfrak{S}_n$

Let $\mathfrak{S}_n$ denote the symmetric group on n elements. An element $\sigma \in \mathfrak{S}_n$ permutes $\{1, \ldots, n\}$ by mapping $i \to \sigma(i)$. This permutation will be written in one-line notation $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$. The length of a permutation $\sigma \in \mathfrak{S}_n$, denoted by $\ell(\sigma)$, is the cardinality of the set of its inversions

$$I(\sigma) = \{(i,j) : 1 \le i < j \le n,\ \sigma(i) > \sigma(j)\}.$$

Example 6. $\sigma = 2314 \in \mathfrak{S}_4$ *then* $I(\sigma) = \{(1,3),(2,3)\}$

The group $\mathfrak{S}_n$ is generated by the simple transpositions $\{s_1, \ldots, s_{n-1}\}$, where s_i is the permutation interchanging i and $i+1$, and fixing all other elements. The simple transpositions satisfy the Coxeter relations

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i, \\ s_i s_j &= s_j s_i && \text{if } |i-j| > 1, \text{ and} \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} && \text{for } 1 \le i \le n-2. \end{aligned} \tag{1.3}$$

The two last are called, respectively, commutation and braid relations. Note that $\ell(s_i) = 1$, for $1 \le i \le n$. Since the symmetric group is generated by simple transpositions and $\ell(s_i\sigma) = \ell(\sigma)\pm 1$, for any $\sigma \in \mathfrak{S}_n$ and s_i, $1 \leq i \leq n$. Any permutation $\sigma \in \mathfrak{S}_n$ can be written as a product of at least $\ell(\sigma)$ simple transpositions. $\sigma = s_{i_\ell} \ldots s_{i_1}$ for some $\{i_1, \ldots, i_\ell\}$, where $\ell \geq \ell(\sigma)$.

Definition 1. *If $\sigma = s_{i_\ell} \ldots s_{i_1}$ where $\ell = \ell(\sigma)$, then $s_{i_\ell} \ldots s_{i_1}$ is a reduced decomposition of σ. In this case, we say that the sequence of indices $i_\ell, \ldots, i_1$ is a reduced word for σ.*

The unique element of maximal length in $\mathfrak{S}_n$ is denoted by $\omega := n \ldots 21$. It is a well known fact that any two reduced decompositions for σ are connected by a sequence of the last two Coxeter relations (1.3). Next example shows a reduced decomposition which will be used in Section 4.1.

Example 7. *Let $1 \leq k \leq m \leq n$ and $n - k \leq m - 1$, consider the decomposition $\sigma = \prod_{i=1}^{k-(n-m)-1}(s_{i+n-k-1} \ldots s_i) \prod_{i=0}^{n-m}(s_{m-1} \ldots s_{k-(n-m)+i})$. It is easy to see that each simple transposition increases one unity the number of inversions. So the number of inversions is equal to the number of simple transpositions and therefore σ is a reduced decomposition.*

The transposition t_{ij} exchanges the integers i and j, where $i < j$. In particular $t_{i,i+1} = s_i$. Let $\sigma \in \mathfrak{S}_n$, then $t_{ij}\sigma$ interchanges the positions of values i and j in the permutation σ and σt_{ij} interchanges the values in positions i and j in σ.

The (strong) Bruhat order in $\mathfrak{S}_n$ is defined by the following primitive relation: given μ and σ in $\mathfrak{S}_n$, μ covers σ (or σ precedes μ) if $\ell(\mu) = \ell(\sigma) + 1$ and there exists a transposition t such that $t\sigma = \mu$ [5]. It can be shown that in this case there is also a transposition t' such that $\sigma t' = \mu$: if $t = t_{ij}$ then $t' = t_{kl}$, where $k < l$ and k, l are the positions of the values i and j.

The Bruhat order in $\mathfrak{S}_n$ is therefore the partial order on $\mathfrak{S}_n$ which is the transitive closure of the relations

$$\theta < t\theta, \text{ if } \ell(\theta) < \ell(t\theta), (t \text{ transposition}, \theta \in \mathfrak{S}_n). \tag{1.4}$$

This definition is equivalent to the subword property of the (strong) Bruhat order in a Coxeter group.

Theorem 1. [5] *Let θ, σ in $\mathfrak{S}_n$ and $i_N \dots i_1$ a reduced word for σ, then $\theta \leq \sigma$ if and only if there exists a subsequence of $i_N \dots i_1$ which is a reduced word for θ.*

Notice that the maximal length element ω is the maximal element of the Bruhat order, $\sigma \leq \omega$, for any $\sigma \in \mathfrak{S}_n$, and it satisfies $\omega^2 = 1$. Besides, its left and right translations $\sigma \to \omega\sigma$ and $\sigma \to \sigma\omega$ are anti automorphisms for the Bruhat order. The action of $\mathfrak{S}_n$ in $\mathbb{N}^n$ is defined by the left action of a permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_n$, written in one-line notation, on a vector $\gamma = (\gamma_1, \gamma_2, \dots, \gamma_n) \in \mathbb{N}^n$, that is, $\sigma(\gamma_1, \gamma_2, \dots, \gamma_n) := (\gamma_{\sigma^{-1}(1)}, \gamma_{\sigma^{-1}(2)}, \dots, \gamma_{\sigma^{-1}(n)})$, each component γ_i ends up at position σ_i in the sequence permutated by σ.

Example 8. *The action of permutation $\sigma = s_1s_2$ on the vector $\alpha = (2, 2, 1)$ is*

$$s_1s_2(2, 2, 1) = s_1(2, 1, 2) = (1, 2, 2).$$

Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be a partition, and $\mathfrak{S}_n\lambda = \{\sigma\lambda : \sigma \in \mathfrak{S}_n\}$, the $\mathfrak{S}_n$-orbit of λ. The stabiliser of λ under the action of $\mathfrak{S}_n$ is the parabolic subgroup $stab_\lambda := \{\sigma \in \mathfrak{S}_n : \sigma\lambda = \lambda\}$. Given $\sigma \in \mathfrak{S}_n$, the class of σ modulo the stabiliser of λ, is the (left) coset $\sigma stab_\lambda = \{\sigma\theta : \theta \in stab_\lambda\}$. Two permutations σ and θ in $\mathfrak{S}_n$ are said to be in the same class modulo the stabiliser of λ if their cosets are equal or, equivalently, $\theta^{-1}\sigma \in stab_\lambda$. Let $\mathfrak{S}_n/stab_\lambda$ be the set of left cosets of $stab_\lambda$ in $\mathfrak{S}_n$. Each coset in $\mathfrak{S}_n/stab_\lambda$ has a unique shortest permutation called a minimal length coset representative for $\mathfrak{S}_n/stab_\lambda$ [5].

The $\mathfrak{S}_n$-orbit of λ is therefore in bijection with the set of cosets of $\mathfrak{S}_n$ modulo the stabiliser of λ, equivalently, with the set of minimal length coset representatives of $\mathfrak{S}_n/stab_\lambda$, whose cardinality is $n!/|stab_\lambda|$ [4, 5, 17]. The minimal length coset representatives for $\mathfrak{S}_n/stab_\lambda$ are characterized in [4, 5, 17] as $\{\sigma \in \mathfrak{S}_n : \ell(\sigma s_i) > \ell(\sigma), s_i \in stab_\lambda\}$.

Example 9. *Consider $\mathfrak{S}_4$ and partition $\lambda = (3, 2, 2, 0)$. Then*

$$stab_\lambda := \{\sigma \in \mathfrak{S}_4 : \sigma\lambda = \lambda\} = \{id, s_2\},$$

the set of cosets is

$\mathfrak{S}_4/stab_\lambda = \{\{id, s_2\}, \{s_1, s_1s_2\}, \{s_3, s_3s_2\}, \{s_2s_1, s_1s_2s_1\}, \{s_3s_1, s_1s_3s_2\},$

$\{s_2s_3, s_2s_3s_2\}, \{s_1s_2s_3, s_1s_2s_3s_2\}, \{s_2s_1s_3, s_2s_1s_3s_2\}, \{s_3s_2s_1, s_1s_3s_2s_1\},$

$\{s_1s_2s_1s_3, s_2s_1s_2s_3s_2\}, \{s_3s_2s_1s_3, s_2s_1s_3s_2s_1\}, \{s_3s_1s_2s_1s_3, s_3s_2s_1s_2s_3s_2\}\}.$

The set of minimal length coset representatives of $\mathfrak{S}_4/stab_\lambda$ *is*

$\{id, s_1, s_3, s_2s_1, s_3s_1, s_2s_3, s_1s_2s_3, s_2s_1s_3, s_3s_2s_1, s_1s_2s_1s_3, s_3s_2s_1s_3, s_3s_1s_2s_1s_3\}$

and the set of all rearrangements of λ *is*

$\mathfrak{S}_4\lambda = \{(3,2,2,0), (2,3,2,0), (3,2,0,2), (2,2,3,0), (2,3,0,2), (3,0,2,2),$

$(0,3,2,2), (2,0,3,2), (2,2,0,3), (0,2,3,2), (2,0,2,3), (0,2,2,3)\}$. *The cardinals of these two sets are both* $4!/|stab_\lambda| = 4!/2 = 12.$

It is known [50, 51] that the restriction of the Bruhat ordering from $\mathfrak{S}_n$ to the set of minimal length coset representatives for $\mathfrak{S}_n/stab_\lambda$ can be converted to an ordering of $\mathfrak{S}_n\lambda$ by taking the transitive closure of the relations

$$\gamma < t_{ij}\gamma, \text{ if } \gamma_i > \gamma_j,\ i < j. \tag{1.5}$$

If α_1 and α_2 are in the $\mathfrak{S}_n$-orbit of λ, and σ_1 and σ_2 are the shortest permutations such that $\sigma_1\lambda = \alpha_1$, $\sigma_2\lambda = \alpha_2$, then we write $\alpha_1 \le \alpha_2$ in the sense of (1.5) which is equivalent to $\sigma_1 \le \sigma_2$ in the Bruhat order. Henceforth, we say that (1.5) defines the Bruhat order on the weak compositions in the $\mathfrak{S}_n$-orbit of λ. Recall that the Hasse diagram of a partially ordered set $(P, \le)$ is the diagram (directed graph) where the vertices are the elements $x \in P$, and there is an upward-directed edge between x and y if y covers x. The bottom of the Hasse diagram of this poset is the partition λ and the top is the reverse of λ. Clearly, $\alpha_1 \le \alpha_2$ only if $\omega\alpha_1 \ge \omega\alpha_2$ since applying ω to the nodes of the Hasse diagram it reverses the upward-directed edges.

If we replace, in (1.4), t with the simple transposition s_i, the transitive closure of a such relations defines the left weak Bruhat order on $\mathfrak{S}_n$. Its restriction to the set of minimal length coset representatives of $\mathfrak{S}_n/stab_\lambda$ is then converted to an ordering in $\mathfrak{S}_n\lambda$ by replacing, in (1.5), j with $i+1$. Consider now the elementary bubble-sorting operation π_i, $1 \le i < n$, on words $\gamma_1\gamma_2\cdots\gamma_n$ of length n (or weak compositions in $\mathbb{N}^n$), which sorts the letters in positions i and $i+1$ in weakly increasing order, that is, it swaps

γ_i and γ_{i+1} if $\gamma_i > \gamma_{i+1}$, or fixes $\gamma_1\gamma_2\cdots\gamma_n$ otherwise. Define the partial order on $\mathfrak{S}_n\lambda$ by taking the transitive closure of the relations: $\gamma < \pi_i\gamma$ when $\gamma_i > \gamma_{i+1}$, ($\gamma \in \mathfrak{S}_n\lambda$ and $1 \leq i < n$). It coincides with the left weak Bruhat ordering of $\mathfrak{S}_n\lambda$.

It can be proved that the elementary bubble-sorting operations π_i, $1 \leq i < n$, satisfy the relations

$$\pi_i^2 = \pi_i,\ \pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i\pi_{i+1},\ \text{ and }\ \pi_i\pi_j = \pi_j\pi_i,\ \text{for } |i-j| > 1. \qquad (1.6)$$

Consider Example 9. Figure 1.1 displays four Hasse diagrams regarding to the Bruhat orders discussed above. The two first are respectively the restrictions of the left weak Bruhat order and the strong Bruhat order to $\mathfrak{S}_4/stab_{(3,2,2,0)}$. The two last are their translation to $\mathfrak{S}_4(3,2,2,0)$.

We recall here a construction of the minimal length coset representatives for $\mathfrak{S}_n/stab_\lambda$ in [30] due to Lascoux, where the notion of key tableau is used. This allows to convert the tableau criterion for the Bruhat order in $\mathfrak{S}_n$ to a tableau criterion for the Bruhat order (1.5) in the orbit $\mathfrak{S}_n\lambda$. The bijection between staircase keys of shape $(n,\dots,1)$ and permutations in $\mathfrak{S}_n$ gives a tableau criterion for the Bruhat order [9, 38]. If σ is a permutation in $\mathfrak{S}_n$, its key tableau, $key(\sigma(n,\dots,1))$, is the semi-standard Young tableau with shape $(n,\dots,1)$, in which the i^{th} column consists of the $n-i+1$ integers $\sigma(1),\dots,\sigma(n-i+1)$, placed in increasing order from bottom to top. Reciprocally any staircase key may be obtained in this way by defining the following permutation: first write the element of the right most column of the key then the new element that appears in the column next to the last, and so on. We have therefore the well-known tableau criterion for the Bruhat order in $\mathfrak{S}_n$.

Example 10. *The permutation corresponding to the stair key tableau.*

$$key(\sigma(4,3,2,1)) = \begin{array}{cccc} 4 & & & \\ 3 & 4 & & \\ 2 & 2 & 2 & \\ 1 & 1 & 1 & 2 \end{array} \qquad \Longleftrightarrow \sigma = 2143$$

Proposition 2. [38] *Let σ, $\beta \in \mathfrak{S}_n$, we have $\sigma \leq \beta$ in the Bruhat order if and only if $key(\sigma(n,\ \dots,\ 1)) \leq key(\beta(n,\dots,1))$ for the entrywise comparison.*

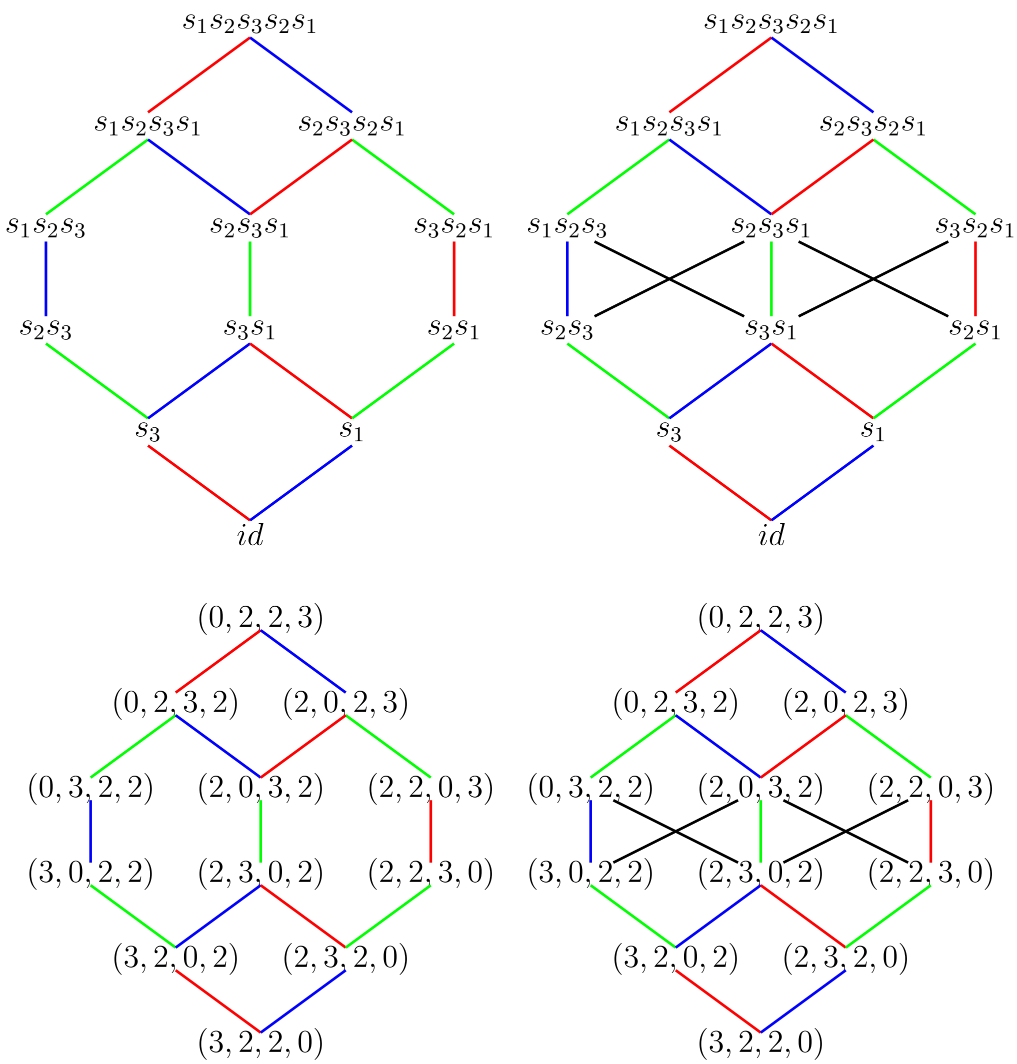


Fig. 1.1: Hasse diagram of the restrictions of the left weak Bruhat order and the strong Bruhat order to $\mathfrak{S}_4/stab_{(3,2,2,0)}$ and their translation to $\mathfrak{S}_4(3,2,2,0)$.

In [30], Lascoux constructs the shortest element in the coset $\sigma stab_\lambda$ such that $\sigma\lambda = \gamma \in \mathbb{N}^n$ using the key tableau of γ as follows: firstly, add the complete column $[n \dots 1]$ as the left most column of $key(\gamma)$, if γ has an entry equal to zero, secondly, write the elements of the right most column of $key(\gamma)$ in increasing order then the new elements that appear in the column next to the last in increasing order and so on until the first column. The resulting word is the desired permutation in $\mathfrak{S}_n$.

Example 11. *Let $\gamma = (1, 3, 0, 1)$ and $key(\gamma) = \begin{array}{lll} 4 & & \\ 2 & & \\ 1 & 2 & 2 \end{array}$. First add the complete column $[4, 3, 2, 1]$, to get $\begin{array}{llll} 4 & & & \\ 3 & 4 & & \\ 2 & 2 & & \\ 1 & 1 & 2 & 2 \end{array}$. Hence, $\sigma = 2143$ is the shortest permutation in the coset $\sigma\, stab_{3110}$, where $stab_{3110} =< s_2 >$.*

Theorem 2. *Let α_1 and α_2 be in the $\mathfrak{S}_n\lambda$ with the Bruhat ordering. Then*

(*a*) *$\alpha_1 \le \alpha_2$ if and only if $key(\alpha_1) \le key(\alpha_2)$.*

(*b*) *$\alpha_1 \le \alpha_2$ if and only if $evac(key(\alpha_2)) \le evac(key(\alpha_1))$.*

Proof. (*a*) Let σ_1 and σ_2 be the shortest length representatives of $\mathfrak{S}_n/stab_\lambda$ such that $\sigma_1\lambda = \alpha_1$, $\sigma_2\lambda = \alpha_2$. Then, $\alpha_1 \le \alpha_2$ if and only if $\sigma_1 \le \sigma_2$ in Bruhat order, and, by Proposition 2, this means $key(\sigma_1(n, \dots, 1)) \le key(\sigma_2(n, \dots, 1))$. Using the constructions of σ_1 and σ_2 explained above this is equivalent to say that $key(\alpha_1) \le key(\alpha_2)$.

(*b*) Recall that $evac(key(\alpha)) = key(\omega\alpha)$. $\square$

1.6 Schur polynomials

Let $x = (x_1, \dots, x_n)$ be a list of indeterminates and $\mathbb{Z}$ the set of integers. The ring of homogenous symmetric polynomials over $\mathbb{Z}$ is denoted by $\Lambda_{\mathbb{Z}}$.

$\Lambda_{\mathbb{Z}}$ is also a module over $\mathbb{Z}$. An important $\mathbb{Z}$-linear basis for this ring is the basis of Schur polynomials, $\{s_\lambda(x) : \lambda \textit{ a partition in } \mathbb{N}^n\}$. To a semi-standard Young tableau T with content $(\alpha_1, \ldots, \alpha_n)$, we associate the monomial $x^T := x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ in the sequence of n variables $x = (x_1, \ldots, x_n)$ called the weight monomial of T. Then for each such partition λ there exists a Schur polynomial $s_\lambda(x)$ which is a homogeneous symmetric polynomial in the variables $(x_1, \ldots, x_n)$ of total degree $|\lambda|$, and it may be defined in terms of SSYTs by

$$s_\lambda(x) = \sum_T x^T,$$

summed over all SSYTs of shape λ with content in $\mathbb{N}^n$.

Example 12. *The SSYTs that appear in the Schur polynomial* $s_\lambda(x_1, x_2, x_3) = s_{311}(x_1, x_2, x_3)$.

$$\begin{array}{|c|cc}\hline 3 \\ \hline 2 \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{1} \\ \hline \end{array} \quad \begin{array}{|c|cc}\hline 3 \\ \hline 2 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|cc}\hline 3 \\ \hline 2 \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{2} \\ \hline \end{array} \quad \begin{array}{|c|cc}\hline 3 \\ \hline 2 \\ \hline 1 & \multicolumn{1}{|c|}{3} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|cc}\hline 3 \\ \hline 2 \\ \hline 1 & \multicolumn{1}{|c|}{2} & \multicolumn{1}{c|}{3} \\ \hline \end{array} \quad \begin{array}{|c|cc}\hline 3 \\ \hline 2 \\ \hline 1 & \multicolumn{1}{|c|}{1} & \multicolumn{1}{c|}{3} \\ \hline \end{array}$$

1.7 The Robinson-Schensted-Knuth correspondence

In this section we follow the notation of [49]. The Robinson-Schensted-Knuth (RSK) correspondence [24, 48] is a combinatorial bijection between matrices A with finitely many non-negative integer entries ($\mathbb{N}$-matrix of finite support) and pairs (P, Q) of semi-standard Young tableaux of the same shape. The content of P is given by the column sums of A, and the content of Q by its row sums.

There are many useful consequences of the RSK. For instance the Cauchy identity, that we will explain in Chapter 4, can be obtained by the RSK correspondence.

Theorem 3. *[24] There exists a bijection between* $\mathbb{N}$*-matrices of finite support and pairs* (P, Q) *of SSYT of the same shape.*

The main operation involved in the RSK correspondence is the Schensted row insertion $P \leftarrow k$ of a positive integer k into a semi-standard Young tableau $P = (P_{ij})$ [48], where the (P_{ij}) is the entry of the cell (i, j) of P. (Here i is the row and j is the column containing the entry P_{ij}.) Let r be the largest integer such that $P_{1,r-1} \leq k$. If there is no such an r, place k at the end of the first row and the procedure is complete. Otherwise, k bumps $P_{1,r} = k'$ into the second row and the procedure is repeated in the second row for k'. Continue this procedure until an element is inserted at the end of a row (possibly creating a new row). The resulting diagram is $P \leftarrow k$.

Let $A = (a_{ij})$ be an $\mathbb{N}$-matrix with finite support. There exists a unique biword, a two-line array, corresponding to A which is defined by the non-zero entries in A. Let a_{ij} be the first non-zero entry encountered when scanning the entries of A from left to right, top to bottom. Place an i in the top line and a j in the bottom line a_{ij} times. When this has been done for each non-zero entry, one obtains the following array

$$w_A = \begin{pmatrix} i_1 & i_2 & \dots \\ j_1 & j_2 & \dots \end{pmatrix}$$

in lexicographic order, that is,

$$(i_t < i_{t+1}) \;\; or \;\; (i_t = i_{t+1} \;\; and \;\; j_t \leq j_{t+1}), \;\; for \; all \; t.$$

Begin with a pair $(P(0), Q(0)) := (\emptyset, \emptyset)$ and let $P(t+1) := P(t) \leftarrow j_{t+1}$. Let $Q(t+1)$ be obtained from $Q(t)$ by placing i_{t+1} at the end of a row of $Q(t)$ so that $Q(t+1)$ has the same shape as $P(t+1)$. The result is the pair (P, Q), where P represents the insertion tableau and Q represents the recording tableau. Notice that we apply Schensted row insertion in the second row of biword from left to right.

Example 13. *RSK correspondence applied to the biword w in the lexicographic order.*

$$w = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 7 & 7 \\ 2 & 7 & 2 & 4 & 1 & 3 & 3 & 1 & 1 \end{pmatrix} \xrightarrow{\underline{RSK}} \begin{array}{cccc} 7 & & & \\ 4 & & & \\ 2 & 2 & 3 & \\ 1 & 1 & 1 & 3 \\ & P & & \end{array} \qquad \begin{array}{cccc} 7 & & & \\ 4 & & & \\ 2 & 4 & 7 & \\ 1 & 1 & 3 & 5 \\ & Q & & \end{array} .$$

The content of the first row of w is $c(Q) = (2,1,1,2,1,0,2)$ and the content of the second row of w is $c(P) = (3,2,2,1,0,0,1)$ and $sh(P) = sh(Q) = (4,3,1,1)$.

The RSK correspondence satisfies the following symmetry property.

Proposition 3. *([24, 49]) Let A be an $\mathbb{N}$-matrix of finite support, and suppose that $A \xrightarrow{RSK} (P,Q)$. Then $A^t \xrightarrow{RSK} (Q,P)$, where ^{t} denotes transpose.*

It means that if we swap the rows of the biword w_A and rearrange it in the lexicographic order, and then we apply the RSK, one obtains the pair (Q,P). This property will become clear when we introduce the growth diagram presentation of RSK in Section 1.10.

1.8 The right key of a tableau

The notion of the right key of tableau T, $K_+(T)$, is due to Lascoux and Schützenberger in [32]. There are now several ways to define the right key of a tableau [12, 32, 40, 53]. For instance, in [53, 44] is provided a new way to calculate the right key of a SSYT. We give in this section the original definition of right key [32]. However for the propose of our work it is more convenient to define it in terms of semi-skyline augmented filling [40], a combinatorial object which will be introduced in the next chapter.

The Knuth or plactic equivalence $\sim$ is defined on the set of all words on the alphabet $[n]$ by the transitive closure of the relations [32, 45]

$$\begin{array}{ll} \mathbf{a}xzy\mathbf{b} \sim \mathbf{a}zxy\mathbf{b} & for \quad x \leq y < z \\ \mathbf{a}yxz\mathbf{b} \sim \mathbf{a}yzx\mathbf{b} & for \quad x < y \leq z \end{array},$$

where $\mathbf{a}$ and $\mathbf{b}$ are words. The set of words congruent modulo the Knuth relations to a given word w is called the Knuth equivalence class of w. There exists a unique word v in each Knuth equivalence class such that $v = col(T)$ for some semi-standard Young tableau T. If $u \sim v$ then the Schensted row insertion of u and v is T and we write $u \sim v \sim T$.

The column form of a word w, denoted colform(w), is the weak composition consisting of the lengths of the maximal strictly decreasing subwords of w from left to right. Let w be an arbitrary word such that $w \sim T$ for SSYT T of shape λ. The word w is said to be column-frank if colform(w) is a rearrangement of the nonzero parts of $\bar{\lambda}$, the conjugate of λ.

Example 14.

$$T = \begin{matrix} 5 & \\ 3 & 4 \\ 1 & 2 \end{matrix}\ , \qquad \lambda = (2,2,1), \qquad \bar{\lambda} = (3,2),$$

$w = 31542 \sim T$ and colform(w) $= (2,3)$, so w is column-frank.

Definition 2. *[32] Let T be a semi-standard Young tableau of shape λ. The* right key *of T, denoted $K_+(T)$, is the unique key of shape λ whose j^{th} column is given by the last column of any column-frank word v such that $v \sim T$ and colform(v) is of the form $(\dots, \lambda'_j)$, where $\bar{\lambda} = (\lambda'_1, \dots, \lambda'_j)$.*

Next example shows the sequence of Knuth equivalent words which gives the right key of tableau T.

Example 15.

$$T = \begin{matrix} 4 & & \\ 2 & 3 & \\ 1 & 2 & 3 \end{matrix}\ ,$$

$$w := col(T) = 421.32.3 \sim 421.3.32 \sim 2.41.3.32 \sim 21.43.32 \sim 21.3.432.$$

The two red words are not column-frank because their colform are $(1,2,1,2)$ and $(2,2,2)$, respectively. We choose the last columns of the black words which are column-frank. Therefore

$$K_+(T) = \begin{matrix} 4 & & \\ 3 & 3 & \\ 2 & 2 & 3 \end{matrix}$$

1.9 Fomin's growth diagrams

In this section we introduce growth diagrams as certain objects which associate sequences of partitions to fillings of matrices with non-negative integer entries or biwords in lexicographic order. Many properties of tableau algorithms such as symmetry become clear when the algorithms are formulated in terms of growth diagrams governed by local rules. Fomin [10, 6] introduced this approach to the Robinson-Schensted correspondence, it was rediscovered by van Leeuwen [52], and Roby [46] developed it further. Krattenthaler also use the growth diagram to treat fillings of diagrams under certain conditions. In this section we follow very close [25].

A 01-filling of a rectangle shape F is obtained by filling the cells of F with 1's and 0's, where we present 1's by $\times$ and suppress the 0's, such that every row and every column contains at most one 1.

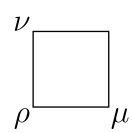
Example 16. *A 01-filling of* 4×4*-square.*

$\times$			
			$\times$
		$\times$	
	$\times$		

A NE-chain of a 01-filling is a sequence of 1's such that any 1 is above and to the right of the preceding 1 in the sequence. A SW-chain of a 01-filling is a sequence of 1's such that any 1 is below and to the left of the preceding 1 in the sequence. The length of a NE-chain or a SW-chain is defined to be the number of 1's in the chain. Consider the upper right corner of the example above: there are two NE-chains, one of lengths 3 and one of length 1, and similarly for SW-chains.

The growth diagram for a 01-filling of a rectangle shape F is obtained by labelling the corners of all the squares in F by partitions in such a way that the partition assigned to any corner is either equal to the partition to its left or is obtained from it by adding a horizontal strip, that is, by a set of squares no two of which are in the same column. The partition assigned to any corner either equals the partition below it or is obtained from this partition by adding a vertical strip, that is, by a set of squares no two of which are in the same row.

We start by assigning the empty partition $\emptyset$ to each corner on the left and bottom edges of F. Then assign the partitions to the other corners inductively by applying the following local rules. Consider the cell below, labeled by the partitions ρ, μ, ν, where $\rho \subseteq \mu$ and $\rho \subseteq \nu$, μ and ρ differ by a horizontal strip, and ν and ρ differ by a vertical strip. Then λ is determined as follows:



- If $\rho = \mu = \nu$, and if there is no cross in the cell, then $\lambda = \rho$.
- If $\rho = \mu \neq \nu$, then $\lambda = \nu$.
- If $\rho = \nu \neq \mu$, then $\lambda = \mu$.
- If ρ, μ, ν are pairwise different, then $\lambda = \mu \cup \nu$.
- If $\rho \neq \mu = \nu$, then λ is formed by adding a square to the $(k+1)$-st row of $\mu = \nu$, given that $\mu = \nu$ and ρ differ in the k-th row.
- If $\rho = \mu = \nu$, and if there is a cross in the cell, then λ is formed by adding a square to the first row of $\rho = \mu = \nu$.

In addition there is a global description of the local rules as a consequence of Greene's theorem [13] as explained in [25].

Theorem 4 (Theorem 2 [25]). *Given a diagram with empty partitions labelling all the corners along the left side and the bottom side of a rectangle shape, the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ labelling corner c satisfies the following property:*

For any k, the maximal cardinality of the union of k NE-chains situated in *the rectangular region to the left and below of c is equal to* $\lambda_1 + \lambda_2 + \cdots + \lambda_k$. *In particular,* λ_1 *is the length of the longest NE-chain in the rectangular region to the left and below of c.*

See the Figure 1.2.

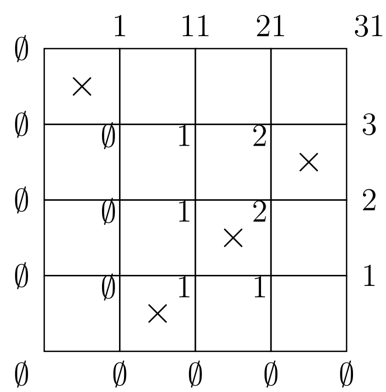


Fig. 1.2: The growth diagram

In the following we extend the construction described above to arbitrary fillings of a rectangle. It means that we allow more than one 1 in each row and in each column, and we allow also arbitrary non-negative entries in our fillings. The construction that we explain next is the same as the first variant construction in [25, 47]. In order to apply the local rules we would like to pass to a 01-filling of the diagram, it means that there will be at most one 1 in each row and each column. To remedy this, we "separate" the entries in the diagram in the following way.

Construct a rectangle diagram with more rows and columns so that entries which are originally in the same column or in the same row are put in different columns and rows in the larger diagram, and that an entry m is replaced by m 1's in the new diagram all of them placed in different rows and columns. Separate the entries in a row from bottom/left to top/right, as before the 1's are represented by $\times$'s and 0's are suppressed. If there should be several entries in a column as well, separate entries in a column from bottom/left to top/right. In the cell with entry m we replace m by a chain of m $\times$'s arranged from bottom/left to top/right.

In the figure, the original columns and rows are indicated by thick lines, whereas the newly created columns and rows are indicated by thin lines. We refer to this process, transforming a filling into a standard filling, as standardization. Figure 1.3 shows an arbitrary filling, its standardization and corresponding growth diagram.

1.10 Growth diagram presentation of Robinson-Schensted-Knuth correspondence

In this section we are going to interpret the RSK in terms of Fomin's growth diagrams. The reader is referred to [10, 12, 24] and [49] for extensive information on the Robinson-Schensted-Knuth correspondence. For what follows it is convenient to consider an equivalent definition of SSYT. Here we see a SSYT of shape λ on the alphabet $[n]$ as a sequence of nested partitions $\emptyset = \lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_n$ such that λ_i/λ_{i-1} is an horizontal strip (possible empty) filled only with $|\lambda_i| - |\lambda_{i-1}|$ letters i for $1 \leq i \leq n$.

Consider a biword w with letters in the alphabet $[n]$ and represent it in a $n \times n$ square as follows: put the number r in the cell (i, j) of the square if the biletter $\begin{pmatrix} j \\ i \end{pmatrix}$ appears r times in the biword w. Then consider the 01-filling associated to that square as explained in the previuos section. Applying the local rules to it leads to a pair of sequences of partitions, one in the right and another in the top of the growth diagram. The partitions of each sequence are related by containment. Let λ_i be the partition associated to the i-th column thick line on the top of the growth diagram when we scan the thick column lines from left to right. Then the top side of the growth diagram is a sequence of partitions with $\emptyset = \lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_l$, where l is the maximum element in the first row of the biword w and λ_i/λ_{i-1} is a horizontal strip.

Let $\underline{\lambda}_i$ be the partition associated to the i-th thick row line on the right side of the growth diagram when we scan the thick row lines from bottom to top. Then the right side of the growth diagram is a sequence of partitions with $\emptyset = \underline{\lambda}_0 \subseteq \underline{\lambda}_1 \subseteq \cdots \subseteq \underline{\lambda}_t$, where t is the maximum element in the second row of the biword w and $\underline{\lambda}_i/\underline{\lambda}_{i-1}$ is a horizontal strip.

Fill with i all the squares in λ_i/λ_{i-1} and $\underline{\lambda}_i/\underline{\lambda}_{i-1}$, for $i \geq 1$. This defines a pair (P, Q) of SSYTs of the same shape, where Q has the content of the first row of w and P has the content of second row of w. In [10, 46] it is shown that this pair of SSYTs is the pair obtained by applying the RSK to the biword w. For example, if we consider the biword

$$w = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 7 & 7 \\ 2 & 7 & 2 & 4 & 1 & 3 & 3 & 1 & 1 \end{pmatrix}$$

then the 01-filling and the growth diagram associated to w are shown in the Figure 1.3. Consider the i-th thick row line on the right side of the growth diagram and the associated partition $\underline{\lambda}_i$ for $i = 1, \ldots, 7$,with $\underline{\lambda}_0 = \emptyset$. Fill with i all the cells that appear in $\underline{\lambda}_i / \underline{\lambda}_{i-1}$, and one gets

$$\begin{array}{ccccccccc}
\emptyset & 1\,1\,1 & \begin{array}{l} 2\,2 \\ 1\,1\,1 \end{array} & \begin{array}{l} 2\,2\,3 \\ 1\,1\,1\,3 \end{array} & \begin{array}{l} 4 \\ 2\,2\,3 \\ 1\,1\,1\,3 \end{array} & & & \begin{array}{l} 7 \\ 4 \\ 2\,2\,3 \\ 1\,1\,1\,3 \end{array} & = P \\
\emptyset & 3 & 32 & 43 & 431 & 431 & 431 & 4311 & \\
\underline{\lambda}_0 & \underline{\lambda}_1 & \underline{\lambda}_2 & \underline{\lambda}_3 & \underline{\lambda}_4 & \underline{\lambda}_5 & \underline{\lambda}_6 & \underline{\lambda}_7 &
\end{array}$$

Consider the i-th thick column line on the top side of the growth diagram and its associated partition λ_i for $i = 1, \ldots, 7$,with $\lambda_0 = \emptyset$. Fill with i all the cells that appear in $\lambda_i / \lambda_{i-1}$, and one gets

$$\begin{array}{ccccccccc}
\emptyset & 1\,1 & \begin{array}{l} 2 \\ 1\,1 \end{array} & \begin{array}{l} 2 \\ 1\,1\,3 \end{array} & \begin{array}{l} 4 \\ 2\,4 \\ 1\,1\,3 \end{array} & \begin{array}{l} 4 \\ 2\,4 \\ 1\,1\,3\,5 \end{array} & & \begin{array}{l} 7 \\ 4 \\ 2\,4\,7 \\ 1\,1\,3\,5 \end{array} & = Q \\
\emptyset & 2 & 21 & 31 & 321 & 421 & 421 & 4311 & \\
\lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 &
\end{array}$$

By applying the RSK to the biword w, one has the pair, as we have seen in the Section 1.7.

$$P = \begin{array}{l} 7 \\ 4 \\ 2\,2\,3 \\ 1\,1\,1\,3 \end{array} \qquad Q = \begin{array}{l} 7 \\ 4 \\ 2\,4\,7 \\ 1\,1\,3\,5 \end{array}$$

which is equal to the pair of SSYTs, coming from the two sequences of partitions.

Another way to find the top and the right sequences of nested partitions is just looking for the k longest NE-chains in the growth diagram using Theorem 4.

1.11 The reverse Robinson-Schensted-Knuth correspondence and its growth diagram presentation

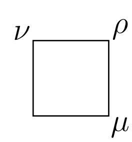
The reverse Robinson-Schensted-Knuth correspondence [49] is a combinatorial bijection between matrices A with non-negative integer entries ($\mathbb{N}$-matrix) or, equivalently, biwords in the lexicographic order, and pairs $(\widetilde{P}, \widetilde{Q})$ of reverse semi-standard Young tableaux of the same shape.

The reverse RSK correspondence is the obvious variant of the RSK correspondence, where we reverse the roles of $\leq$ and $\geq$ in the Schensted row insertion, called reverse Schensted row insertion. Equivalently, applying the RSK to the biword $\widetilde{w} = \begin{pmatrix} -i_n & \cdots & -i_1 \\ -j_n & \cdots & -j_1 \end{pmatrix}$ instead of $w = \begin{pmatrix} i_1 & \cdots & i_n \\ j_1 & \cdots & j_n \end{pmatrix}$ (whose entries are now negative integers) and then change the sign back to positive of all entries of the pair of tableaux. We obtain then a pair $(\widetilde{P}, \widetilde{Q})$ of RSSYTs. For example applying the reverse RSK correspondence to the biword

$$w = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 7 & 7 \\ 2 & 7 & 2 & 4 & 1 & 3 & 3 & 1 & 1 \end{pmatrix}$$

gives the pair of RSSYT in Figure 1.4.

To find an interpretation of the reverse RSK in terms of growth diagrams, we consider the diagram corresponding to $\widetilde{w}$, that is, the diagram of w by reflecting each cross with respect to the origin. Equivalently, we can just change the origin to the upper right corner and apply backward the local rules from there in the diagram of w, see Figure 1.5. The backward local rules are the same as local rules for the following square,



Applying backward the local rules leads to the pair of sequences of partitions in the left and in the bottom of growth diagram. The partitions of each sequence are related by containment.

Let λ_i be the partition associated to the i-th thick column line on the bottom of the growth diagram when we scan the thick column lines from right to left. Then the bottom side of the growth diagram is a sequence of partitions with $\emptyset = \lambda_0 \subseteq \lambda_1 \subseteq \cdots \subseteq \lambda_l$, where l is the maximum element in the first row of the biword w and λ_i/λ_{i-1} is a horizontal strip.

Let $\underline{\lambda}_i$ be the partition associated to the i-th row thick line on the left of the growth diagram when we scan the thick row lines from top to bottom. Then the right side of the growth diagram is a sequence of partitions with $\emptyset = \underline{\lambda}_0 \subseteq \underline{\lambda}_1 \subseteq \cdots \subseteq \underline{\lambda}_t$, where t is the maximum element in the second row of the biword w and $\underline{\lambda}_i/\underline{\lambda}_{i-1}$ is a horizontal strip. Fill with $n+1-i$ all the squares in λ_i/λ_{i-1} and $\underline{\lambda}_i/\underline{\lambda}_{i-1}$, for $i \geq 1$. This pair of nested sequences of partitions defines a pair $(\widetilde{P}, \widetilde{Q})$ of RSSYTs of the same shape with contents respectively, of the second row and the first row of w. The growth diagram corresponding to the reverse RSK for

$$w = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 7 & 7 \\ 2 & 7 & 2 & 4 & 1 & 3 & 3 & 1 & 1 \end{pmatrix}$$

is shown in the Figure 1.6.

The pair of RSSYTs that one obtains from the left and the bottom sequences of partitions is as follows:

$\emptyset$	$7\ 7$		$\begin{array}{l}5\\7\ 7\end{array}$	$\begin{array}{l}5\ 4\\7\ 7\ 4\end{array}$	$\begin{array}{l}3\\5\ 4\\7\ 7\ 4\end{array}$	$\begin{array}{l}3\\5\ 4\ 2\\7\ 7\ 4\end{array}$	$\begin{array}{l}1\\3\\5\ 4\ 2\\7\ 7\ 4\ 1\end{array}$	$= \widetilde{Q}$
$\emptyset$	2	2	21	32	321	331	4311	
$\underline{\lambda}_0$	$\underline{\lambda}_1$	$\underline{\lambda}_2$	$\underline{\lambda}_3$	$\underline{\lambda}_4$	$\underline{\lambda}_5$	$\underline{\lambda}_6$	$\underline{\lambda}_7$	

and,

$\emptyset$	7			$\begin{array}{l}4\\7\end{array}$	$\begin{array}{l}3\\4\\7\ 3\end{array}$	$\begin{array}{l}3\\4\\7\ 3\ 2\ 2\end{array}$	$\begin{array}{l}1\\3\\4\ 1\ 1\\7\ 3\ 2\ 2\end{array}$	$= \widetilde{P}$
$\emptyset$	1	1	1	11	211	411	4311	
λ_0	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7	

This pair $(\widetilde{P}, \widetilde{Q})$ of RSSYTs is the same as the pair that we have obtained by applying the reverse RSK to the biword w in Figure 1.4. Another way to find the nested sequences of partitions on the bottom and on the left of diagram is just looking for the k SW-chains by using the following natural version of the Theorem 4.

Theorem 5. *Given a diagram with empty partitions labelling all the corners along the right side and the top side of a rectangle shape, which has been completed according to the reverse RSK, the partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ *labelling corner* c *satisfies the following property:*
For any k*, the maximal cardinality of the union of* k *SW-chains situated in the rectangular region to the right and above of* c *is equal to* $\lambda_1 + \lambda_2 + \cdots + \lambda_k$*. In particular,* λ_1 *is the length of the longest SW-chain in the rectangular region to the right and above* c*.*

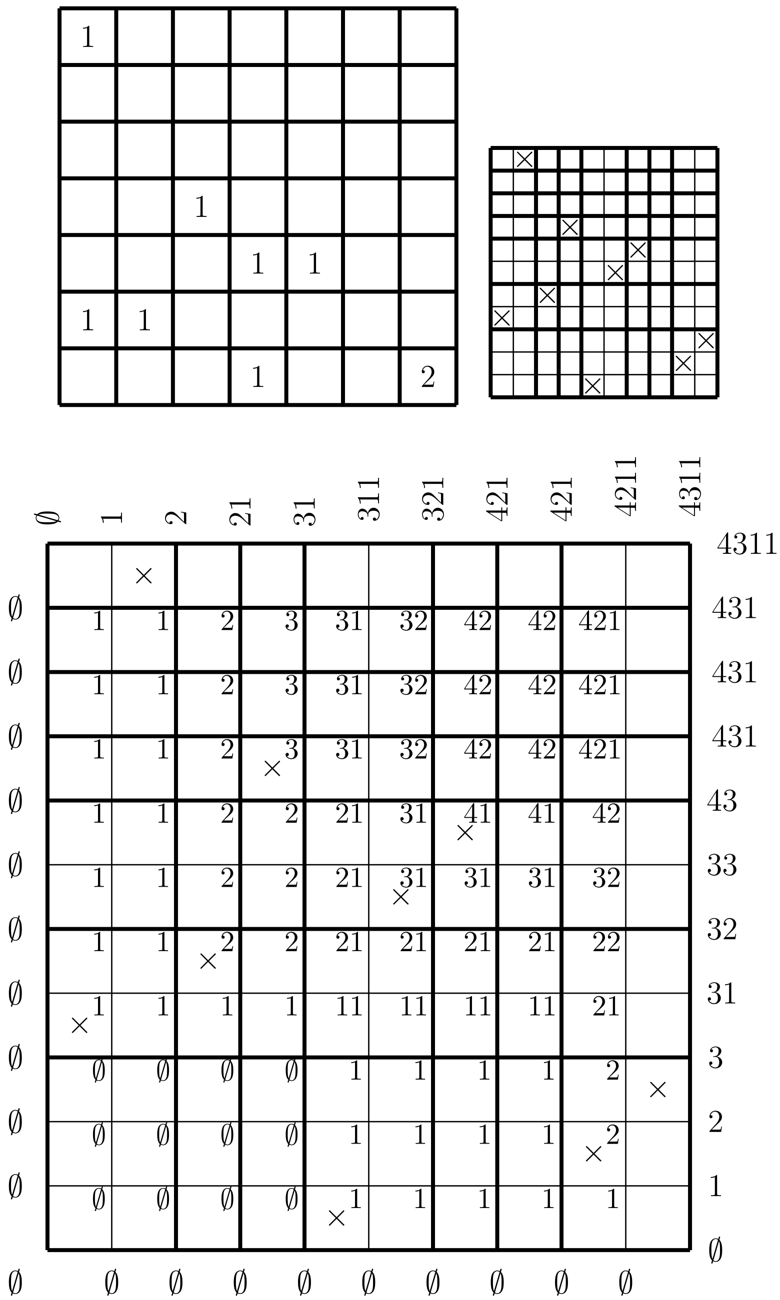


Fig. 1.3: An arbitrary filling, 01-filling and the growth diagram

$$
\widetilde{P} = \begin{array}{llll} 1 & & & \\ 3 & & & \\ 4 & 1 & 1 & \\ 7 & 3 & 2 & 2 \end{array}
\qquad
\widetilde{Q} = \begin{array}{llll} 1 & & & \\ 3 & & & \\ 5 & 4 & 2 & \\ 7 & 7 & 4 & 1 \end{array}
$$

Fig. 1.4: The pair of RSSYT

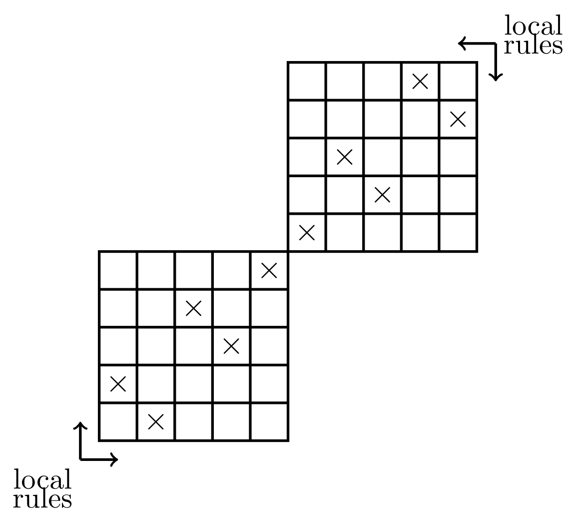


Fig. 1.5: Reflection with respect to the origin

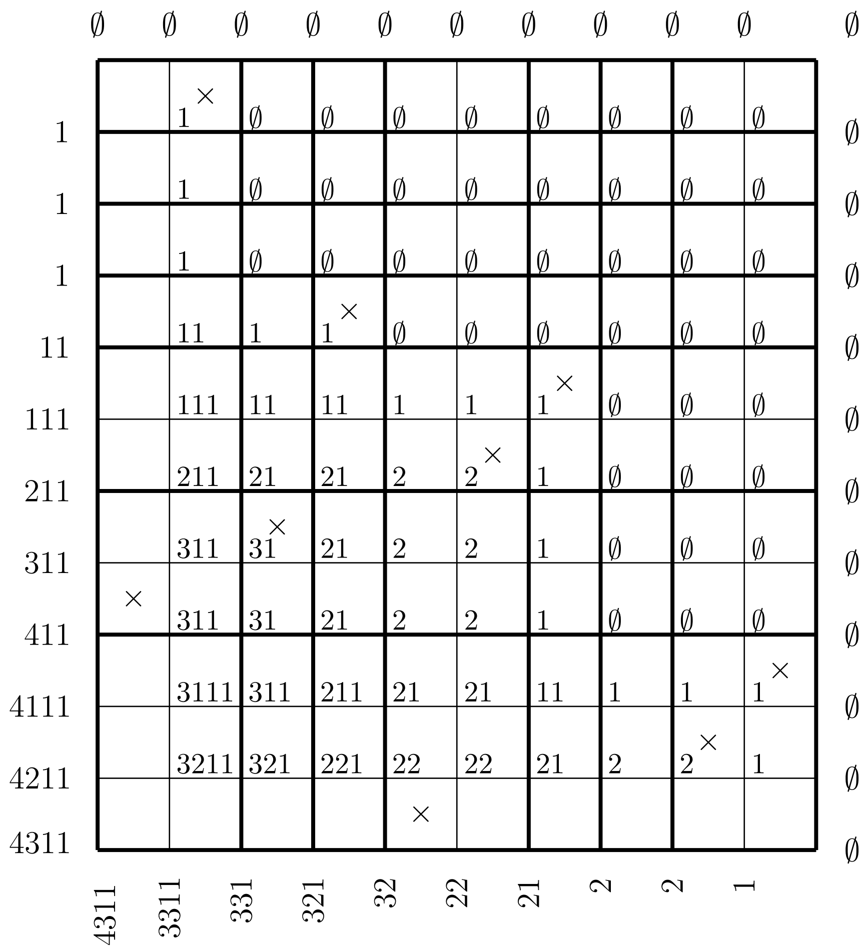


Fig. 1.6: Growth diagram with backward local rules

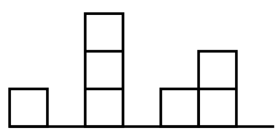
2. SEMI-SKYLINE AUGMENTED FILLINGS

In this chapter we introduce the combinatorial objects called semi-skyline augmented fillings (SSAF), which were defined to describe combinatorially non-symmetric Macdonald polynomials [15]. This is done in sections 2.1 and 2.2 where, in particular, the relationship with Schur polynomials is discussed. In sections 2.3, 2.4, 2.5, 2.6 and 2.7 we study useful properties of SSAF and its relation with RSSYT and SSYT. In sections 2.8 and 2.9 we study an analogue of RSK and its relation with reverse RSK. In Section 2.10 we make an interpretation of the analogue of RSK in terms of Fomin's growth diagram. Finally in the last section we bring the main theorem which will be used to find a bijective proof for non-symmetric Cauchy kernel expansions over staircases and truncated staircases.

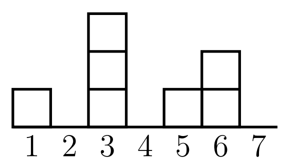
2.1 Combinatorial description of non-symmetric Macdonald polynomials

The following combinatorial statistics defined on fillings are used to describe the Macdonald polynomials [14] and non-symmetric Macdonald polynomials [15]. The Macdonald polynomials were introduced by Macdonald in [35]. The theory of the non-symmetric Macdonald polynomials, $\widehat{E}_\gamma(x;q,t)$, where $\gamma \in \mathbb{N}^n$ and $x = (x_1,\ldots,x_n)$ a list of indeterminates, is developed by Cherednik [7], Macdonald [37], and Opdam [41]. Haglund, Haiman, and Loehr [14, 15] recently found a combinatorial description for the Macdonald polynomials and the non-symmetric Macdonald polynomials. We follow most of the time the conventions and terminology in [14, 15, 39, 40].

A weak composition $\gamma = (\gamma_1,\ldots,\gamma_n)$ is visualised as a diagram consisting of n columns, with γ_j boxes in column j, for $1 \le j \le n$. Formally, the *column diagram* of γ is the set $dg'(\gamma) = \{(i,j) \in \mathbb{N}^2 : 1 \le j \le n, 1 \le i \le \gamma_j\}$ where the coordinates are in French convention, i indicates the vertical coordinate, indexing the rows, and j the horizontal coordinate, indexing the columns. (The prime reminds that the components of γ are the columns.) The number of cells in a column is called the height of that column and a cell a in a column diagram is denoted $a = (i, j)$, where i is the row index and j is the column index. The augmented diagram of γ, $\widehat{dg}(\gamma) = dg'(\gamma) \cup \{(0, j) : 1 \leq j \leq n\}$, is the column diagram with n extra cells in row 0. This adjoined row is called the basement and it always contains the numbers 1 through n in strictly increasing order. The shape of $\widehat{dg}(\gamma)$ is defined to be γ. For example, column diagram and the augmented diagram for $\gamma = (1, 0, 3, 0, 1, 2, 0)$ are



$dg'(\gamma)$



$\widehat{dg}(\gamma)$

A filling of a diagram γ is a function $\sigma : dg'(\gamma) \to [n]$, and an augmented filling is the filling $\widehat{\sigma} : \widehat{dg}(\gamma) \to [n]$, of the augmented diagram such that $\widehat{\sigma}$ agrees with σ on $dg'(\gamma)$. In the other words an augmented filling can be pictured as an assignment of positive integer entries to the non-basement cells of $\widehat{dg}(\gamma)$. Two cells a and b are called attacking if either they are in the same row or they are in adjacent rows, such that the entry in the higher row is strictly to the right of the entry in the lower row. It means $a = (i_1, j_1)$ and $b = (i_2, j_2)$ are attacking if either $i_1 = i_2$, or $i_2 - i_1 = 1$ and $j_1 < j_2$, or $i_1 - i_2 = 1$ and $j_2 < j_1$.

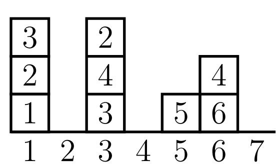


Fig. 2.1: An augmented filling $\widehat{\sigma}$

A non-attacking filling is a filling such that $\widehat{\sigma}(a) \neq \widehat{\sigma}(b)$ for every pair of attacking cells a and b. As in [14], a descent of $\widehat{\sigma}$ is a pair of entries

$\widehat{\sigma}(u) > \widehat{\sigma}(v)$, where the cell u is directly above v. In other words, $v = (i, j)$ and $u = (i+1, j)$, where i is the row of cell v and j is the column of cell v. (We include pairs u, v such that $v = (0, j)$ and $u = (1, j)$.) Define $Des(\widehat{\sigma}) = \{u \in dg'(\gamma) : \widehat{\sigma}(u) > \widehat{\sigma}(v) \text{ is a descent}\}$. In Figure 2.1, we have $Des(\widehat{\sigma}) = \{(2,1), (3,1), (2,3)\}$. Let leg of u be the number of cells above u in the column of $\widehat{dg}(\gamma)$, which contains u, denoted $l(u)$. If we consider cell $u = (1,3)$ in Figure 2.1, then $l(u) = 2$. Define $maj(\widehat{\sigma}) = \sum_{u \in Des(\widehat{\sigma})} (l(u)+1)$. In Figure 2.1, $maj(\widehat{\sigma}) = 2+1+2 = 5$.

The reading order of a shape $dg'(\gamma)$ is the total ordering of the cells given by reading the rows from left to right, top to bottom. A cell $a = (i, j)$ precedes a cell $b = (i', j')$ in the reading order if either $i' < i$ or $i' = i$ and $j' > j$. The word derived from a filling $\widehat{\sigma}$ by reading the non-basement entries in reading order is called the reading word of $\widehat{\sigma}$, denoted $read(\widehat{\sigma})$. In Figure 2.1 we have $read(\widehat{\sigma}) = 322441356$. The content of an augmented filling $\widehat{\sigma}$, denoted by $c(\widehat{\sigma})$, is the weak composition $(\alpha_1, \ldots, \alpha_n)$ where α_i is the number of non-basement cells in $\widehat{dg}(\gamma)$ (or the number of cells in $dg'(\gamma)$) with entry i, where n is the number of basement elements. The content of $\widehat{\sigma}$ in Figure 2.1 is $c(\widehat{\sigma}) = (1,2,2,2,1,1,0)$.

The standardization of $\widehat{\sigma}$ is the unique augmented filling that one obtains by sending the i-th occurrence of j in the reading order to $i + \sum_{m=1}^{j-1} \alpha_m$, where $c(\widehat{\sigma}) = (\alpha_1, \ldots, \alpha_n)$. Figure 2.2 is a standardization of $\widehat{\sigma}$ in Figure 2.1.

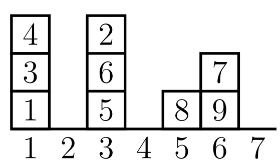
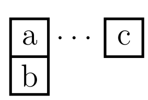
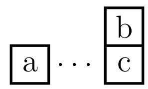


Fig. 2.2: A standardization of $\widehat{\sigma}$

Let three cells $a, b, c \in \widehat{dg}(\gamma)$, where a and c are in the same row. Then (a, b, c) is said to be a triple. Let $a, b, c \in \widehat{dg}(\gamma)$ three cells situated as follows,



where a and c are in the same row, possibly the first row, possibly with cells between them, and the height of the column containing a and b is greater than or equal to the height of the column containing c. Then the triple (a, b, c) is a triple of type *I*. The triple (a, b, c) is said to be an inversion triple of type *I* if and only if after standardization the ordering from smallest to largest of the entries in cells a, b, c induces a counterclockwise orientation. Similarly, consider three cells $a, b, c \in \widehat{dg}(\gamma)$ situated as follows,



where a and c are in the same row (possibly the basement) and the column containing b and c has strictly greater height than the column containing a. Then the triple (a, b, c) is a triple of type *II*. The triple (a, b, c) is said to be an inversion triple of type *II* if and only if after standardization the ordering from smallest to largest of the entries in cells a, b, c induces a clockwise orientation. Figure 2.3 shows inversion triples of type *I* and *II*. In fact the blue entries define a type *I* inversion triple and the red entries a type *II* inversion triple.

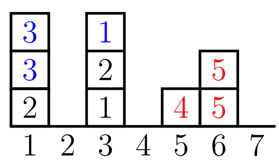


Fig. 2.3: Inversion triples of type *I* and *II* in $\widehat{\sigma}$

For an augmented filling $\widehat{\sigma}$ of a weak composition γ, define [39]

$$Inv(\widehat{\sigma}) = \{ \text{ inversion triples of } \widehat{\sigma}\}, \text{ and } inv(\widehat{\sigma}) = |Inv(\widehat{\sigma})|.$$

Let $coinv(\widehat{\sigma})$ be the number of triples of type *I* and *II* which are not an inversion triples [39]. For example, in Figure 2.3 we have $inv(\widehat{\sigma}) = 4$ and $coinv(\widehat{\sigma}) = 6$.

The last combinatorial notion that we need to introduce, in order to give a combinatorial description of non-symmetric Macdonald polynomials, is the notion of arm of a cell $u \in dg'(\gamma)$. Define the arm of cell u, denoted $a(u)$, to be the number of cells to the right of u in row i appearing in columns whose height is weakly less than the height, h, of the column containing u plus the number of cells to the left of u in row $i-1$ appearing in columns whose height is strictly less than h. For example, if we consider $u=(2,3)$ in Figure 2.3, then $a(u)=1$, that is, there is only the cell $v=(2,6)$ satisfying the definition.

Theorem 6. *[15] The non-symmetric Macdonald polynomials $\widehat{E}_\gamma$ are given by the formula*

$$\widehat{E}_\gamma(x;q,t) = \sum_{\substack{\sigma:dg'(\gamma)\to[n]\\ non-attacking}} x^\sigma q^{maj(\widehat{\sigma})} t^{coinv(\widehat{\sigma})} \prod_{\substack{u\in dg'(\gamma)\\ \widehat{\sigma}(u)\neq\widehat{\sigma}(d(u))}} \frac{1-t}{1-q^{l(u)+1}t^{a(u)+1}},$$

where $\gamma \in \mathbb{N}^n$ and $x=(x_1,\dots,x_n)$ a list of indeterminates, $x^\sigma = \prod_{u\in dg'(\gamma)} x_{\sigma(u)}$ and $d(u)$ is the cell directly below u.

Notice that we have used the notation of Mason in [39] for non-symmetric Macdonald polynomial, $\widehat{E}_\gamma(x;q,t)$, which is equal to the notation in [15], $E_\gamma(x;q,t)$, when we replace in latter q and t with $1/q$ and $1/t$ respectively. In [15] it is shown that the Macdonald polynomial monic form $P_\lambda(x;q,t)$ [36] can be written in terms of non-symmetric Macdonald polynomials as in the next proposition, where $\lambda=\gamma^+$.

Proposition 4. *[15] Let λ be a partition and $\omega\lambda$ be the rearrangement of λ in weakly increasing order. Then*

$$P_\lambda(x;q,t) = \prod_{u\in dg'(\omega\lambda)} (1-q^{l(u)+1}t^{a(u)}). \sum_{\substack{\gamma\\ \gamma^+=\lambda}} \frac{\widehat{E}_\gamma(x;q,t)}{\prod_{u\in dg'(\gamma)}(1-q^{l(u)+1}t^{a(u)})},$$

where the sum is over all rearrangements γ of λ.

It is well known [36] that $s_\lambda(x) = lim_{q,t\to 0} P_\lambda(x;q,t)$, so letting $q,t\to 0$ we get the following combinatorial expansion of the Schur polynomials [15],

$$\begin{aligned} s_\lambda(x) &= \prod_{u\in dg'(\omega\lambda)} (1-0^{l(u)+1}0^{a(u)}). \sum_{\substack{\gamma\\ \gamma^+=\lambda}} \frac{\widehat{E}_\gamma(x;0,0)}{\prod_{u\in dg'(\gamma)}(1-0^{l(u)+1}0^{a(u)})} \\ &= \sum_{\substack{\gamma\\ \gamma^+=\lambda}} \widehat{E}_\gamma(x;0,0). \end{aligned}$$

Therefore, in Theorem 6, setting $q = t = 0$ and assuming $q^0 = t^0 = 1$

$$\widehat{E}_\gamma(x; 0, 0) = \sum_{\substack{\sigma : dg'(\gamma) \to [n] \\ non-attacking \\ maj(\widehat{\sigma}) = coinv(\widehat{\sigma}) = 0}} x^\sigma. \tag{2.1}$$

Hence,

$$s_\lambda(x) = \sum_{\substack{\gamma \\ \gamma^+ = \lambda}} \widehat{E}_\gamma(x; 0, 0) = \sum_{\substack{\gamma^+ = \lambda \\ \sigma : dg'(\gamma) \to [n] \\ non-attacking \\ maj(\widehat{\sigma}) = coinv(\widehat{\sigma}) = 0}} x^\sigma. \tag{2.2}$$

2.2 Semi-skyline augmented fillings

The conditions $maj(\widehat{\sigma}) = coinv(\widehat{\sigma}) = 0$ are equivalent to say every columns should be weakly decreasing from bottom to top and every triple should be an inversion triple. Mason in [39] has shown that these two conditions imply a non-attacking condition. Let us define the next object called semi-skyline augmented filling [39].

Definition 3. *A* semi-skyline augmented filling *(SSAF) of an augmented diagram $\widehat{dg}(\gamma)$ is an augmented filling F such that every triple is an inversion triple and columns are weakly decreasing from bottom to top. The shape of the semi-skyline augmented filling is γ and is denoted by $sh(F)$.*

Figure 2.4 is an example of a semi-skyline augmented filling with $sh(F) = (1, 0, 3, 2, 0, 1)$, reading word $read(F)$=1321346, and content $c(F)$=$(2, 1, 2, 1{,}0, 1)$. Corollary 2.4 in [39] allows us to write the next proposition.

Proposition 5. *Every semi-skyline augmented filling F is a non-attacking filling.*

Therefore the combinatorial interpretation of $\widehat{E}_\gamma(x; 0, 0)$ (2.1) is as follows

$$\widehat{E}_\gamma(x; 0, 0) = \sum_{\substack{F\ SSAF \\ sh(F) = \gamma}} x^F. \tag{2.3}$$

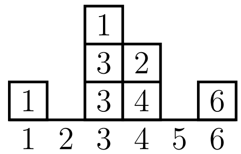
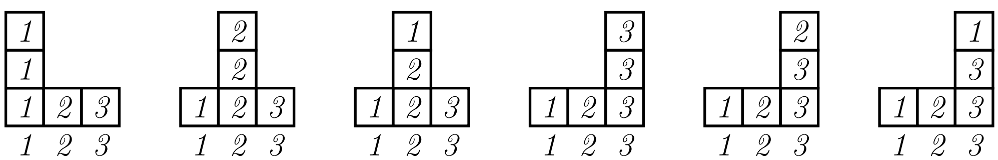


Fig. 2.4: A SSAF F of shape $(1,0,3,2,0,1)$

Thereby the decomposition of the Schur polynomials (2.2) can be written in terms of SSAF [39], as a generating function of all SSAFs of shape γ with entries $\leq n$,

$$s_\lambda(x) = \sum_{\gamma^+=\lambda} \widehat{E}_\gamma(x;0,0) = \sum_{\substack{F\ SSAF\\ sh(F)=\gamma\\ \gamma^+=\lambda}} x^F. \tag{2.4}$$

Example 17. *Decomposition of the Schur polynomial $s_\lambda(x)$ where $\lambda = 311$ and $x = (x_1, x_2, x_3)$, in terms of SSAFs.*



$s_\lambda(x_1,x_2,x_3) = x_1^3x_2x_3 + x_1x_2^3x_3 + x_1^2x_2^2x_3 + x_1x_2x_3^3 + x_1x_2^2x_3^2 + x_1^2x_2x_3^2.$

2.3 Properties of semi-skyline augmented fillings

We collect in this section the properties of SSAF in [39] which are important for our purpose in sections 2.8 and 2.11. The two first properties are consequence of the definition of Semi-skyline augmented filling.

Property 1. The entry of a cell in the first row of a SSAF is equal to the basement element where it sits and, thus, in the first row the cell entries increase from left to the right.

Proof. By contrary suppose that there is a cell b on the top of the basement

a with $F(b) < a$, as follows

$$\begin{array}{ccccccc} & & \boxed{\mathrm{c}} & & \boxed{\mathrm{b}} & & \\ \hline 1 & \cdots & \mathrm{e} & \cdots & \mathrm{a} & \cdots & \mathrm{n} \end{array}.$$

Since $F(b) < a$, there is a basement $e = F(b)$ to the left of the basement a. If the height of the column with the basement a is strictly bigger than the height of column with the basement e, then the triple (a, e, b) is not a type II inversion triple. If the height of column with the basement a is weakly less than the height of column with the basement e, since there is no attacking cell in SSAF, then the cell on the top of the basement e should be different from $F(b)$. As the columns are weakly decreasing from bottom to top the cell on the top of e is , c with $F(c) < F(b)$, and therefore the triple (b, c, e) is not a type I inversion triple. This is a contradiction. So the entries of the first row are equal to their basement. $\square$

Property 2. For any weak composition γ in $\mathbb{N}^n$, there is at least one SSAF with shape γ, by putting γ_i cells with entries i in the top of the basement element i.

Proof. This construction implies decreasing property of columns. So we only need to prove that every triple is an inversion triple. Consider triple,

$$\begin{array}{ccc} & & \boxed{c} \\ \boxed{a} & \cdots & \boxed{b} \end{array}$$

with $F(a) = a,\ F(b) = F(c) = b$ and $a < b$, after standardization we get, $F(a) = 1, F(c) = 2$ and $F(b) = 3$ that is clockwise orientation and so it is type II inversion triple. Now consider triple,

$$\begin{array}{ccc} \boxed{a} & \cdots & \boxed{b} \\ \boxed{c} & & \end{array}$$

with $F(a) = F(c) = a$, $F(b) = b$ and $a < b$, after standardization one gets, $F(a) = 1, F(c) = 2$ and $F(b) = 3$ that is counterclockwise orientation and so it is type I inversion triple. □

In [39] a sequence of lemmas provides several conditions on triples of cells in a SSAF. We recall a property regarding an inversion triple of type II which will be used in the proof of our main theorem in Section 2.11.

Property 3. Consider the type II inversion triple (a, b, c) as below

$$\begin{array}{ccc} & & \boxed{\mathrm{b}} \\ \boxed{\mathrm{a}} & \cdots & \boxed{\mathrm{c}} \end{array}$$

then $F(a) < F(b) \leq F(c)$.

2.4 A weight preserving bijection between reverse semi-standard Young tableaux and semi-skyline augmented fillings

There is a weight preserving bijection between semi-skyline augmented fillings (SSAFs) and reverse semi-standard Young tableaux (RSSYTs) in [39]. Construct the RSSYT, $\rho(F)$, from SSAF, F, by putting the i-th row of SSAF in decreasing order from bottom to top as a i-th column of RSSYT. In [39] it is proved that $\rho(F)$ is a RSSYT. It means that the columns are strictly decreasing from bottom to top and rows are weakly decreasing from left to right. (Note that in [39] the rows of SSAF go to the rows of RSSYT and therefore the RSSYT defined in [39] is the transpose of ours. It means that in there RSSYT is a weakly decreasing from bottom to top and strictly decreasing from left to right.)

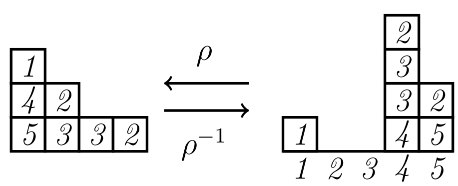
To find the map ρ^{-1}, consider RSSYT $\widetilde{P}$, and call C_i the column i of $\widetilde{P}$. Consider C_1, and put the largest element of C_1 in the leftmost possible place in the first row of an empty augmented diagram to have a decreasing from bottom to top, then put the next largest element of C_1 in the leftmost possible place in the first row of the new augmented diagram, continue this manner to put all the elements of C_1 in the first row. Then put in the same way all the elements of C_2 in the second row of augmented filling and so on. It means that the elements in column i of RSSYT go to the row i of the augmented filling, write $F = \rho^{-1}(\widetilde{P})$. Next theorem says that $\rho^{-1}(\widetilde{P})$ is a SSAF and therefore it can be seen as a map from a collection $\{C_i\}$ of columns to a SSAF.

Theorem 7. *[39] The augmented filling F that has been constructed above is a semi-skyline augmented filling.*

To have a bijection between SSAF and RSSYT, it remains to prove that F is the unique SSAF with row collection $\{C_1, \ldots, C_k\}$ where C_i consists of the elements in column i of the RSSYT $\widetilde{P}$, $1 \leq i \leq k$.

Theorem 8. *[39] The SSAF F is the only SSAF with row entries given by the collection $\{C_i\}_{i=1}^k$, where C_i is the set of entries of column i of the RSSYT $\widetilde{P}$.*

Example 18. *The SSAF corresponding to the RSSYT $\widetilde{P}$*



$\widetilde{P}$ F

$$sh(\widetilde{P}) = (4, 2, 1) \qquad sh(F) = (1, 0, 0, 4, 2)$$

$$c(\widetilde{P}) = c(F) = (1, 2, 2, 1, 1)$$

2.5 An analogue of Schensted insertion

As we have seen in Section 1.7, the fundamental operation of the Robinson-Schensted-Knuth (RSK) correspondence is Schensted insertion which is a procedure for inserting a positive integer k into a semi-standard Young tableau T. In [39] it is defined a similar procedure for inserting a positive integer

k into a SSAF F, which is used to describe an analogue of the RSK correspondence. If F is a SSAF of shape γ, we set $F := (F(j))$, where $F(j)$ is the entry in the j-th cell in reading order, with the cells in the basement included, and j goes from 1 to $n + \sum_{i=1}^{n} \gamma_i$. If $\hat{j}$ is the cell immediately above j and the cell is empty, set $F(\hat{j}) = 0$. The operation $k \to F$, for $k \leq n$, is defined as follows.

Procedure. The insertion $k \to F$:

1. Set $i := 1$, set $x_1 := k$, set $p_0 = \emptyset$, and set $j := 1$.

2. If $F(j) < x_i$ or $F(\hat{j}) \geq x_i$, then increase j by 1 and repeat this step. Otherwise, set $x_{i+1} := F(\hat{j})$ and set $F(\hat{j}) := x_i$. Set $p_i = (b+1, a)$, where (b, a) is the j-th cell in reading order. (This means that the entry x_i "*bumps*" the entry x_{i+1} from the cell p_i.)

3. If $x_{i+1} \neq 0$ then increase i by 1, increase j by 1, and repeat step 2.

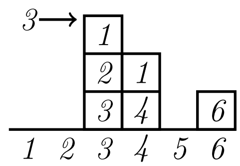
4. Set t_k equal to p_i, which is the termination cell, and terminate the algorithm.

Another way to explain this algorithm is as follows. Scan cells through the reading word, stop at the first cell c with $F(c) \geq k$. If the top of c is empty, then k sits in the top of c and the resulting figure is $k \to F$. Otherwise let a be the cell on the top of c. If $F(a) < k$ then k bumps $F(a)$. In other words, k replaces $F(a)$ and we continue the same manner with the cell immediately to the right of c in $read(F)$. If $F(a) \geq k$ then continue scanning $read(F)$ to find the cell greater than or equal $F(a)$ such that the entry in the top of that, is less than k. Continue this scanning and bumping process until an entry is placed on top of a column. Then the resulting diagram is $k \to F$. See Example 19.

Proposition 6. *[39] The procedure terminates in finitely many steps and the result $k \to F$ is a SSAF.*

Notice that if the insertion element, k, is bigger than the number of elements in basement, n, then the insertion of k to F increases the number of basement elements until k, and k sits in the top of basement k.

Example 19. *Insertion* $3 \to F$



F

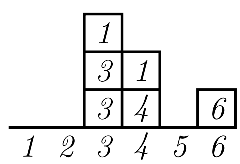
$$i = 1, \quad x_1 := 3, \quad p_0 = \emptyset, \quad j = 1,$$

$$F(1) < x_1 \to j = 2,$$

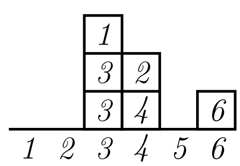
$$F(2) < x_1 \to j = 3,$$

$$F(3) < x_1 \to j = 4,$$

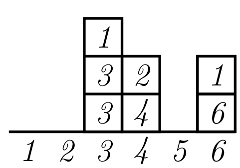
$$x_2 := F(\hat{4}) = 2, \quad F(\hat{4}) := 3, \quad p_1 = (2, 3),$$



$$i = 2, \quad j = 5, \quad x_3 := F(\hat{5}) = 1, \quad F(\hat{5}) := 2, \quad p_2 = (2, 4),$$



$$i = 3, \quad j = 6, \quad x_4 := F(\hat{6}) = 0, \quad F(\hat{6}) := 1, \quad p_3 = t_3 = (2, 6).$$



$3 \to F$

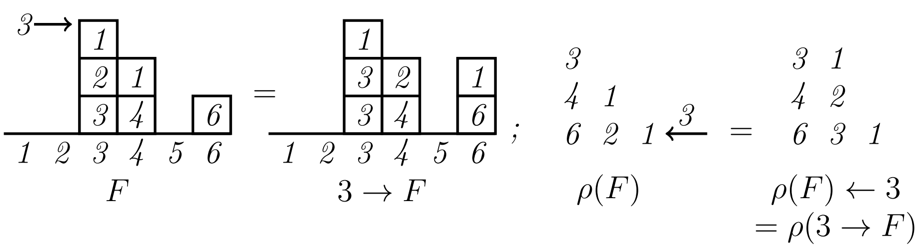
Note that in [39] the analogue of Schensted insertion comes from joining the reverse Schensted row insertion in the RSSYT and map ρ. The relation between the analogue of Schensted insertion $k \to F$ and reverse Schensted row insertion of k into $\rho(F) = \widetilde{P}$, $\widetilde{P} \leftarrow k$, is as follows:

Proposition 7. *[39] If F is a semi-skyline augmented filling and k is an arbitrary positive integer, then the figure $k \to F$ is a semi-skyline augmented filling and the insertion procedure commutes with the map ρ in the sense that*

$$\rho(k \to F) = \rho(F) \leftarrow k.$$

To see it, consider the insertion in Example 19 and the reverse Schensted row insertion for $\rho(F)$.

Example 20. *The map ρ commutes with Schensed insertions.*



2.6 A weight preserving, shape rearranging bijection between semi-standard Young tableaux and semi-skyline augmented fillings

Based on the analogue of Schensted insertion, it is given a weight preserving, shape rearranging bijection Ψ between SSYT and SSAF over the alphabet $[n]$ in [39]. The bijection Ψ is defined to be the insertion, from right to left, of the column word of a SSYT into the empty SSAF with basement $1, \ldots, n$. Consider the SSYT T with column word $col(T) = k_1 \ldots k_t k_{t+1}$ then $F := \Psi(T) = (k_1 \to \cdots (k_t \to (k_{t+1} \to \emptyset)))$. See the Example 21.

Example 21. *The bijection Ψ between SSYT and SSAF on the alphabet* $[5]$

$$T = \begin{array}{ccc} 4 & & \\ 2 & 5 & \\ 1 & 3 & 3 \end{array} \longrightarrow \quad col(T) = 421533$$

$$F_1 := (3 \to \emptyset) = \begin{array}{ccccc} & & \boxed{3} & & \\ \hline 1 & 2 & 3 & 4 & 5 \end{array} \qquad F_2 := (3 \to F_1) = \begin{array}{ccccc} & & \boxed{3} & & \\ & & \boxed{3} & & \\ \hline 1 & 2 & 3 & 4 & 5 \end{array}$$

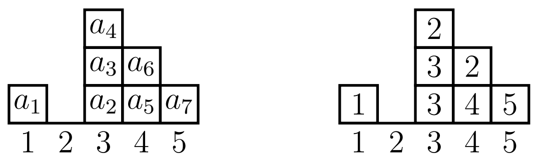
$$F_3 := (5 \to F_2) = \begin{array}{ccccc} & & \boxed{3} & & \\ & & \boxed{3} & & \boxed{5} \\ \hline 1 & 2 & 3 & 4 & 5 \end{array} \qquad F_4 := (1 \to F_3) = \begin{array}{ccccc} & & \boxed{1} & & \\ & & \boxed{3} & & \\ & & \boxed{3} & & \boxed{5} \\ \hline 1 & 2 & 3 & 4 & 5 \end{array}$$

$$F_5 := (2 \to F_4) = \begin{array}{ccccc} & & \boxed{2} & & \\ & & \boxed{3} & & \boxed{1} \\ & & \boxed{3} & & \boxed{5} \\ \hline 1 & 2 & 3 & 4 & 5 \end{array} \qquad \Psi(T) := (4 \to F_5) = \begin{array}{ccccc} & & \boxed{2} & & \\ & & \boxed{3} & & \boxed{4} \\ \boxed{1} & & \boxed{3} & & \boxed{5} \\ \hline 1 & 2 & 3 & 4 & 5 \end{array}$$

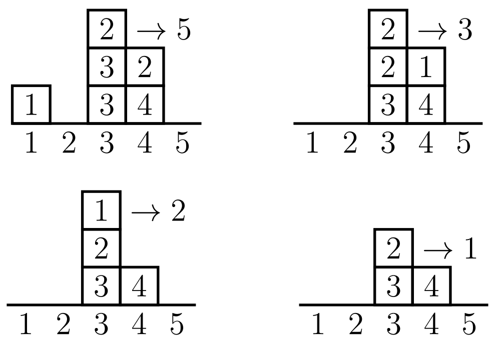
$sh(T) = (3, 2, 1, 0, 0)$, $sh(\Psi(T)) = (1, 0, 3, 0, 2)$, $c(T) = c(\Psi(T)) = (1, 1, 2, 1, 1)$.

Since the analogue of the Schensted insertion is related to the reverse Schensted row insertion via map ρ, if we apply the reverse Schensted row insertion to the $col(T)$, in Example 21, from right to left, the RSSYT that we obtain is $\rho(F)$. To prove that Ψ is a bijection it is shown in [39] that the map Ψ is invertible. Since the analogue of the Schensted insertion is related to the reverse Schensted row insertion via map ρ, and the reverse Schensted row insertion is invertible, then the map Ψ is invertible too. Let F be a SSAF. First consider all the topmost cells of each column and put $A_1 := \{a_1, a_2, \ldots, a_l\}$. Start with the cell in A_1 with smallest height, say a_1, if there are more than one column with the same height consider rightmost as a smaller. Let $F(a_1)$ be a filling of cell a_1, delete $F(a_1)$ and scan the elements in the backward $read(F)$ from a_1 to find the cell b_1 with $F(b_1) > F(a_1)$ and $F(\hat{b}_1) \leq F(a_1)$ or $F(\hat{b}_1) = \emptyset$, where $\hat{b}_1$ is the cell immediately above the cell b_1. Replace $F(b_1)$ with $F(a_1)$ and continue the searching for $F(b_1)$, continue this manner until the first letter of $col(T)$ is reached. Then choose the second elements in A_1 and do the same manner to get the second element of $col(T)$. When we delete all the elements of A_1, consider all the topmost cells of the new columns as a set A_2, and do the same for set A_2. Continue this manner until all the elements of $col(T)$ are reached. We illustrate with the following example.

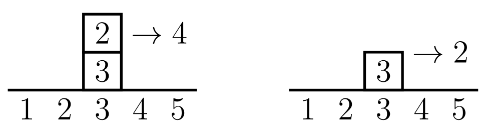
Example 22. *The left picture shows the name of cells in the SSAF F, and on the right we have SSAF F.*



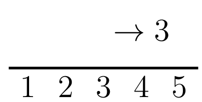
Then $A_1 = \{a_7, a_1a_6, a_4\}$, *delete* a_7 *and continue deleting, one gets,*



Let $A_2 = \{a_5, a_3\}$, *delete first* a_5 *and then* a_3, *one has,*



Finally $A_3 = \{a_2\}$, *after deleting* a_2 *one has,*



Therefore $col(\Psi^{-1}(F)) = 5321.42.3$ *and*

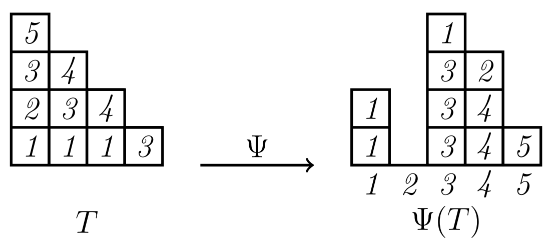
$$T = \Psi^{-1}(F) = \begin{array}{lll} 5 & & \\ 3 & & \\ 2 & 4 & \\ 1 & 2 & 3 \end{array}$$

2.7 The right key of a tableau via semi-skyline augmented filling

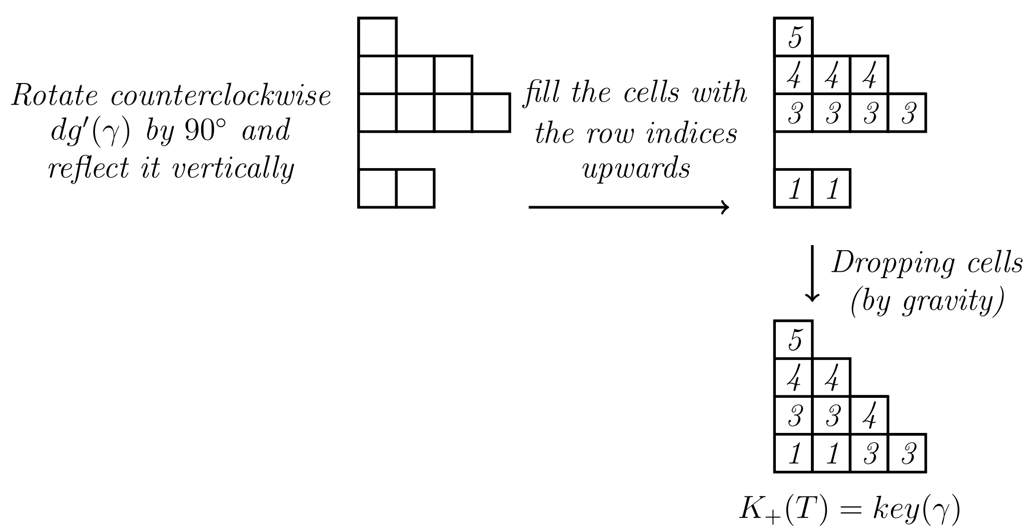
We have seen in Section 1.8, the original definition of the right key. In [40] it is shown that the right key can be defined in terms of SSAF. The bijection Ψ provides the right key of tableau T, $K_+(T)$, in the following theorem.

Theorem 9. *[40] Given an arbitrary SSYT T, let γ be the shape of $\Psi(T)$. Then $K_+(T) = key(\gamma)$.*

Example 23.



Consider $dg'(sh(\Psi(T)))$, where $sh(\Psi(T)) = \gamma = (2, 0, 4, 3, 1)$.



Since the map Ψ is a weight preserving and shape rearranging bijection between SSYTs and SSAFs, one has

$$\{T \in SSYT_n : sh(T) = \lambda\} = \biguplus_{\gamma^+=\lambda} \{T \in SSYT_n : K_+(T) = key(\gamma)\}$$

and therefore

$$\sum_{\substack{F\ SSAF \\ sh(F)=\gamma}} x^F = \sum_{\substack{T\ SSYT \\ sh(T)=\gamma^+ \\ K_+(T)=key(\gamma)}} x^T,$$

for all $\gamma^+ = \lambda$.

From this equation and (2.4), we have the following combinatorial expansions for Schur polynomials

$$s_\lambda(x) = \sum_{\gamma^+=\lambda} \widehat{E}_\gamma(x;0,0) = \sum_{\substack{F\ SSAF \\ sh(F)=\gamma \\ \gamma^+=\lambda}} x^F = \sum_{\substack{T\ SSYT \\ sh(T)=\gamma^+ \\ K_+(T)=key(\gamma) \\ \gamma^+=\lambda}} x^T.$$

Mason has proved in [40] that $\widehat{E}_\gamma(x,0,0)$ are equivalent to Demazure atoms to be discussed in the next chapter.

2.8 An analogue of Robinson-Schensted-Knuth correspondence

We have introduced the RSK correspondence in the first chapter. Given the alphabet $[n]$, the RSK correspondence is a bijection between biwords in lexicographic order on the alphabet $[n]$ and pairs of SSYTs of the same shape over $[n]$. Equipped with the analogue of Schensted insertion, Mason finds in [39] an analogue Φ of the RSK yielding a pair of SSAFs. This bijection has an advantage over the classical RSK because the pair of SSAFs comes along with the extra pair of right keys. The map Φ defines a bijection between the set $\mathbb{A}_n$ of all biwords w in lexicographic order in the alphabet $[n]$ or, equivalently, $\mathbb{N}$-matrices of finite support, and pairs of SSAFs whose shapes are rearrangements of the same partition in $\mathbb{N}^n$, and the contents are, respectively, those of the second and first rows of w. Let $SSAF_n$ be the set of all semi skylines augmented fillings with basement $[n]$.

Procedure. The map $\Phi : \mathbb{A}_n \longrightarrow SSAF_n \times SSAF_n$. Let $w \in \mathbb{A}_n$.

1. Set $r := l$, where l is the number of biletters in w and the index of rightmost biletter in w. Let $F = \emptyset = G$, where $\emptyset$ is the empty $SSAF$.

2. Set $F := (j_r \to F)$. Let h_r be the height of the column in $(j_r \to F)$ at which the insertion procedure $(j_r \to F)$ terminates.

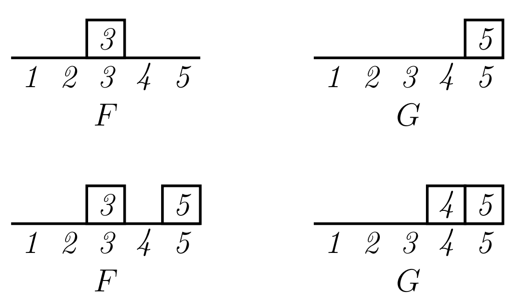
3. Place i_r on top of the leftmost column of height $h_r - 1$ in G such that doing so preserves the decreasing property of columns from bottom to top. Set G equal to the resulting figure.

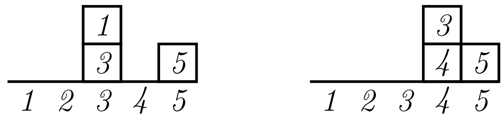
4. If $r - 1 \neq 0$, repeat step 2 for $r := r - 1$. Else terminates the algorithm.

Remark 1. 1. *The entries in the top row of the biword are weakly increasing when read from left to right. Henceforth, if $h_r > 1$, placing i_r on top of the leftmost column of height $h_r - 1$ in G preserves the decreasing property of columns. If $h_r = 1$, the i_r^{th} column of G does not contain an entry from a previous step. It means that number i_r sits on the top of basement i_r.*
2. *Let h be the height of the column in F at which the insertion procedure $(j \to F)$ terminates. The condition on inversion triples of type II, in Property 1 implies that there is no column of height $h + 1$ in F to the right.*

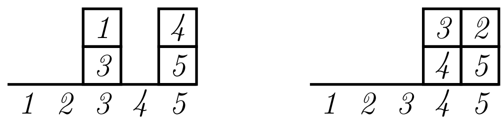
The Example 24 shows the action of map Φ.

Example 24. *Consider biword* $w = \begin{pmatrix} 1 & 2 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 & 3 \end{pmatrix}$, *starting from rightmost biletter and applying map* Φ *one has,*

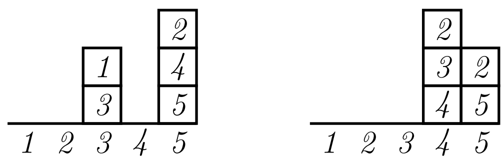




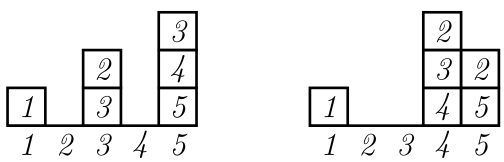
F $\qquad$ G



F $\qquad$ G



F $\qquad$ G



F $\qquad$ G

The map Φ is invertible, in [39] there is a description of the inverse of Φ.

2.9 A triangle of Robinson-Schensted-Knuth correspondences

There are two propositions in [39] that give relations between the analogue of RSK, the reverse RSK, and RSK. One is a relation between RSK and the analogue of RSK as below.

Proposition 8. [39, 40] *The RSK correspondence commutes with the above analogue* Φ. *That is, if* (P, Q) *is the pair of SSYTs produced by RSK correspondence applied to biword* w, *then* $(\Psi(P), \Psi(Q)) = \Phi(w)$, *and* $K_+(P) = key(sh(\Psi(P)))$, $K_+(Q) = key(sh(\Psi(Q)))$.

$$\begin{tikzcd} & w \arrow[dl, dash, "\mathrm{RSK}"'] \arrow[dr, dash, "\Phi"] & \\ (P,Q) \arrow[rr, dash, "\Psi"'] & & (F,G) \end{tikzcd}$$

$$\begin{gathered} c(P) = c(F), \quad c(Q) = c(G), \\ sh(F)^{+} = sh(G)^{+} = sh(P) = sh(Q), \\ K_{+}(P) = key(sh(F)), \ K_{+}(Q) = key(sh(G)). \end{gathered}$$

Fig. 2.5: The relation between RSK and the analogue of RSK

This result is summarised in the Figure 2.5 from which, in particular, it is clear the RSK analogue Φ also shares the symmetry of RSK.

The other relation is between the reverse RSK and the analogue of RSK as below.

Proposition 9. *[39] If $(\widetilde{P}, \widetilde{Q})$ is the pair of reverse semi-standard Young tableaux obtained by applying the reverse RSK correspondence to the biword w, then*

$$(\rho^{-1}(\widetilde{P}), \rho^{-1}(\widetilde{Q})) = \Phi(w) = (F, G).$$

Putting together these two propositions and also the relations between SSAF, SSYT and RSSYT, one obtains Figure 2.6.

2.10 Growth diagram presentation of an analogue of Robinson-Schensted-Knuth corrspondence

Figure 2.6 shows the relation between RSK, reverse RSK and analogue of RSK. Since there are presentations of RSK and reverse RSK in terms of Fomin's growth diagrams, it is interesting to find a similar presentation of the analogue of RSK. The map ρ gives a nice relation between SSAF and RSSYT, and from that we are able to find the pair of SSAFs using a Fomin's growth diagram. Let w be a biword in the lexicographic order. Consider the growth diagram associated to the biword w. As we have seen before,

$$\begin{tikzcd}[arrows=dash]
 & w \arrow[dl, "\text{reverse RSK}"'] \arrow[d, "\Phi"] \arrow[dr, "\text{RSK}"] & \\
(\widetilde{P}, \widetilde{Q}) \arrow[r, "\rho"'] & (F, G) \arrow[r, "\Psi"'] & (P, Q)
\end{tikzcd}$$

$$sh(F)^+ = sh(G)^+ = sh(P) = sh(Q) = sh(\widetilde{P}) = sh(\widetilde{Q})$$

$$key(sh(F)) = K_+(P),\ key(sh(G)) = K_+(Q)$$

$$c(F) = c(P) = c(\widetilde{P}),\ c(G) = c(Q) = c(\widetilde{Q})$$

Fig. 2.6: The relation between RSK, reverse RSK and analogue of RSK

applying backward local rules we are led to a pair of sequences of partitions respectively, on the left and on the bottom of the growth diagram. The partitions of each sequence are related by containment and they give a pair of RSSYTs. Now we are going to use these sequences of partitions to get a pair of SSAFs as follows.

Let λ_i be the partition associated to the i-th thick column line on the bottom of the growth diagram when we scan the thick column lines from right to left, and consider the sequence of partitions $\{\lambda_{i_j}\}$, where $\lambda_i = \lambda_{i_1} \subseteq \cdots \subseteq \lambda_{i_{l_i}}$, associated with the thick column line i and the $l_i - 1$ thin column lines strictly in between the two thick column lines i and $i+1$ counted from right to left. Start with the rightmost partition of the bottom sequence in the growth diagram and also with an empty SSAF. When we arrive to the partition λ_{i_j} we put a cell with filling $n+1-i$ in the leftmost possible place of the SSAF with basement 1 through n, such that the shape of the new SSAF becomes a rearrangement of the partition λ_{i_j} and the decreasing property on the columns of SSAF, from bottom to top, is preserved.

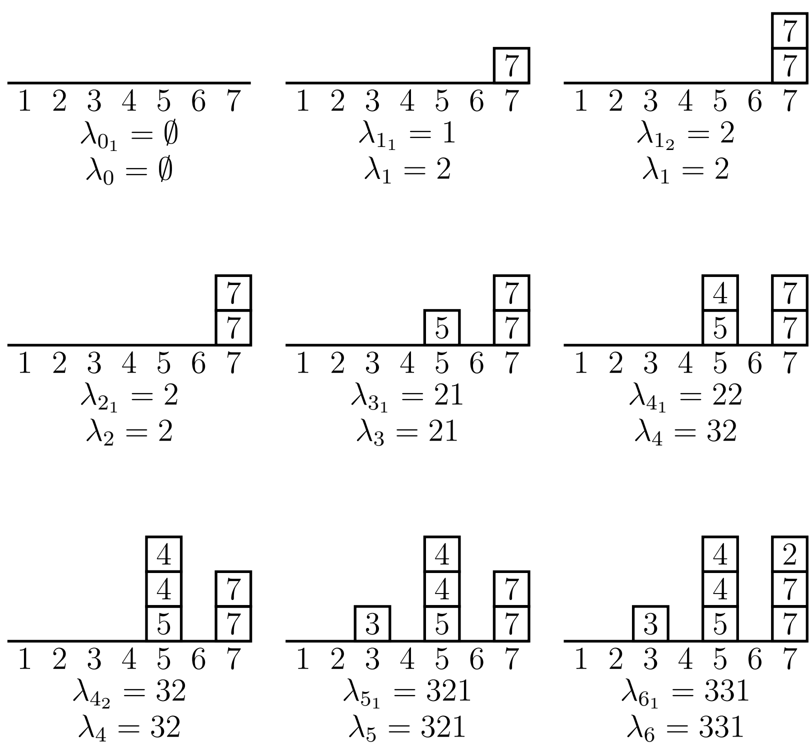
Similarly let $\underline{\lambda}_i$ be the partition associated to the i-th thick row line in the left of the growth diagram when we scan the thick row lines from top to bottom, and consider the sequence of partitions $\{\underline{\lambda}_{i_j}\}$, where $\underline{\lambda}_i = \underline{\lambda}_{i_1} \subseteq$

$\cdots \subseteq \underline{\lambda}_{i_{e_i}}$, associated with the thick row line i and the $e_i - 1$ thin row lines strictly in between the two thick row lines i and $i+1$ counted from top to bottom. Start with the topmost partition of the left sequence in the growth diagram and also with an empty SSAF. When we arrive to the partition $\underline{\lambda}_{i_j}$ we put a cell with filling $n+1-i$ in the leftmost possible place of the SSAF with basement 1 through n, such that the shape of the new SSAF becomes a rearrangement of the partition $\underline{\lambda}_{i_j}$ and the decreasing property on the columns of SSAF, from bottom to top, is preserved.

The growth diagram corresponding to the reverse RSK for

$$w = \begin{pmatrix} 1 & 1 & 2 & 3 & 4 & 4 & 5 & 7 & 7 \\ 2 & 7 & 2 & 4 & 1 & 3 & 3 & 1 & 1 \end{pmatrix}$$

is shown in the Figure 2.7. The SSAF that is obtained from the bottom sequence of partitions is constructed below.



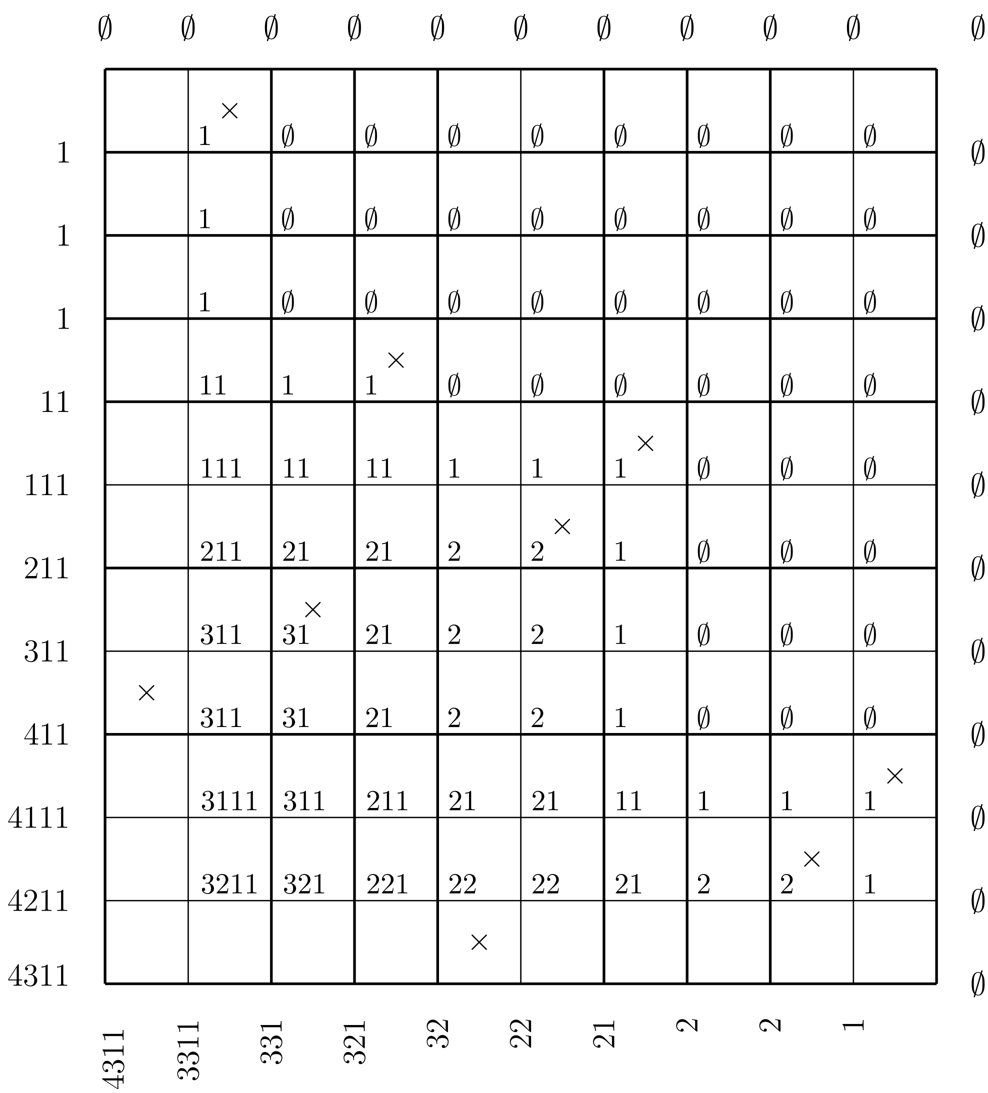
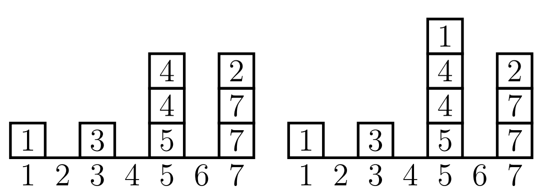


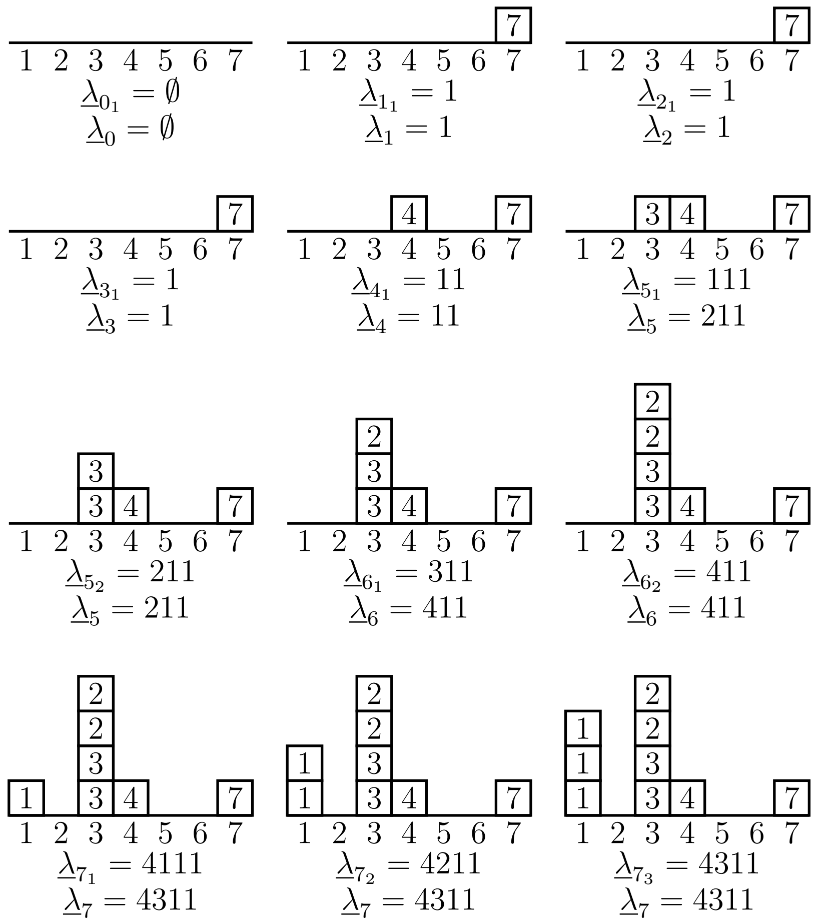
Fig. 2.7: Growth diagram with backward local rules



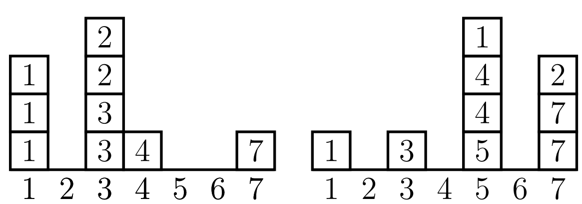
$\lambda_{7_1} = 3311$ $\lambda_{7_2} = 4311$

$\lambda_7 = 4311$ $\lambda_7 = 4311$

The same procedure for the left side of growth diagram gives,



Therefore the pair of SSAF corresponding to the growth diagram in Figure 2.7 is,



which through map ρ^{-1} corresponds to the pair $(\widetilde{P}, \widetilde{Q})$ of RSSYT in the growth diagram of Figure 1.6.

2.11 Main theorem

We prove a restriction of the bijection Φ to multisets of cells in a staircase or truncated staircase of length n, such that the staircases of length $n-k$ on the upper left corner, or of length $n-m$ on the bottom right corner, with $1 \leq m \leq n$, $1 \leq k \leq n$ and $k+m \geq n+1$, are erased. The restriction to be imposed on the pairs of SSAFs is that the pair of shapes in a same $\mathfrak{S}_n$-orbit, satisfy an inequality in the Bruhat order, where one shape is bounded by the reverse of the other. Equivalently, pairs of SSYTs whose right keys are such that one is bounded by the Schützenberger's evacuation of the other.

The following lemma gives sufficient conditions to preserve the Bruhat order relation between two weak compositions when one box is added to a column of each column diagram.

Lemma 1. *Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ be two weak compositions in $\mathbb{N}^n$, rearrangements of each other, with key$(\beta) \leq$ key(α). Given $k \in \{1, \ldots, n\}$, let $k' \in \{1, \ldots, n\}$ be such that $\beta_{k'}$ is the left most entry of β satisfying $\beta_{k'} = \alpha_k$. Then if $\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_k + 1, \ldots, \alpha_n)$ and $\tilde{\beta} = (\beta_1, \beta_2, \ldots, \beta_{k'} + 1, \ldots, \beta_n)$, it holds key$(\tilde{\beta}) \leq$ key$(\tilde{\alpha})$.*

Proof. Let $k,\ k' \in \{1, \ldots, n\}$ as in the lemma, and put $\alpha_k = \beta_{k'} = m \geq 1$. (The proof for $m = 0$ is left to the reader. The case of interest for our problem is $m > 0$ which is related with the procedure of map Φ.) This means that k appears exactly in the first m columns of *key*(α), and k' is the smallest number that does not appear in column $m+1$ of *key*(β) but appears exactly in the first m columns. Let t be the row index of the cell with entry k' in column m of *key*(β). Every entry less than k' in column m of *key*(β) appears in column $m+1$ as well, and since in a key tableau each column is contained in the previous one, this implies that the first t rows of columns m and $m+1$ of *key*$(\tilde{\beta})$ are equal. The only difference between *key*$(\tilde{\beta})$ and *key*(β) is in columns $m+1$, from row t to the top. Similarly if z

is the row index of the cell with entry k in column $m+1$ of $key(\tilde{\alpha})$, the only difference between $key(\tilde{\alpha})$ and $key(\alpha)$ is in columns $m+1$ from row z to the top. To obtain column $m+1$ of $key(\tilde{\beta})$, shift in the column $m+1$ of $key(\beta)$ all the cells with entries $> k'$ one row up, and add to the position left vacant (of row index t) a new cell with entry k'. The column $m+1$ of $key(\tilde{\alpha})$ is obtained similarly, by shifting one row up in the column $m+1$ of $key(\alpha)$ all the cells with entries $> k$ and adding a new cell with entry k in the vacant position. See Figure 2.8. Put $p := \min\{t, z\}$ and $q := \max\{t, z\}$. We divide

$$
\begin{array}{cccccc}
\mathit{column} & \mathit{column} & \qquad & \mathit{column} & \mathit{column} \\
m & m+1 & & m & m+1 \\
key(\beta) & key(\beta) & & key(\tilde{\beta}) & key(\tilde{\beta}) \\
 & & & & \\
b_l & & & b_l & \\
\vdots & & & \vdots & \\
b_{q+1} & & & b_{q+1} & d_q \\
b_q & d_q & & b_q & \vdots \\
\vdots & \vdots & & \vdots & \\
b_{p+1} & d_{p+1} & & b_{p+1} & d_p \\
k' & d_p & & k' & k' \\
b_{p-1} & d_{p-1} & & b_{p-1} & d_{p-1} \\
\vdots & \vdots & & \vdots & \vdots \\
b_1 & d_1 & & b_1 & d_1
\end{array}
$$

Fig. 2.8: .

the columns $m+1$ in each tableau pair $key(\beta)$, $key(\tilde{\beta})$ and $key(\alpha)$, $key(\tilde{\alpha})$ into three parts: the first, from row one to row $p-1$; the second, from row p to row q; and the third, from row $q+1$ to the top row. The first parts of column $m+1$ of $key(\tilde{\beta})$ and $key(\beta)$ are the same, equivalently, for $key(\tilde{\alpha})$ and $key(\alpha)$. The third part of column $m+1$ of $key(\tilde{\beta})$ consists of row q plus the third part of $key(\beta)$, equivalently, for $key(\tilde{\alpha})$ and $key(\alpha)$. As columns

$m+1$ of $key(\beta)$ and $key(\alpha)$ are entrywise comparable, the same happens to the first and third parts of columns $m+1$ in $key(\tilde{\beta})$ and $key(\tilde{\alpha})$. It remains to analyse the second parts of the pair $key(\tilde{\beta})$, $key(\tilde{\alpha})$ which we split into two cases according to the relative magnitude of p and q.

Case 1. $p = t < q = z$. Let $k' < b_t < \cdots < b_{z-1}$ and $d_t < \cdots < d_{z-1} < k$ be respectively the cell entries of the second parts of columns $m+1$ in the pair $key(\tilde{\beta})$, $key(\tilde{\alpha})$. By construction $k' < b_t \leq d_t < d_{t+1}$, $b_i < b_{i+1} \leq d_{i+1}$, $t < i < z-2$, and $b_{z-1} \leq d_{z-1} < k$, and, therefore, the second parts are also comparable.

$$
\begin{array}{cc}
\begin{array}{rcc}
 & \begin{array}{c}\textit{column}\\ m+1\\ key(\tilde{\beta})\end{array} & \begin{array}{c}\textit{column}\\ m+1\\ key(\tilde{\alpha})\end{array} \\[2ex]
\text{part 3}\left[\begin{array}{c} b_{l'}\\ \vdots\\ b_z\end{array}\right. & \begin{array}{c} \\ \\ \\ \end{array} & \begin{array}{c} d_{l'}\\ \vdots\\ d_z\end{array} \\
\text{part 2}\left[\begin{array}{c} b_{z-1}\\ b_{z-2}\\ \vdots\\ b_t\\ \color{red}{k'}\end{array}\right. & & \begin{array}{c} \color{red}{k}\\ d_{z-1}\\ \vdots\\ d_{t+1}\\ d_t\end{array} \\
\text{part 1}\left[\begin{array}{c} b_{t-1}\\ \vdots\\ b_1\end{array}\right. & & \begin{array}{c} d_{t-1}\\ \vdots\\ d_1\end{array}
\end{array}
&
\begin{array}{ccl}
\begin{array}{c}\textit{column}\\ m+1\\ key(\tilde{\beta})\end{array} & & \begin{array}{c}\textit{column}\\ m+1\\ key(\tilde{\alpha})\end{array} \\[2ex]
\begin{array}{c} b_{l'}\\ \vdots\\ b_t\end{array} & & \left.\begin{array}{c} d_{l'}\\ \vdots\\ d_t\end{array}\right]\text{part 3} \\
\begin{array}{c} \color{red}{k'}\\ b_{t-1}\\ \vdots\\ b_{z+1}\\ b_z\end{array} & & \left.\begin{array}{c} d_{t-1}\\ d_{t-2}\\ \vdots\\ d_z\\ \color{red}{k}\end{array}\right]\text{part 2} \\
\begin{array}{c} b_{z-1}\\ \vdots\\ b_1\end{array} & & \left.\begin{array}{c} d_{z-1}\\ \vdots\\ d_1\end{array}\right]\text{part 1}
\end{array}
\\[2ex]
\text{Case 1} & \text{Case 2}
\end{array}
$$

Case 2. $p = z \leq q = t$. In this case, the assumption on k' implies that the first q rows of columns m and $m+1$ of $key(\tilde{\beta})$ are equal. On the other hand, since column m of $key(\beta)$ is less or equal than column m of $key(\alpha)$, which is equal to the column m of $key(\tilde{\alpha})$ and in turn is less or equal to column $m+1$ of $key(\tilde{\alpha})$, forces by transitivity that the second part of column $m+1$

of $key(\tilde{\beta})$ is less or equal than the corresponding part of $key(\tilde{\alpha})$. □

Example 25. *We illustrate the lemma with* $\beta = (3, 2^2, 1, 0^2, 1)$,
α=$(2, 0, 3, 0, 1, 2, 1)$, $\tilde{\beta} = (3, 2^3, 0^2, 1)$, *and* $\tilde{\alpha} = (2, 0, 3, 0, 2^2, 1)$,

$$key(\beta) = \begin{array}{|c|c|c|}\cline{1-1} 7 \\ \cline{1-1} 4 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \leq key(\alpha) = \begin{array}{|c|c|c|}\cline{1-1} 7 \\ \cline{1-1} 6 \\ \cline{1-2} 5 & 6 \\ \cline{1-2} 3 & 3 \\ \hline 1 & 1 & 3 \\ \hline \end{array} \qquad key(\tilde{\beta}) = \begin{array}{|c|c|c|}\cline{1-1} 7 \\ \cline{1-2} 4 & 4 \\ \cline{1-2} 3 & 3 \\ \cline{1-2} 2 & 2 \\ \hline 1 & 1 & 1 \\ \hline \end{array} \leq key(\tilde{\alpha}) = \begin{array}{|c|c|c|}\cline{1-1} 7 \\ \cline{1-2} 6 & 6 \\ \cline{1-2} 5 & 5 \\ \cline{1-2} 3 & 3 \\ \hline 1 & 1 & 3 \\ \hline \end{array}.$$

We are now ready to state and prove the main theorem.

Theorem 10. *Let w be a biword in lexicographic order in the alphabet $[n]$, and let $\Phi(w) = (F, G)$. For each biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in w one has $i+j \leq n+1$ if and only if $key(sh(G)) \leq key(\omega sh(F))$, where ω is the longest permutation of $\mathfrak{S}_n$. Moreover, if the first row of w is a word in the alphabet $[k]$, with $1 \leq k \leq n$, and the second row is a word in the alphabet $[m]$, with $1 \leq m \leq n$, the shape of G has the last $n-k$ entries equal to zero, and the shape of F the last $n-m$ entries equal to zero.*

Proof. *"Only if part".* We prove by induction on the number of biletters of w. If w is the empty word then F and G are the empty semi-skyline augmented filling and there is nothing to prove. Let $w' = \begin{pmatrix} i_{p+1} & i_p & \cdots & i_1 \\ j_{p+1} & j_p & \cdots & j_1 \end{pmatrix}$ be a biword in lexicographic order such that $p \geq 0$ and $i_t + j_t \leq n+1$ for all $1 \leq t \leq p+1$, and $w = \begin{pmatrix} i_p \cdots i_1 \\ j_p \cdots j_1 \end{pmatrix}$ such that $\Phi(w) = (F, G)$. Let $F' := (j_{p+1} \to F)$ and h the height of the column in F' at which the insertion procedure terminates. There are two possibilities for h which the third step of the algorithm procedure of Φ requires to consider.

• $h = 1$. It means j_{p+1} is sited on the top of the basement element j_{p+1} in F and therefore i_{p+1} goes to the top of the basement element i_{p+1} in G. Let G' be the semi-skyline augmented filling obtained after placing i_{p+1} in G. As $i_{p+1} \leq i_t$, for all t, i_{p+1} is the bottom entry of the first column in $key(sh(G'))$ whose remain entries constitute the first column of $key(sh(G)$. Suppose $n+1-j_{p+1}$ is added to the row z of the first column in $key(\omega sh(F))$ by shifting all the entries above it one row up. Let $i_{p+1} < a_1 < \cdots < a_z < a_{z+1} < \cdots < a_l$ and $b_1 < b_2 < \cdots < n+1-j_{p+1} < b_z < \cdots < b_l$ be respectively the cell entries of the first columns in the pair $key(sh(G'))$, $key(\omega sh(F'))$, where $a_1 < \cdots < a_z < \cdots < a_l$ and $b_1 < \cdots < b_z < \cdots < b_l$ are respectively the cell entries of the first columns in the pair $key(sh(G))$, $key(\omega sh(F))$. If $z = 1$, as $i_{p+1} \leq n+1-j_{p+1}$ and $a_i \leq b_i$ for all $1 \leq i \leq l$, then $key(sh(G')) \leq key(\omega sh(F'))$. If $z > 1$, as $i_{p+1} < a_1 \leq b_1 < b_2$, we have $i_{p+1} \leq b_1$ and $a_1 \leq b_2$. Similarly $a_i \leq b_i < b_{i+1}$, and $a_i \leq b_{i+1}$, for all $2 \leq i \leq z-2$. Moreover $a_{z-1} \leq b_{z-1} < n+1-j_{p+1}$, therefore $a_{z-1} \leq n+1-j_{p+1}$. Also $a_i \leq b_i$ for all $z \leq i \leq l$. Hence. $key(sh(G')) \leq key(\omega sh(F'))$.

• $h > 1$. Place i_{p+1} on the top of the leftmost column of height $h-1$. This means by Lemma 1 $key(sh(G')) \leq key(\omega sh(F'))$.

"*If part*". We prove the contrapositive statement. If there exists a biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in w such that $i+j > n+1$, then at least one entry of $key(sh(G))$ is strictly bigger than the corresponding entry of $key(\omega sh(F))$.

Let $w = \begin{pmatrix} i_p \cdots i_1 \\ j_p \cdots j_1 \end{pmatrix}$ be a biword in lexicographic order on the alphabet

$[n]$, and $\left(\begin{array}{c} i_t \\ j_t \end{array}\right)$ the first biletter in w, from right to left, with $i_t+j_t>n+1$. Set $F_0=G_0:=\emptyset$, and for $d\geq 1$, let $(F_d,G_d):=\Phi\left(\begin{array}{c} i_d\cdots i_1 \\ j_d\cdots j_1 \end{array}\right)$. First apply the map Φ to the biword $\left(\begin{array}{c} i_{t-1}\cdots i_1 \\ j_{t-1}\cdots j_1 \end{array}\right)$ to obtain the pair (F_{t-1},G_{t-1}) of SSAFs whose right keys satisfy, by the "only if part" of the theorem, $key(sh(G_{t-1}))\leq key(\omega sh(F_{t-1}))$. Now insert j_t to F_{t-1}. As $i_k+j_k\leq n+1$, for $1\leq k\leq t-1$, $i_k+j_k\leq n+1<i_t+j_t$, and $i_t\leq i_k$, $1\leq k\leq t-1$, then $j_t>j_k$, $1\leq k\leq t-1$, and, since w is in lexicographic order, this implies $i_t<i_{t-1}$. Therefore, j_t sits on the top of the basement element j_t in F_{t-1} and i_t sits on the top of the basement element i_t in G_{t-1}. It means that $n+1-j_t$ is added to the first row and first column of $key(\omega sh(F_{t-1}))$ and all entries in this column are shifted one row up. Similarly, i_t is added to the first row and first column of $key(sh(G_{t-1}))$, and all the entries in this column are shifted one row up. As $i_t>n+1-j_t$ then the first columns of $key(sh(G_t))$ and $key(\omega sh(F_t))$ respectively, are not entrywise comparable, and we say that we have a "*problem*" in the key-pair $(key(sh(G_t)),key(\omega sh(F_t)))$. From now on, "*problem*" means $i_t>n+1-j_t$ in some row of a pair of columns in the key-pair $(key(sh(G_d)),key(\omega sh(F_d)))$, with $d\geq t$. Let $d\geq t$ and denote by J the column with basement j_t in F_d, and by I the column with basement i_t in G_d. Let $|J|$ and $|I|$ denote, respectively, the height of J and I, and let r_i and k_i denote the number of columns of height $\geq i\geq 1$, respectively, to the right of J and to the left of I. The classification of the *"problem"* will follow from a sequence of four claims below.

Claim 1: Let (F_d, G_d), with $d \geq t$. Then $k_i \geq r_i \geq 0$, for all $i \geq 1$.

Proof. By induction on $d \geq t$. For $d = t$, one has, $k_i = r_i = 0$, for all $i \geq 1$. Let $d \geq t$, and suppose (F_d, G_d) satisfies $k_i \geq r_i \geq 0$, for all $i \geq 1$. Let us prove for (F_{d+1}, G_{d+1}). If the insertion of j_{d+1} terminates on a column of height l to the left or on the top of J, then $r_i := r_i$, for all i, $k_i := k_i$, for all $i \neq l+1$, and $k_{l+1} := k_{l+1} + 1$, or k_{l+1}. Thus, $k_i \geq r_i$, for all $i \geq 1$. On the other hand, if the insertion of j_{d+1} terminates to the right of J, then in F_d one has $r_l > r_{l+1}$, and two cases have to be considered for placing i_{d+1} in G_d. First, i_{d+1} sits on the left of I and, hence, $k_{l+1} := k_{l+1} + 1 \geq r_{l+1} := r_{l+1} + 1$, $k_i := k_i \geq r_i := r_i$, for $i \neq l+1$. Second, either i_{d+1} sits on the top of I or to the right of I, in both cases, (F_d, G_d) satisfy $k_{l+1} = k_l \geq r_l > r_{l+1}$, and, therefore, $k_{l+1} > r_{l+1}$. This implies for (F_{d+1}, G_{d+1}), $r_{l+1} := r_{l+1} + 1$, and $k_{l+1} := k_{l+1} \geq r_{l+1}$, $k_i := k_i \geq r_i := r_i$, for $i \neq l+1$. The next scheme shows all the possibilities after inserting new biletters to (F_d, G_d).

$$\begin{array}{c} j_{d+1} \text{ inserts to the left} \\ \text{of } J \text{ with height } l : \end{array} \left[\begin{array}{l} 1.\ i_{d+1} \text{ sits on the left of } I : \\ \quad k_{l+1} := k_{l+1} + 1, \text{ so } k_i \geq r_i.\checkmark \\ \\ 2.\ i_{d+1} \text{ sits on the top of } I,\ k_i \geq r_i.\checkmark \\ \\ 3.\ i_{d+1} \text{ sits on the right of } I : k_i \geq r_i.\ \checkmark \end{array}\right.$$

j_{d+1} inserts to the top of J with height l :

1. i_{d+1} sits on the left of I :
   $k_{l+1} := k_{l+1} + 1$, so $k_i \geq r_i$.✓
2. i_{d+1} sits on the top of I : $k_i \geq r_i$.✓
3. i_{d+1} sits on the right of I : $k_i \geq r_i$. ✓

j_{d+1} inserts to the right of J with height l :

1. i_{d+1} sits on the left of I :
   $k_{l+1} := k_{l+1} + 1, r_{l+1} := r_{l+1} + 1$ so $k_i \geq r_i$.✓
2. i_{d+1} sits on the top of I : $k_l = k_{l+1}$, $r_l > r_{l+1}$, so $k_{l+1} = k_l \geq r_l > r_{l+1}$, $k_{l+1} > r_{l+1}$, therefore $k_{l+1} > r_{l+1}$, $r_{l+1} := r_{l+1} + 1$, so $k_{l+1} \geq r_{l+1}$.✓
3. i_{d+1} sits on the right of I : $k_l = k_{l+1}$, $r_l > r_{l+1}$, so $k_{l+1} = k_l \geq r_l > r_{l+1}$, $k_{l+1} > r_{l+1}$, therefore $k_{l+1} > r_{l+1}$, $r_{l+1} := r_{l+1} + 1$, so $k_{l+1} \geq r_{l+1}$.✓

□

Claim 2. Let (F_d, G_d), with $d \geq t$. If $|J| > |I|$, then $k_i > r_i \geq 0$, $i = |I| + 1, \ldots, |J|$.

Proof. Since, for $d = t$, it holds $|I| = |J|$, there is a $d > t$ where for the first time one has $|J| = |I| + 1$. We assume that, for some $d > t$, one has (F_d, G_d) with $|J| - |I| \geq 1$. Then, either (F_{d-1}, G_{d-1}) has $|I| = |J|$ or $|J| > |I|$. In the first case, it means that the insertion of j_d has terminated on the top of J and the cell i_d sits on the left of I on a column of height $|J| = |I|$, otherwise, it would sit on the top of I. Then, by the previous claim, $k_{|J|+1} := k_{|J|+1} + 1 > r_{|J|+1} := r_{|J|+1}$. In the second case, we suppose that, (F_{d-1}, G_{d-1}) satisfies $k_i > r_i \geq 0$, for $i = |I| + 1, \ldots, |J|$. Put $z := |I|$ and $h := |J|$. Let us prove for (F_d, G_d), when $|J| > |I|$. If the insertion of j_d terminates in a column of height l $(\neq h-1)$ to the left of J then $r_i := r_i$, for all $i \geq 1$, $k_{l+1} := k_{l+1} + 1$, or k_{l+1} and $k_i := k_i$, for $i \neq l+1$, and $z \leq |I| < |J| = h$, $|J| - |I| \geq 1$. Therefore, $k_i > r_i \geq 0$, for $i = |I|+1, \dots, |J|$. If the insertion terminates on the top of J, then $|J| = h+1$, $|I| = z$, $r_i := r_i$, for all $i \geq 1$, $k_i := k_i$, for $i = z+1, \dots, h$, and $k_{h+1} := k_{h+1} + 1 > r_{h+1}$ or $k_{h+1} := k_h > r_h \geq r_{h+1}$. Again $k_i > r_i$, for $i = |I|+1, \dots, |J| = h+1$. Finally, if the insertion terminates to the right of J, $|J| = h$ and three cases for the height l have to be considered. When $l < z$, or $l \geq h$, $r_i := r_i < k_i := k_i$, for $i = z+1, \dots, |J|$; when $l = z$, then either $|I| = z$ and $k_{z+1} := k_{z+1} + 1 > r_{z+1} := r_{z+1} + 1$, $k_i := k_i > r_i := r_i$, $z < i \leq |J|$, or $z+1 = |I| \leq |J|$ and $k_i := k_i > r_i := r_i$, $i = z+2, \dots, |J|$; and when $z < l < h$, then $|I| = z$ and $r_i := r_i$, $i \neq l+1$, and either $k_{l+1} := k_{l+1} + 1 > r_{l+1} := r_{l+1} + 1$ or $k_{l+1} = k_l > r_l \geq r_{l+1} := r_{l+1} + 1$. Henceforth $k_i > r_i$, for $i = |I|+1, \dots, |J|$. The next scheme shows all the possibilities after inserting new biletters to (F_d, G_d).

j_{d+1} inserts to the left of J with height l :

1. i_{d+1} sits on the left of I : $k_{l+1} := k_{l+1} + 1 > r_{l+1}$, so, $k_i > r_i$ for $i = z+1, \dots h$.✓
2. i_{d+1} sits on the top of I $|I| := z+1$,
   - 2.1. $z = h-1$; $|I| = |J|$.
   - 2.2. $z < h-1$; $k_i > r_i$, for $i = z+2, \dots h$.✓
3. i_{d+1} sits on the right of I : $k_i > r_i$ for $i = z+1, \dots h$. ✓

j_{d+1} inserts to the top of J with height h :

1. i_{d+1} sits on the left of I :
$|J| = h + 1$, by claim 1.1, $k_{h+1} \geq r_{h+1}$,
put $k_{h+1} := k_{h+1} + 1$, so $k_{h+1} > r_{h+1}$,
therefore $k_i > r_i$ for $i = z + 1, \ldots, h + 1$. ✓

2. i_{d+1} sits on the top of I : impossible,
because $|I| < h$.✓

3. i_{d+1} sits on the right of I :
$|J| = h + 1$, as $k_{h+1} = k_h$, and $k_h > r_h$, so
$k_{h+1} = k_h > r_h \geq r_{h+1}$, so $k_{h+1} > r_{h+1}$,
hence $k_i > r_i$ for $i = z + 1, \ldots h + 1$. ✓

j_{d+1} inserts to the right of J with height l :

- 1. $l < z : r_{l+1} := r_{l+1} + 1$,
  $k_i > r_i$ for $i = z+1, \dots h.$✓
- 2. $l = z$
  - 2.1. i_{d+1} sits on the left of I :
    $r_{z+1} := r_{z+1} + 1$, $k_{z+1} := k_{z+1} + 1$,
    so $k_i > r_i$ for $i = z+1, \dots h.$✓
  - 2.2. i_{d+1} sits on the top of I
    - 2.2.1. $z = h - 1$.
      $|I| = z + 1$, $|I| = |J|$.
    - 2.2.2. $z < h - 1$.
      $r_{z+1} := r_{z+1} + 1$,
      $|I| := |I| + 1$, $k_i > r_i$,
      for $i = z+2, \dots h.$✓
- 3. $z < l \leq h$
  - 3.1. i_{d+1} sits on the left of I :
    $r_{l+1} := r_{l+1} + 1, k_{l+1} := k_{l+1} + 1$,
    $k_i > r_i$ for $i = z+1, \dots h.$✓
  - 3.2. i_{d+1} sits on the top of I :
    impossible, because $l \neq z$.
  - 3.3. i_{d+1} sits on the right of I :
    $k_{l+1} = k_l > r_l \geq r_{l+1} + 1$, so
    $k_{l+1} > r_{l+1} + 1$, $r_{l+1} := r_{l+1} + 1$,
    so $k_{l+1} > r_{l+1}$, and $k_i > r_i$
    for $i = z+1, \dots h.$✓
- 4. $l > h : k_i > r_i$, for $i = z+1, \dots, h.$✓

□

Claim 3: Let (F_d, G_d), with $d \geq t$, be such that, for some $s \geq 1$, one has $|I|, |J| \geq s$ and $k_s = r_s > 0$. Then, for (F_{d+1}, G_{d+1}) there exists also a $s \geq 1$ with the same properties.

Proof. Observe that, from the previous claim, $k_{s+1} = r_{s+1}$ and $|J| \geq s+1$ only if $|I| \geq s+1$. If the insertion of j_{d+1} terminates on the top of a column of height $l \neq s-1$, then still $|I|, |J| \geq s$ and $k_s = r_s > 0$. It remains to analyse when $l = s-1$ which means that the insertion of j_{d+1} either terminates to the left or to the right of J. In the first case, (F_d, G_d) satisfies $|J| \geq s+1$ (using Remark 1), $r_s = r_{s+1}$, and, therefore, $k_s \geq k_{s+1} \geq r_{s+1} = r_s = k_s \geq k_{s+1}$. It implies for (F_{d+1}, G_{d+1}) that $k_{s+1} = r_{s+1} > 0$, $|J|, |I| \geq s+1$, and thus the claim is true for $s+1$. In the second case, (F_d, G_d) satisfies $k_{s-1} \geq r_{s-1} > r_s = k_s$ and thus $k_{s-1} > k_s$. Thereby the cell i_{d+1} sits to the left of I and $r_s := r_s + 1 = k_s := k_s + 1$, with $|I|, |J| \geq s$. The claim is true for s. The next scheme shows all the possibilities after inserting new biletters to (F_d, G_d).

- 1.j_{d+1} inserts to the left of J with height l :
  - 1.1. $l = s-1$
    - 1.1.1. i_{d+1} sits to the left of I: by Remark 1 $r_s = r_{s+1}$, $k_{s+1} \leq k_s = r_s = r_{s+1} \leq k_{s+1}$, so $k_s = k_{s+1} = r_{s+1}$, by contrapositive of claim 1.2, $|I| \geq |J| \geq s+1$ therefore claim is true for $s+1$.✓
    - 1.1.2. i_{d+1} sits to the top of I: impossible, because $|I| \geq s$,
    - 1.1.3.i_{d+1} sits to the right of I: $k_s = r_s$.✓
  - 1.2. $l \neq s-1$: $r_s = k_s$.✓

2.j_{d+1} inserts to the top of J with height l, as $|J| \geq s$, so $l \neq s-1$, and therefore $k_s = r_s$.✓

3.j_{d+1} inserts to the right of J with height l

- 3.1. $l = s - 1$
  - 3.1.1. i_{d+1} sits to the left of I: $k_s := k_s + 1$, $r_s := r_s + 1$, therefore $k_s = r_s$.✓
  - 3.1.2. i_{d+1} sits to the top of I: impossible, because $r_{s-1} > r_s$, $k_{s-1} = k_s = r_s$, so $r_{s-1} > k_{s-1}$, contradiction by claim 1.1.
  - 3.1.3.i_{d+1} sits to the right of I: impossible, the same reason as above.
- 3.2. $l \neq s - 1 : r_s = k_s$.✓

□

Next claim describes the pair (F_d, G_d) of SSAFs, for $d \geq t$, when it does not fit the conditions of Claim 3.

Claim 4. Let (F_d, G_d), with $d \geq t$, be a pair of SSAFs such that, for all $i = 1, \ldots, \min\{|I|, |J|\}$, $k_i = r_i > 0$ never holds. Then, $|J| \leq |I|$ and, there is $1 \leq f \leq |J|$, such that $k_i > r_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$.

Proof. We show by induction on $d \geq t$ that (F_d, G_d) either satisfy the conditions of the Claim 3 or, otherwise, $|J| \leq |I|$ and, there is $1 \leq f \leq |J|$, such that $r_i < k_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$. For $d = t$, we have $|I| = |J| = 1$, and $k_i = r_i = 0$, $i \geq 1$. Put $f := 1$. Let (F_d, G_d), with $d \geq t$. If (F_d, G_d) fits the conditions of Claim 3, then (F_{d+1}, G_{d+1}) does it as well. Otherwise, assume for (F_d, G_d), $|J| \leq |I|$, and, there exists $1 \leq f \leq |J|$, such that $r_i < k_i$, for $1 \leq i < f$, and $k_i = r_i = 0$, for $i \geq f$. We show next that (F_{d+1}, G_{d+1}) either fits the conditions of the previous Claim 3, or, otherwise, it is as described in the present claim. If the insertion of j_{d+1} terminates to the left of J, and i_{d+1} sits on the top or to the right of I, still $|I| \geq |J|$ and there is nothing to prove. If i_{d+1} sits on the top of a column of height l, to the left of I, then, since $k_f = 0$, one has $l < f$, and two cases have to be considered. When $l = f - 1$, it implies $|I| \geq |J| \geq f + 1$, $r_f = 0$ and $k_f := 1$, and (F_{d+1}, G_{d+1}) satisfies the claim for $f + 1$; in the case of $l < f - 1$, $r_{l+1} < k_{l+1} := k_{l+1} + 1$ and still, for the same f, $k_i > r_i$, $1 \leq i < f$, $k_i = r_i = 0$, $i \geq f$. If the insertion of j_{d+1} terminates on the top of J, since $k_{|J|} = 0$ and $|I| \geq |J|$, then i_{d+1} either sits on the top of I when $|I| = |J|$, and still for the same f, $k_i > r_i$, $1 \leq i < f$, $k_i = r_i = 0$, $i \geq f$, or sits to the right of I, when $|I| > |J|$, and still $|I| \geq |J| + 1$, and, for the same f, $k_i > r_i$, $1 \leq i < f$, $k_i = r_i = 0$, $i \geq f$. If the insertion of j_{d+1} terminates to the right of J on the top of a column of height $l < f$ (recall that $r_f = 0$), then, since $|I| > f$, i_{d+1} either sits on the left of I or to the right of I. In the first case, if $l = f - 1$, one has $r_f := r_f + 1 = k_f := k_f + 1 = 1$, and, therefore, we are in the conditions of Claim 3, with $s = f < |J| \leq |I|$; if $l < f - 1$, still $r_{l+1} := r_{l+1} + 1 < k_{l+1} := k_{l+1} + 1$, so $k_i > r_i$, for $1 \leq i < f$ and $r_i = k_i = 0$, for $i \geq f$. In the second case, it means $k_{l+1} = k_l > r_l \geq r_{l+1} := r_{l+1} + 1$ and hence $k_{l+1} > r_{l+1} := r_{l+1} + 1$, with $l + 1 < f$. Similarly, $k_i > r_i$, for $1 \leq i < f$ and $k_i = r_i = 0$, $i \geq f$. The next scheme shows all the possibilities after inserting new biletters to (F_d, G_d).

j_{d+1} inserts to the left of J with height l :

- 1. i_{d+1} sits to the left of I
  - 1.1. $l < f-1$, $r_{l+1} < k_{l+1} := k_{l+1}+1$, $|J| \leq |I|$. ✓
  - 1.2. $l = f-1$, $|J| > f$, $r_f = 0, k_f = 1$, put $f := f+1$, $f \leq |J| \leq |I|$. ✓
  - 1.3. $l \geq f$, impossible.
- 2. i_{d+1} sits to the top of $I : k_i > r_i$, $1 \leq i < f$, $k_f = \ldots k_{|J|} = 0$, $|J| \leq |I|$. ✓
- 3. i_{d+1} sits to the right of $I : k_i > r_i$, $1 \leq i < f$, $k_f = \ldots k_{|J|} = 0$, $|J| \leq |I|$. ✓

j_{d+1} inserts to the top of J

- 1. i_{d+1} sits on the left of I : impossible, because $k_{|J|} = 0$.
- 2. i_{d+1} sits on the top of $I : k_i > r_i$, $1 \leq i < f$, $|J| := |J|+1 \leq |I| := |I|+1$, $k_{|J|+1} = 0$. ✓
- 3. i_{d+1} sits on the right of I : $|I| > |J|$, $k_i > r_i$, $1 \leq i < f$, $|J| := |J|+1 \leq |I|$, $k_{|J|+1} = 0$. ✓

$$
\begin{array}{c} j_{d+1} \text{ inserts to the right} \\ \text{of } J \text{ with height } l<f, \\ \text{because } k_f=0: \end{array}
\left[\begin{array}{l}
\begin{array}{c} 1.\ i_{d+1} \text{ sits to} \\ \text{the left of } I \end{array}
\left[\begin{array}{l}
1.1.\ l=f-1, \\
r_f:=r_f+1=k_f:=k_f+1=1, \\
\text{put } s=f,\ k_f=r_f. \\
\\
1.2.\ l<f-1, \\
r_{l+1}:=r_{l+1}+1<k_{l+1}:=k_{l+1}+1, \\
k_i>r_i,\ 1\le i<f, \\
k_f=\cdots=k_{|J|}=0.\ \checkmark
\end{array}\right. \\
\\
\begin{array}{l} 2.\ i_{d+1} \text{ sits to the top of } I: \text{impossible,} \\ \quad \text{because } l<f<|J|\le|I|. \end{array} \\
\\
\begin{array}{c} 3.\ i_{d+1} \text{ sits to} \\ \text{the right of } I \end{array}
\left[\begin{array}{l}
3.1.\ r_{l+1}+1=k_{l+1}, \text{ impossible,} \\
\text{because } k_l=k_{l+1}=r_{l+1}+1\le r_l+1 \\
\text{so } k_l<r_l. \\
\\
3.2.\ r_{l+1}+1<k_{l+1}, \\
r_{l+1}:=r_{l+1}+1<k_{l+1}, \\
k_i>r_i,\ 1\le i<f, \\
k_f=\cdots=k_{|J|}=0.\ \checkmark
\end{array}\right.
\end{array}\right.
$$

$\square$

Classification of the "problem": For any $d \ge t$, either there exists $s \ge 1$ such that $|J|, |I| \ge s$, $r_s = k_s > 0$; or $1 \le |J| \le |I|$, and there exists $1 \le f \le |J|$, such that $k_i > r_i$, for $1 \le i < f$, and $k_i = r_i = 0$, for $i \ge f$. In the first case, one has a *"problem"* in the $(r_s+1)^{th}$ rows of the s^{th} columns in the key-pair $(key(sh(G_d)), key(\omega sh(F_d)))$. In the second case, one has a *"problem"* in the bottom of the $|J|^{th}$ columns.

Finally, if the second row of w is over the alphabet $[m]$, there is no cell on the top of the basement of F greater than m. Therefore, the shape of F has the last $n-m$ entries equal to zero and thus its decreasing rearrangement is a partition of length $\le m$. Using the symmetry of Φ, the other case is similar. □

Remark 2. *1. Given $\nu \in \mathbb{N}^n$ and $\beta \leq \omega\nu$, there exists always a pair (F, G) of SSAFs with shapes ν and β respectively. Construct F and G as it is explained in Lemma 2.3.*

2. If the rows in w are swapped, one obtains the biword $\tilde{w}$ such that $\Phi(\tilde{w}) = (G, F)$ with $key(sh(F) \leq key(\omega sh(G)))$. It comes from Proposition 3 and scheme in the Figure 2.6.

Using the bijection Ψ between $SSYT$ and $SSAF$ and Proposition 1 one has,

Corollary 1. *Let w be a biword in lexicographic order in the alphabet $[n]$, and let $w \xrightarrow{RSK} (P, Q)$. For each biletter $\begin{pmatrix} i \\ j \end{pmatrix}$ in w we have $i + j \leq n + 1$ if and only if $K_+(Q) \leq evac(K_+(P))$.*

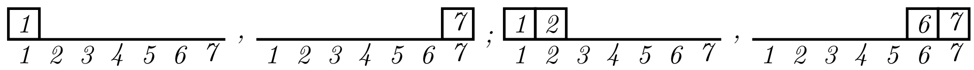
This result can be described in a picture.

$$\begin{array}{ccccc} \multicolumn{5}{c}{\overbrace{\hphantom{w \quad\quad (F,G) \quad\quad (P,Q)}}^{\text{RSK}}} \\ w & \overset{\Phi}{\longleftrightarrow} & (F,G) & \overset{\Psi}{\longleftrightarrow} & (P,Q) \\ i+j \leq n+1 & & sh(F) = \alpha,\ sh(G) = \beta & & K_+(P) = key(\alpha) \\ & & \beta \leq \omega\alpha & & K_+(Q) = key(\beta) \\ & & & & \beta \leq \omega\alpha \\ & & & & K_+(Q) \leq evac(K_+(P)). \end{array}$$

Two examples are given to illustrate Theorem 10.

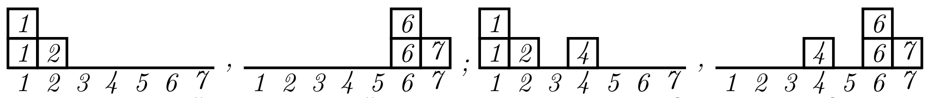
Example 26. *1. Given $w = \begin{pmatrix} 4 & 6 & 6 & 7 \\ 4 & 1 & 2 & 1 \end{pmatrix}$, $\Phi(w)$ and the key-pair*

$key(sh(G)) \leq key(\omega sh(F))$ are calculated.



$sh(F_1) = (1, 0^6)$ $sh(G_1) = (0^6, 1)$ $sh(F_2) = (1^2, 0^5)$ $sh(G_2) = (0^5, 1^2)$

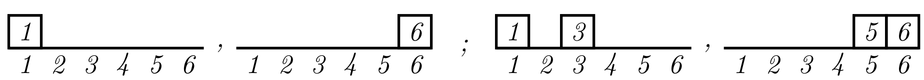
$key(sh(G_1)) = 7 = key(\omega sh(F_1))$ $key(sh(G_2)) = 67 = key(\omega sh(F_2))$



$sh(F_3) = (2, 1, 0^5)$ $sh(G_3) = (0^5, 2, 1)$ $sh(F_4) = (2, 1, 0, 1, 0^3)$ $sh(G_4) = (0^3, 1, 0, 2, 1)$

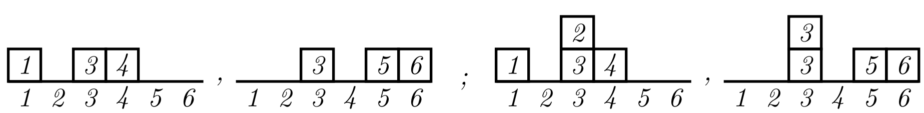
$$key(sh(G_3)) = \begin{array}{ll} 7 & \\ 6 & 6 \end{array} \leq \begin{array}{ll} 7 & \\ 6 & 7 \end{array} = key(\omega sh(F_3)) \qquad key(sh(G_4)) = \begin{array}{ll} 7 & \\ 6 & \\ 4 & 6 \end{array} \leq \begin{array}{ll} 7 & \\ 6 & \\ 4 & 7 \end{array} = key(\omega sh(F_4))$$

2. Let $w = \begin{pmatrix} 1 & 2 & 3 & 3 & 5 & 6 \\ 6 & 3 & 2 & 4 & 3 & 1 \end{pmatrix}$, *with* $n = 6$, $i_2 = 5 > 6 + 1 - 3$. *We calculate* $\Phi(w)$ *whose key-pair* $key(sh(G))$, $key(\omega sh(F))$ *is not entrywise comparable.*



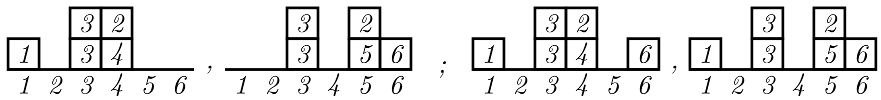
$sh(F_1) = (1, 0^5)$ $sh(G_1) = (0^5, 1)$ $sh(F_2) = (1, 0, 1, 0^3)$ $sh(G_2) = (0^4, 1^2)$

$$key(sh(G_1)) = 6 = key(\omega sh(F_1)) \qquad key(sh(G_2)) = \begin{array}{l} 6 \\ 5 \end{array} \not\leq \begin{array}{l} 6 \\ 4 \end{array} = key(\omega sh(F_2))$$



$sh(F_3) = (1, 0, 1^2, 0^2)$ $sh(G_3) = (0^2, 1, 0, 1^2)$ $sh(F_4) = (1, 0, 2, 1, 0^2)$ $sh(G_4) = (0^2, 2, 0, 1^2)$

$$key(sh(G_3)) = \begin{array}{l} 6 \\ 5 \\ 3 \end{array} \not\leq \begin{array}{l} 6 \\ 4 \\ 3 \end{array} = key(\omega sh(F_3)) \qquad key(sh(G_4)) = \begin{array}{ll} 6 & \\ 5 & \\ 3 & 3 \end{array} \not\leq \begin{array}{ll} 6 & \\ 4 & \\ 3 & 4 \end{array} = key(\omega sh(F_4))$$



$sh(F_5) = (1, 0, 2^2, 0^2)$ $sh(G_5) = (0^2, 2, 0, 2, 1)$ $sh(F_6) = (1, 0, 2^2, 0, 1)$ $sh(G_6) = (1, 0, 2, 0, 2, 1)$

$$key(sh(G_5)) = \begin{array}{cc} 6 & \\ \color{red}{5} & \color{red}{5} \\ 3 & 3 \end{array} \not\leq \begin{array}{cc} 6 & \\ \color{red}{4} & \color{red}{4} \\ 3 & 3 \end{array} = key(\omega sh(F_5)) \quad key(sh(G_6)) = \begin{array}{cc} 6 & \\ \color{red}{5} & \\ 3 & \color{red}{5} \\ 1 & 3 \end{array} \not\leq \begin{array}{cc} 6 & \\ \color{red}{4} & \\ 3 & \color{red}{4} \\ 1 & 3 \end{array} = key(\omega sh(F_6))$$

3. DEMAZURE CHARACTER AND DEMAZURE ATOM

Demazure characters (or key polynomials) can be defined through Demazure operators (or isobaric divided differences). They were introduced by Demazure [8] for all Weyl groups and were studied combinatorially, in the case of $\mathfrak{S}_n$, by Lascoux and Schützenberger [28, 32] who produce a crystal structure. They have also decomposed Demazure characters into non-intersecting pieces called Demazure atoms or standard basis [32].

3.1 Isobaric divided differences and the generators of the 0-Hecke algebra

Let $\mathbb{Z}[x_1, \ldots, x_n]$ be the set of all polynomials in the indeterminates $x_1, \ldots, x_n$ and coefficients over $\mathbb{Z}$. The action of the simple transpositions $s_i = (i\ i{+}1)$ of $\mathfrak{S}_n$ on weak compositions $v = (v_1, \ldots, v_n) \in \mathbb{N}^n$, $s_i v := (v_1, \ldots, v_{i+1}, v_i, \ldots, v_n)$, for $1 \leq i < n$, induces an action of $\mathfrak{S}_n$ on the polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$ by considering vectors v as exponents of monomials $x^v := x_1^{v_1} x_2^{v_2} \cdots x_n^{v_n}$ [29]. That is, $x^{s_i v}$ induces the simple transposition of x_i and x_{i+1} in the monomial x^v and, therefore, if $f \in \mathbb{Z}[x_1, \ldots, x_n]$, $s_i f$ indicates the result of the action of s_i in each monomial of f.

For $i = 1, \ldots, n{-}1$, define the linear operators ∂_i, π_i and $\widehat{\pi}_i$ on $\mathbb{Z}[x_1, \ldots, x_n]$ [45] by

$$\partial_i = \frac{1 - s_i}{x_i - x_{i+1}}, \tag{3.1}$$

$$\pi_i = \partial_i x_i \ \ \& \ \ \widehat{\pi}_i := \pi_i - 1, \tag{3.2}$$

where 1 is the identity operator on $\mathbb{Z}[x_1, \ldots, x_n]$. So

$$\pi_i : f \mapsto \pi_i f := \frac{1 - s_i}{x_i - x_{i+1}}(x_i f) = \frac{x_i f - x_{i+1} s_i f}{x_i - x_{i+1}}, \quad \widehat{\pi}_i f = \pi_i f - f,\ 1 \leq i < n. \tag{3.3}$$

The operators (3.2) are called *isobaric divided differences* [27, 42], where the first is the Demazure operator [8, 29, 18] for the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$. The operators (3.1) and (3.2) satisfy the relations,

$$\begin{aligned} \partial_i^2 &= 0 \\ \partial_i\partial_j &= \partial_j\partial_i \quad \text{for } |i-j|>1 \\ \partial_i\partial_{i+1}\partial_i &= \partial_{i+1}\partial_i\partial_{i+1} \end{aligned} \tag{3.4}$$

$$\begin{aligned} \pi_i^2 &= \pi_i \\ \pi_i\pi_j &= \pi_j\pi_i \quad \text{for } |i-j|>1 \\ \pi_i\pi_{i+1}\pi_i &= \pi_{i+1}\pi_i\pi_{i+1} \end{aligned} \tag{3.5}$$

$$\begin{aligned} \widehat{\pi}_i^2 &= -\widehat{\pi}_i \\ \widehat{\pi}_i\widehat{\pi}_j &= \widehat{\pi}_j\widehat{\pi}_i \qquad \text{for } |i-j|>1 \\ \widehat{\pi}_i\widehat{\pi}_{i+1}\widehat{\pi}_i &= \widehat{\pi}_{i+1}\widehat{\pi}_i\widehat{\pi}_{i+1}, \end{aligned} \tag{3.6}$$

where the last two relations in (3.4), (3.5) and (3.6) are the commutation and braid relations of the symmetric group, and the first are called quadratic relations.

Let $w \in \mathfrak{S}_n$ and let $w = s_{i_N}\cdots s_{i_2}s_{i_1}$ be a reduced decomposition of w. Define

$$\pi_w = \pi_{i_N}\cdots\pi_{i_2}\pi_{i_1}, \qquad \widehat{\pi}_w = \widehat{\pi}_{i_N}\cdots\widehat{\pi}_{i_2}\widehat{\pi}_{i_1}. \tag{3.7}$$

Since any two reduced decompositions are connected by a sequence of braid and commutation relations the operators π_w and $\widehat{\pi}_w$ are well defined.

Isobaric divided difference operators π_i and $\widehat{\pi}_i$, $1 \le i < n$, (3.2), have an equivalent definition

$$\pi_i(x_i^a x_{i+1}^b m) = \begin{cases} x_i^a x_{i+1}^b m + (\sum_{j=1}^{a-b} x_i^{a-j} x_{i+1}^{b+j})m, & \text{if } a>b, \\ x_i^a x_{i+1}^b m, & \text{if } a=b, \\ x_i^a x_{i+1}^b m - (\sum_{j=0}^{b-a-1} x_i^{a+j} x_{i+1}^{b-j})m, & \text{if } a<b, \end{cases} \tag{3.8}$$

and

$$\widehat{\pi}_i(x_i^a x_{i+1}^b m) = \begin{cases} (\sum_{j=1}^{a-b} x_i^{a-j} x_{i+1}^{b+j})m, & \text{if } a>b, \\ 0, & \text{if } a=b, \\ -(\sum_{j=0}^{b-a-1} x_i^{a+j} x_{i+1}^{b-j})m, & \text{if } a<b, \end{cases} \tag{3.9}$$

where m is a monomial not containing x_i nor x_{i+1}. It follows from the definition that $\pi_i(f) = f$ and $\widehat{\pi}_i(f) = 0$ if and only if $s_i f = f$.

The 0-Hecke algebra $H_n(0)$ of $\mathfrak{S}_n$, a deformation of the group algebra of $\mathfrak{S}_n$, is an associative $\mathbb{C}$-algebra generated by $T_1, \ldots, T_{n-1}$ satisfying the commutation and the braid relations of the symmetric group $\mathfrak{S}_n$, and the quadratic relation $T_i^2 = T_i$ for $1 \leq i < n$. Setting $\widehat{T}_i := T_i - 1$, for $1 \leq i < n$, one obtains another set of generators of the 0-Hecke algebra $H_n(0)$.

The sets $\{T_\sigma : \sigma \in \mathfrak{S}_n\}$ and $\{\widehat{T}_\sigma : \sigma \in \mathfrak{S}_n\}$ are both linear basis for $H_n(0)$, where $T_\sigma := T_{i_N} \cdots T_{i_2} T_{i_1}$ and $\widehat{T}_\sigma := \widehat{T}_{i_N} \cdots \widehat{T}_{i_2} \widehat{T}_{i_1}$, for any reduced expression $s_{i_N} \cdots s_{i_2} s_{i_1}$ in $\mathfrak{S}_n$ [5]. Demazure operators (3.8) and bubble sort operators (1.6) satisfy the same relations as T_i, and, similarly, isobaric divided difference operators (3.9) and $\widehat{T}_i$. Thus the 0-Hecke algebra $H_n(0)$ of $\mathfrak{S}_n$ may be viewed as an algebra of operators realised either by any of the two isobaric divided differences (3.2), or by bubble sort operators (1.6), swapping entries i and $i+1$ in a weak composition α, if $\alpha_i > \alpha_{i+1}$, and doing nothing, otherwise. The two families $\{\pi_\sigma : \sigma \in \mathfrak{S}_n\}$ and $\{\widehat{\pi}_\sigma : \sigma \in \mathfrak{S}_n\}$ are therefore both linear basis for $H_n(0)$, and from the relation $\widehat{\pi}_i = \pi_i - 1$, the change of basis from one to the other is given by a sum over the Bruhat order in $\mathfrak{S}_n$, precisely [26, 43],

$$\pi_\sigma = \sum_{\theta \leq \sigma} \widehat{\pi}_\theta, \quad \widehat{\pi}_\sigma = \sum_{\theta \leq \sigma} (-1)^{\ell(\theta) - \ell(\sigma)} \pi_\theta. \tag{3.10}$$

3.2 Demazure characters, Demazure atoms and sorting operators

Let $\lambda \in \mathbb{N}^n$ be a partition and α a weak composition in the $\mathfrak{S}_n$-orbit of λ. Write $\alpha = \sigma\lambda$, where σ is a minimal length coset representative of $\mathfrak{S}_n / stab_\lambda$. The key polynomial [32, 45] or Demazure character [8, 18] in type A, corresponding to the dominant weight λ and permutation σ, is the polynomial in $\mathbb{Z}[x_1, \ldots, x_n]$ indexed by the weak composition $\alpha \in \mathbb{N}^n$, defined by

$$\kappa_\alpha := \pi_\sigma x^\lambda. \tag{3.11}$$

If β is a rearrangement of α, the set of monomials which appears in the Demazure characters κ_α and κ_β may in general intersect nontrivially. If $\beta \leq \alpha$ then the set of monomials in κ_β is a subset of the monomials in κ_α. For example when $n = 4$ and $\lambda = (2, 1, 1, 0)$ one has, $(2, 1, 1, 0) < (2, 1, 0, 1) <$

$(2, 0, 1, 1)$, in the Bruhat order and

$$\kappa_{2011} = \pi_2\pi_3(x^{2110}) = \pi_2\kappa_{2101} = \pi_2(x^{2110} + x^{2101}) = x^{2110} + x^{2101} + x^{2011},$$

and

$$\kappa_{2101} = \pi_3(x^{2110}) = x^{2110} + x^{2101}.$$

The monomials in κ_{2101} also appear in κ_{2011}. This motivates replacing the operator π_i with $\widehat{\pi}_i = \pi_i - 1$ in the previous example $\widehat{\pi}_2\widehat{\pi}_3(x^{2110}) = x^{2011}$ and leads to the following definition.

The standard basis [28, 32] or Demazure atom [40] is

$$\widehat{\kappa}_\alpha := \widehat{\pi}_\sigma x^\lambda. \tag{3.12}$$

Due to (3.10), the Demazure atom $\widehat{\kappa}_\alpha$ consists of all monomials in κ_α which do not appear in κ_β for any $\beta < \alpha$. Thereby, key polynomials (3.11) are decomposed into Demazure atoms [32, 29],

$$\kappa_\alpha = \sum_{\beta \leq \alpha} \widehat{\kappa}_\beta. \tag{3.13}$$

Key polynomials $\{\kappa_\alpha : \alpha \in \mathbb{N}^n\}$ and Demazure atoms $\{\widehat{\kappa}_\alpha : \alpha \in \mathbb{N}^n\}$ form a linear $\mathbb{Z}$-basis for $\mathbb{Z}[x_1, \ldots, x_n]$ [45]. The change of basis from the first to the second is expressed in (3.13). The operators π_i act on key polynomials κ_α via elementary bubble sorting operators on the entries of the weak composition α [45],

$$\pi_i\kappa_\alpha = \begin{cases} \kappa_{s_i\alpha} & \text{if } \alpha_i > \alpha_{i+1} \\ \kappa_\alpha & \text{if } \alpha_i \leq \alpha_{i+1} \end{cases}. \tag{3.14}$$

The general description of the action of the Isobaric divided differences on κ_α and $\widehat{\kappa}_\alpha$ [42] is given by,

$$\widehat{\pi}_i\widehat{\kappa}_\alpha = \begin{cases} \widehat{\kappa}_{s_i\alpha} & \text{if } \alpha_i > \alpha_{i+1} \\ 0 & \text{if } \alpha_i = \alpha_{i+1} \\ -\widehat{\kappa}_\alpha & \text{if } \alpha_i < \alpha_{i+1} \end{cases} \tag{3.15}$$

$$\widehat{\pi}_i\kappa_\alpha = \begin{cases} \kappa_{s_i\alpha} - \kappa_\alpha & \text{if } \alpha_i > \alpha_{i+1} \\ 0 & \text{if } \alpha_i \leq \alpha_{i+1} \end{cases} \tag{3.16}$$

$$\pi_i\widehat{\kappa}_\alpha = \begin{cases} \widehat{\kappa}_{s_i\alpha} + \widehat{\kappa}_\alpha & \text{if } \alpha_i > \alpha_{i+1} \\ \widehat{\kappa}_\alpha & \text{if } \alpha_i = \alpha_{i+1} \\ 0 & \text{if } \alpha_i < \alpha_{i+1} \end{cases} \tag{3.17}$$

The property (3.14) allows a recursive definition of key polynomials [29]. For $\alpha \in \mathbb{N}^n$, the key polynomial κ_α is $\kappa_\alpha = x^\alpha$, if α is a partition. Otherwise, $\kappa_\alpha = \pi_i\kappa_{s_i\alpha}$, where $\alpha_{i+1} > \alpha_i$, for some i. The key polynomial κ_α is symmetric in x_i and x_{i+1} if and only if $\alpha_{i+1} \geq \alpha_i$, and therefore it lifts the Schur polynomial $s_{\alpha^+}(x)$, $\kappa_\alpha = s_{\alpha^+}(x)$, when $\alpha_1 \leq \cdots \leq \alpha_n$.

3.3 Crystals and combinatorial descriptions of Demazure characters and Demazure atoms

In [32] Lascoux and Schützenberger have given a combinatorial version for Demazure operators π_i and $\widehat{\pi}_i$ in terms of crystal (or coplactic) operators f_i, e_i to produce a crystal graph on $\mathfrak{B}^\lambda$, the set of SSYTs with entries $\leq n$ and shape λ [19, 20, 31].

A SSYT can be uniquely recovered from its column word. To describe the action of the crystal operators f_i and e_i, $1 \leq i < n$, on $T \in \mathfrak{B}^\lambda$, change all i, in the column word of T, to right parentheses ")" and $i+1$ to left parentheses "(". Ignore all other entries and match the parentheses in the usual manner to construct a subword $)^r\ (^s$ of unmatched parentheses. If there is no unmatched right parentheses, that is, $r = 0$, then f_i is not defined in T and put $f_i(T) = 0$; if there is no unmatched left parentheses, that is, $s = 0$, then e_i is not defined in T, and put $e_i(T) = 0$. Otherwise, either $r > 0$ and replace the rightmost unmatched right parenthesis by a left parenthesis to construct $)^{r-1}\ (^{s+1}$, or $s > 0$ and replace the leftmost unmatched left parenthesis by a right parenthesis to construct $)^{r+1}\ (^{s-1}$. Next, in either cases, convert the parentheses back to i and $i+1$ and recover the ignored entries. The resulting word defines the semi-standard Young tableau $f_i(T)$ or $e_i(T)$. For convenience, we extend f_i and e_i to $\mathfrak{B}^\lambda \cup \{0\}$ by setting them to map 0 to 0.

Example 27. *Let T be a SSYT with col$(T) = 5321\ 431\ 42\ 43\ 4$. Ignoring all the entries different from 3 and 4. One gets 3434434. Applying twice the operator e_3 gives,*

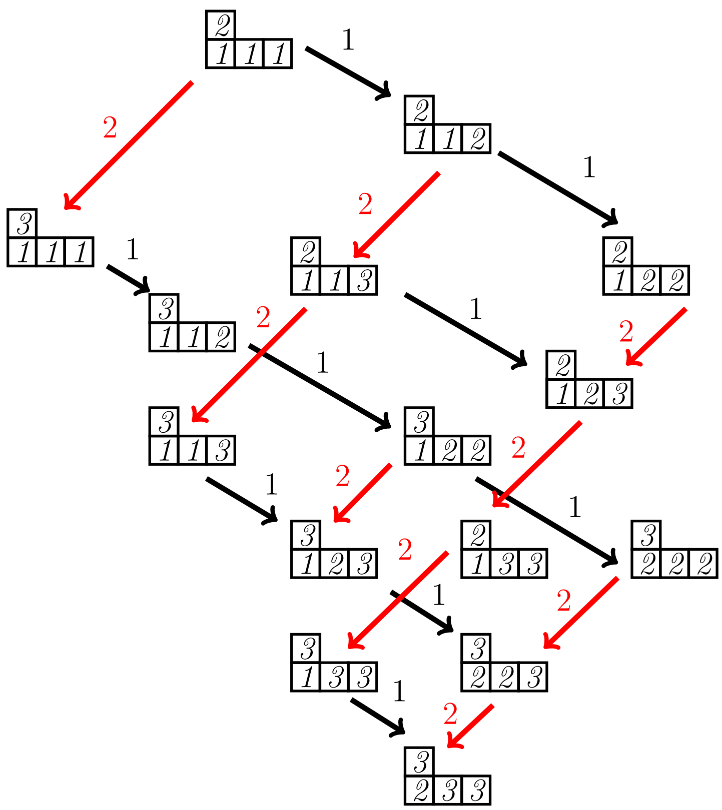
$$\begin{array}{ccccc} & e_3 & & e_3 & \\ 3434434 & \longrightarrow & 343\mathbf{3}434 & \longrightarrow & 3433433\mathbf{3} \\ )()(()(& &)()))()(& &)()))()) \end{array}.$$

Hence $e_3^2(col(T)) = 532143132433$.

Kashiwara and Nakashima [20, 23] have given to $\mathfrak{B}^\lambda$ a $U_q(\mathfrak{gl}_n)$-crystal (quantum group of the $\mathfrak{gl}_n(\mathbb{C})$) structure. We view crystals as special graphs. The crystal graph on $\mathfrak{B}^\lambda$ is a coloured directed graph whose vertices are the elements of $\mathfrak{B}^\lambda$, and the edges are coloured with a colour i, for each pair of crystal operators f_i, e_i, such that there exists a coloured i-arrow from the vertex T to T' if and only if $f_i(T) = T'$, equivalently, $e_i(T') = T$. We refer to [22, 16, 33] for details. Start with the Yamanouchi tableau $Y := key(\lambda)$ and apply all the crystal operators f_i's until each unmatched i has been converted to $i+1$, for $1 \leq i < n$ [20, 22] (see Example 28 and Figure 3.1). The resulting set is $\mathfrak{B}^\lambda$.

From the definition of this graph, in each vertex there is at most one incident arrow of colour i, and at most one outgoing arrow of colour i. Hence, the crystal $\mathfrak{B}^\lambda$ is the disjoint union of connected components of colour i, $P_1 \xrightarrow{i} \cdots \xrightarrow{i} P_k$, called i-strings, of lengths $k-1 \geq 0$, for any i, $1 \leq i < n$. A SSYT P_1, satisfying $e_i(P_1) = 0$, is said to be the head of the i-string, and, in the case of $f_i(P_k) = 0$, P_k is called the end of the i-string.

Example 28. *The crystal graph* $\mathfrak{B}^{\lambda=310}$. *The* 1 *and* 2*-strings are represented in black and red colours respectively.*



Given α in the $\mathfrak{S}_n$-orbit of λ, the Demazure crystal $\mathfrak{B}_\alpha$ is viewed as a certain subgraph of the crystal $\mathfrak{B}^\lambda$ which can be defined inductively [21, 34] as $\mathfrak{B}_\alpha = \{Y\}$ if $\alpha = \lambda$, otherwise

$$\mathfrak{B}_\alpha = \{f_i^k(T) : T \in \mathfrak{B}_{s_i\alpha}, k \geq 0, e_i(T) = 0\} \setminus \{0\}, \quad \text{if } \alpha_{i+1} > \alpha_i. \tag{3.18}$$

(When α is the reverse of λ, one has $\mathfrak{B}_{\omega\lambda} = \mathfrak{B}^\lambda$.) In fact $\mathfrak{B}_\alpha$ (3.18) is well defined, it does not depend on the reduced expression for σ. More generally, write $\alpha = s_{i_N} \dots s_{i_2} s_{i_1} \lambda$, with $(i_N, \dots, i_2, i_1)$ a reduced word, then apply the crystal operator f_{i_1} to Y until each unmatched i_1 has been converted to i_1+1, then apply similarly f_{i_2} to each of the previous Young tableaux until each unmatched i_2 has been converted to $i_2 + 1$, and continue this procedure with $f_{i_3}, \dots, f_{i_N}$. Therefore, $\mathfrak{B}_\alpha = \{f_{i_N}^{m_N} \dots f_{i_1}^{m_1}(Y) : m_k \geq 0\} \setminus \{0\}$. See Example 29.

Let $T \in \mathfrak{B}^\lambda$, and $f_{s_i}(T) := \{f_i^m(T) : m \geq 0\} \setminus \{0\}$. (If $f_i(T) = 0$, $f_{s_i}(T) = \{T\}$.) If P is the head of an i-string $S \subseteq \mathfrak{B}^\lambda$, $S = f_{s_i}(P)$. We abuse notation and say the Demazure operator π_i (3.8) sends the head of an i-string to the sum of all elements of the string [32, 21],

$$\pi_i(x^P) = \sum_{T\in S} x^T, \text{ and } \pi_i(\sum_{T\in S} x^T) = \pi_i(x^P). \tag{3.19}$$

(Since $e_i(P) = 0$, the number of unmatched i is equal to the difference between the number of $i+1$ and the number of i in P. So S contains all the tableaux which appear in the $\pi_i(x^P)$ by using equation 3.8. For example if we consider the lowest 2-string of Example 28, then $\pi_2(x^{031}) = x^{031}+x^{022}+x^{013} = \sum_{T\in S} x^T$ and since $\pi_i\pi_i = \pi_i$, then $\pi_i(\sum_{T\in S} x^T) = \pi_i(x^P)$.)

If $\beta \le \alpha$, then $\mathfrak{B}_\beta \subseteq \mathfrak{B}_\alpha$. Let $s_i\alpha < \alpha$, equivalently, $\alpha_i < \alpha_{i+1}$. For any i-string $S \subseteq \mathfrak{B}^\lambda$, either $\mathfrak{B}_{s_i\alpha}\cap S = \mathfrak{B}_\alpha\cap S$ is empty, or $\mathfrak{B}_{s_i\alpha}\cap S = \mathfrak{B}_\alpha\cap S = S$, or $\mathfrak{B}_{s_i\alpha}\cap S$ is only the head of S in which case $S \subseteq \mathfrak{B}_\alpha$. Since $\mathfrak{B}^\lambda$ is the disjoint union of i-strings, from these string properties, and (3.19), one has for any i-string S

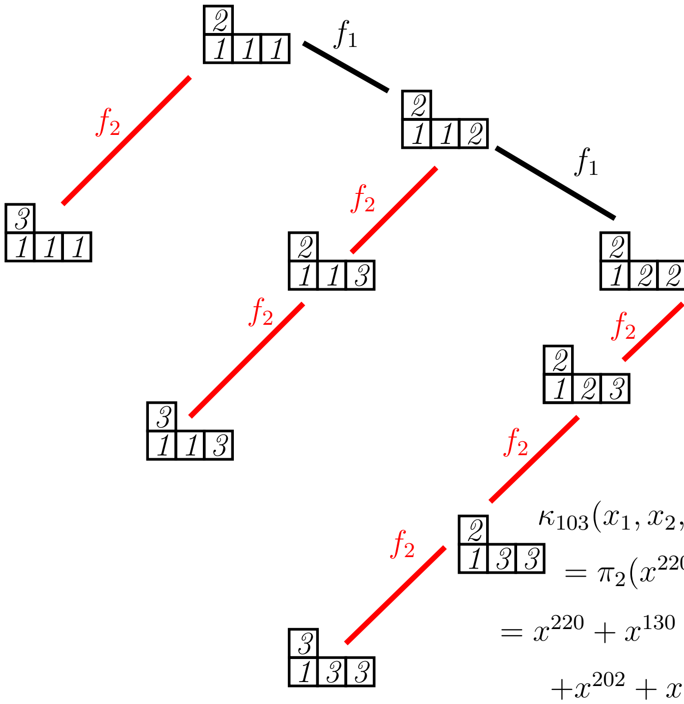
$$\sum_{T\in\mathfrak{B}_\alpha\cap S} x^T = \pi_i(\sum_{T\in\mathfrak{B}_{s_i\alpha}\cap S} x^T); \text{ and } \sum_{T\in\mathfrak{B}_\alpha} x^T = \pi_i(\sum_{T\in\mathfrak{B}_{s_i\alpha}} x^T). \tag{3.20}$$

Henceforth, $\kappa_\alpha = \pi_i\kappa_{s_i\alpha}$, if $\alpha_i < \alpha_{i+1}$, and $\kappa_\alpha = \pi_{i_N}\cdots\pi_{i_1}x^\lambda$ for any reduced word $(i_N,\dots,i_1)$ such that $s_{i_N}\cdots s_{i_1}\lambda = \alpha$.

Next proposition is a consequence of the properties of i-strings.

Proposition 10. *Let T be a SSYT, $K_+(T) = key(s_i\gamma)$, and $\gamma_i \neq \gamma_{i+1}$. If $f_i(T) \neq 0$, either $K_+(f_i(T)) = key(s_i\gamma)$ or $K_+(f_i(T)) = key(\gamma)$. Moreover $K_+(f_i(T)) = key(\gamma)$ only if $\gamma_i < \gamma_{i+1}$.*

Example 29. *The Demazure crystal $\mathfrak{B}_{s_2s_1\lambda}$ with $\lambda = (3,1,0)$.*



$$
\begin{aligned}
\kappa_{103}(x_1, x_2, x_3) &= \pi_2\pi_1(x^{310}) \\
&= \pi_2(x^{220} + x^{130} + x^{310}) \\
&= x^{220} + x^{130} + x^{310} + x^{301} + x^{211} \\
&\quad + x^{202} + x^{121} + x^{112} + x^{103} \\
&= \textstyle\sum_{T\in\mathfrak{B}_{1,0,3}} x^T
\end{aligned}
$$

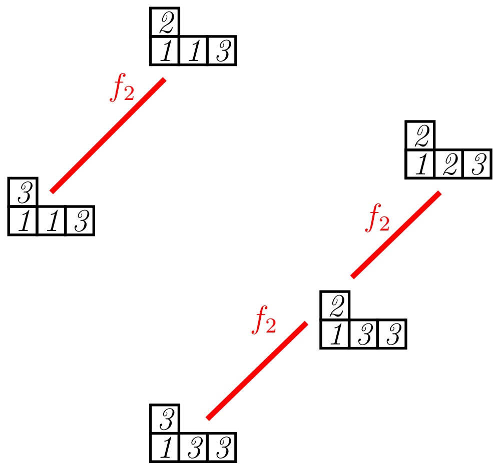
Set $\widehat{\mathfrak{B}}_\alpha := \mathfrak{B}_\alpha \setminus \bigcup_{\beta<\alpha} \mathfrak{B}_\beta$. Then $\mathfrak{B}_\alpha = \biguplus_{\beta\leq\alpha} \widehat{\mathfrak{B}}_\beta$.

In Example 29, with $\alpha = (1,0,3) = s_2s_1(3,1,0)$, the component $\widehat{\mathfrak{B}}_{s_2s_1(3,1,0)} = \mathfrak{B}_{s_2s_1(3,1,0)} \setminus (\mathfrak{B}_{s_1(3,1,0)} \cup \mathfrak{B}_{s_2(3,1,0)})$ consists of the two lowest red strings, starting in the black string, minus their heads.

Again, we abuse notation and say the action of the Demazure operator $\widehat{\pi}_i$ (3.9) on the head P of an i-string S is the same as π_i minus the head of S, and, thus, $\widehat{\pi}_i(x^P) = 0$ if $S = \{P\}$, and $\widehat{\pi}_i(x^P) = \sum_{T\in f_{s_i}(P)\setminus\{P\}}(x^T)$. From the string property, one still has, $\sum_{T\in\widehat{\mathfrak{B}}_\alpha} x^T = \widehat{\pi}_i(\sum_{T\in\widehat{\mathfrak{B}}_{s_i\alpha}} x^T)$.

Henceforth, $\widehat{\kappa}_\alpha = \widehat{\pi}_i\widehat{\kappa}_{s_i\alpha}$, if $\alpha_i < \alpha_{i+1}$, and $\widehat{\kappa}_\alpha = \widehat{\pi}_{i_N}\cdots\widehat{\pi}_{i_1}x^\lambda$ with $s_{i_N}\cdots s_{i_1}$ a minimal length representative modulo the stabiliser of α. For instance, $\widehat{\kappa}_{(1,0,3)} = \sum_{T\in\widehat{\mathfrak{B}}_{(1,0,3)}} x^T = \widehat{\pi}_2\widehat{\pi}_1x^{(3,1,0)} = \widehat{\pi}_2x^{(2,2,0)} + \widehat{\pi}_2x^{(1,3,0)} = x^{(2,1,1)} + x^{(2,0,2)} + x^{(1,2,1)} + x^{(1,1,2)} + x^{(1,0,3)}$. See Example 30.

Example 30. *The component* $\widehat{\mathfrak{B}}_{s_2s_1(3,1,0)} = \mathfrak{B}_{s_2s_1(3,1,0)} \setminus (\mathfrak{B}_{s_1(3,1,0)} \cup \mathfrak{B}_{s_2(3,1,0)})$.



Lascoux and Schützenberger have characterised the SSYTs in $\widehat{\mathfrak{B}}_\alpha$ [32] as those whose right key is $key(\alpha)$, precisely the unique key tableau in $\widehat{\mathfrak{B}}_\alpha$. The Demazure crystal $\mathfrak{B}_\alpha$ consists of all Young tableaux in $\mathfrak{B}^\lambda$ with right key bounded by $key(\alpha)$.

Theorem 11. (Lascoux, Schützenberger [28, 32]) *The Demazure atom $\widehat{\kappa}_{\sigma\lambda} = \widehat{\pi}_\sigma x^\lambda$ is the sum of the weight monomials of all SSYTs with entries $\leq n$ whose right key is equal to $key(\sigma\lambda)$, with σ a minimal length coset representative modulo the stabiliser of λ.*

We may put together the three combinatorial interpretations of Demazure characters and Demazure atoms

$$\widehat{\kappa}_\alpha = \sum_{T \in \widehat{\mathfrak{B}}_\alpha} x^T = \sum_{\substack{T \in SSYT_n \\ K_+(T) = key(\alpha)}} x^T = \sum_{\substack{F \in SSAF_n \\ sh(F) = \alpha}} x^F,$$

$$\kappa_\alpha = \sum_{T \in \mathfrak{B}_\alpha} x^T = \sum_{\substack{T \in SSYT_n \\ K_+(T) \leq key(\alpha)}} x^T = \sum_{\substack{F \in SSAF_n \\ sh(F) \leq \alpha}} x^F.$$

In particular, the sum of the weight monomials over all crystal graph $\mathfrak{B}^\lambda$ gives the Schur polynomial s_λ, and thus Demazure atoms decompose Schur polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$. Notice that the Demazure atom $\widehat{\kappa}_\alpha$ is precisely the $\widehat{E}_\alpha(x,0,0)$ defined in (2.3).

$$\widehat{\kappa}_\alpha = \sum_{\substack{F \in SSAF_n \\ sh(F)=\alpha}} x^F = \widehat{E}_\alpha(x,0,0)$$

Example 28 exhibits a crystal graph associated to the partition $\lambda = (3,1,0)$ which does not have repeated components. To have a general crystal graph we can consider the crystal graph associated to the partition $\lambda = (3,2,2,0)$ with two equal components. Figures 3.1 shows the crystal graph corresponding to partition $\lambda = (3,2,2,0)$. In this Figure key tableaux are defined by all rearrangements of $(3,2,2,0)$ and its number is $|\mathfrak{S}_4|/|stab_\lambda|$, one is in the top of the crystal graph and the others are located at the end of each thick string. Consider the thick strings of $\mathfrak{B}^\lambda$ without the tableaux which are not the head or the tail of such strings. Replace the key tableaux with their contents which are rearrangements of $(3,2,2,0)$. Flipping up side down, the resulting figure is the bubble sorting graph or left weak Bruhat graph on $\mathfrak{S}_n\lambda$ shown in Figure 1.1.

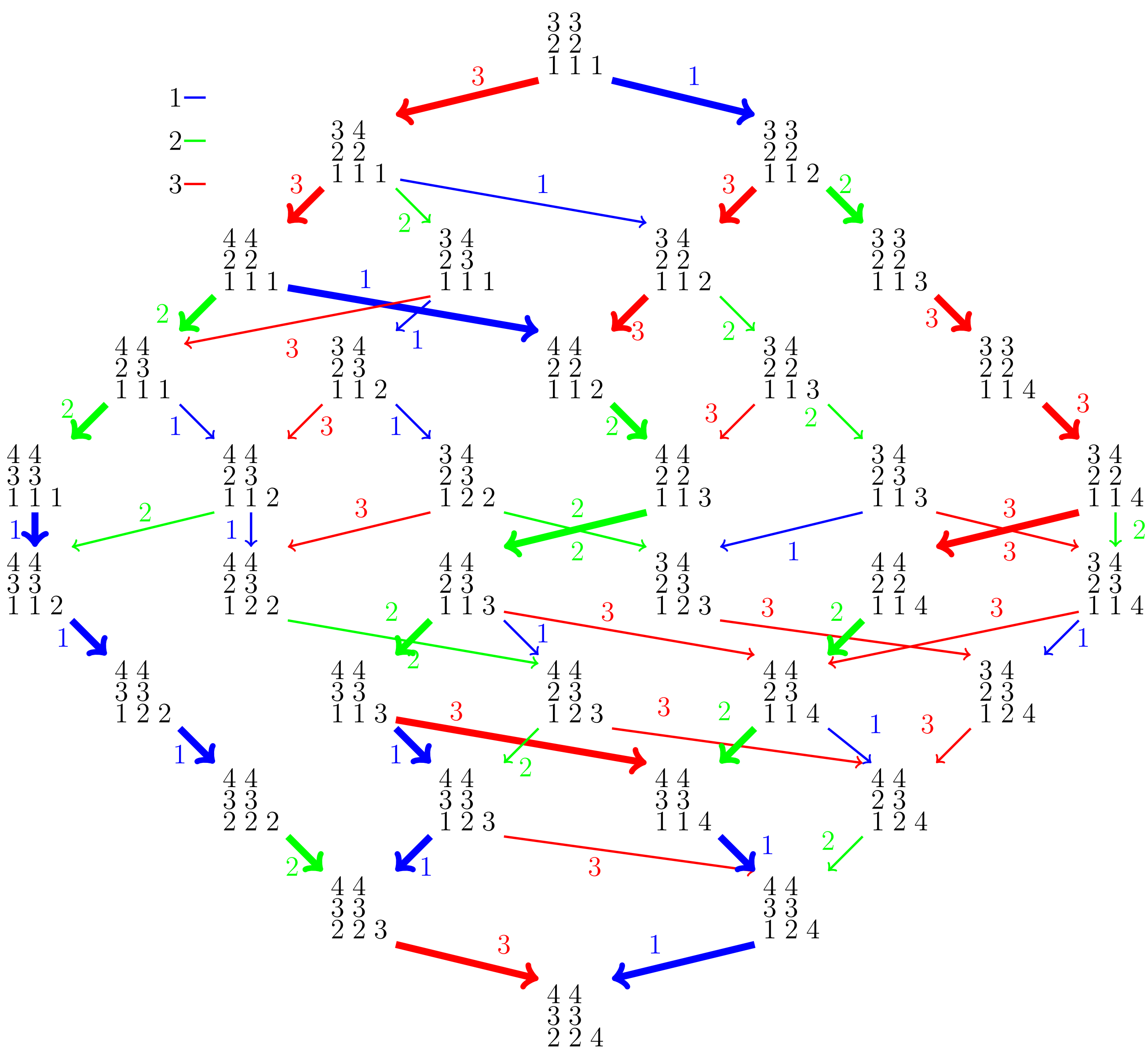


Fig. 3.1: The crystal graph $\mathfrak{B}^{\lambda}$ corresponding to the partition $\lambda = (3, 2, 2, 0)$

3.4 An analogue of crystal operator

In [40] there is an interpretation of crystal operator f_i for SSAF from which we can derive a crystal graph where the vertices are SSAFs. Mason has defined the map $\Theta_i : SSAF_n \to SSAF_n$, $1 \leq i \leq n$, such that the following diagram commutes for all semi-standard Young tableaux in SSYT_n

$$\begin{tikzcd} T \arrow[r, "f_i"] \arrow[d, "\Psi"'] & T' \arrow[d, "\Psi"] \\ F \arrow[r, "\Theta_i"] & F' \end{tikzcd}$$

where $F, F' \in SSAF_n$ and $T, T' \in SSYT_n$. The map Θ_i acts in the following way [Proposition 4.1, [40]]. First match any pair i and $i+1$ which occur in the same row of F and remove these entries from the reading word of F. Next apply the parenthetical matching procedure of [32] as described in the previous section, to the $read(F)$ to determine which of the remaining occurrences of i and $i+1$ are unmatched. In other words, replace each $i+1$ by a left (open) parenthesis "(" and each i by a right (closed) parenthesis ")" and match left and right parenthesis. Pick the rightmost unmatched i. Convert it to an $i+1$. (If there is no unmatched i, then $\Theta_i(F) = 0$. The result is a collection of row entries which differ from those of $read(F)$ in precisely one entry. Now apply procedure ρ^{-1} described in Section 2.4 to the rows from lowest to highest. For example if we consider the SSAF F and the operator Θ_1 as below

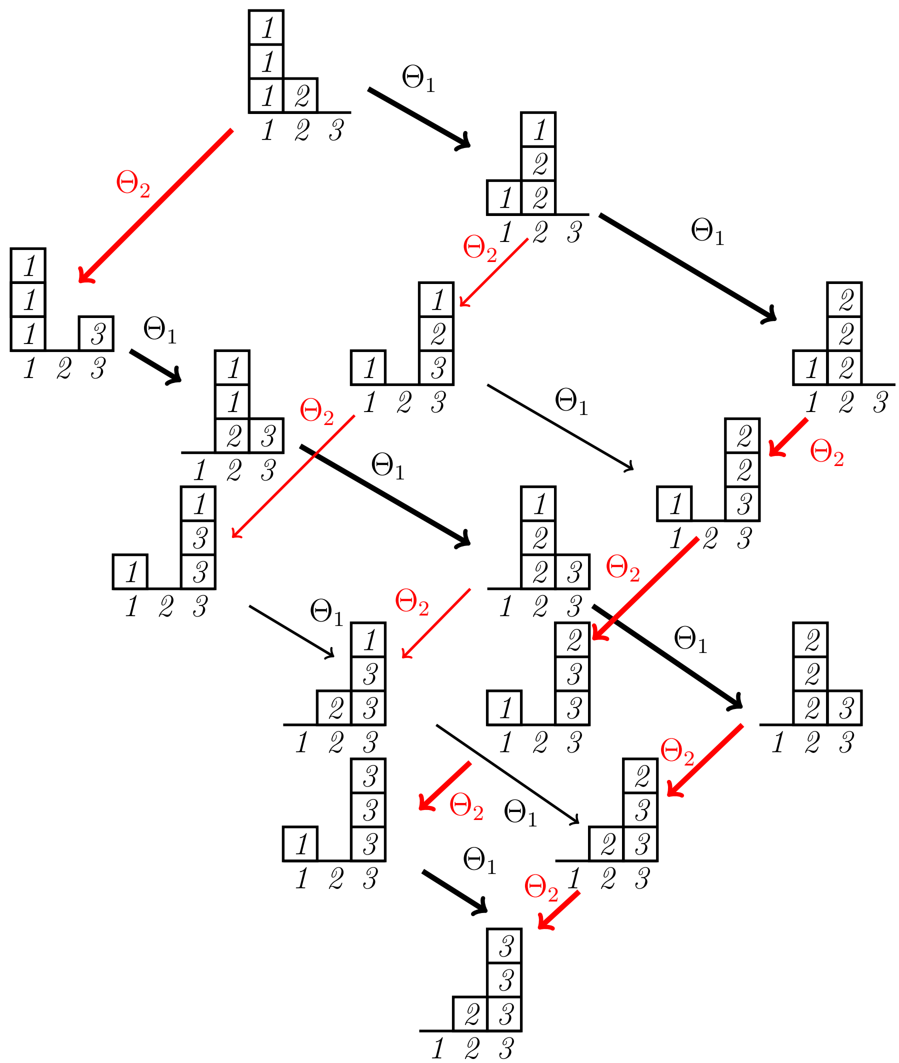
$$\begin{array}{ccc} \boxed{1} & & \\ \boxed{1} & & \\ \boxed{1} & \boxed{2} & \\ \hline 1 & 2 & 3 \end{array}$$

then $read(F) = 1\ 1\ 12$ after matching 1 and 2 we get, 1 2 12 and then applying procedure ρ^{-1} to the collection of row entries gives

$$\begin{array}{ccc} & \boxed{1} & \\ & \boxed{2} & \\ \boxed{1} & \boxed{2} & \\ \hline 1 & 2 & 3 \end{array}$$

Next example shows the crystal graph of the Example 28 translated to operators Θ_is and SSAFs. For convenience, we extend Θ_i to $\mathfrak{B}^\lambda \cup \{0\}$ by setting it to map 0 to 0.

Example 31. *The crystal graph in terms of operators Θ_i, corresponding to partition $\lambda = (3, 1, 0)$.*



Next proposition is the translation of the Proposition 10 in terms of SSAF. The first part of the proposition also appears in [40].

Proposition 11. *For each SSAF F with $sh(F) = \gamma$, and $\gamma_i \neq \gamma_{i+1}$. If $\Theta_i(F) \neq 0$, either $sh(\Theta_i(F)) = \gamma$ or $sh(\Theta_i(F)) = s_i\gamma$. Moreover $sh(\Theta_i(F)) = s_i\gamma$ only if $\gamma_i > \gamma_{i+1}$.*

4. EXPANSIONS OF CAUCHY KERNELS OVER TRUNCATED STAIRCASES

In this chapter we use Theorem 10 to give an expansion of the Cauchy kernel $\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}$, when λ is a staircase or a truncated staircase, in the basis of the Demazure characters and the basis of the Demazure atoms. This expansion, in particular, covers the Cauchy identity, when λ is a rectangle. The expansion implies a Lascoux's expansion formula [30], when specialised to staircases or truncated staircases, and make explicit, in the latter, the tableaux in the Demazure crystal defining the Demazure characters and the Demazure atoms in the expansion.

4.1 Cauchy identity and Lascoux's non-symmetric Cauchy kernel expansions

Given $n \in \mathbb{N}$ positive, let m and k be fixed positive integers where $1 \leq m \leq n$, $1 \leq k \leq n$. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ be two sequences of indeterminates. The well-known Cauchy identity expresses the Cauchy kernel $\prod_{i=1}^{k}\prod_{j=1}^{m}(1-x_iy_j)^{-1}$ as a sum of products of Schur polynomials s_{μ^+} in $(x_1, x_2, \dots, x_k)$ and $(y_1, y_2, \dots, y_m)$,

$$\prod_{(i,j)\in(m^k)}(1-x_iy_j)^{-1} = \prod_{i=1}^{k}\prod_{j=1}^{m}(1-x_iy_j)^{-1} = \sum_{\mu^+} s_{\mu^+}(x_1,\dots,x_k)s_{\mu^+}(y_1,\dots,y_m), \tag{4.1}$$

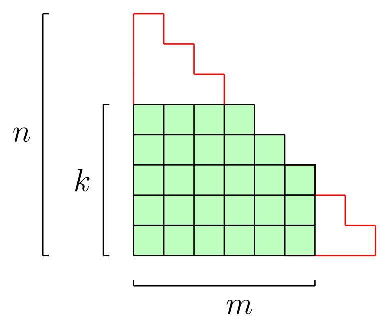
over all partitions μ^+ of length $\leq \min\{k, m\}$. Using either the RSK correspondence [24] or the Φ correspondence, the Cauchy formula (4.1) can be interpreted as a bijection between monomials on the left hand side and pairs of SSYTs or SSAFs on the right. As the basis of key polynomials lifts the

Schur polynomials w.r.t. the same list of indeterminates, the expansion (4.1) can also be expressed in the two bases of key polynomials. Assuming $k \le m$, we may write (4.1) as

$$\begin{aligned}&\sum_{\mu^+\in\mathbb{N}^k} s_{\mu^+}(x_1,\dots,x_k)s_{(\mu^+,0^{m-k})}(y_1,\dots,y_m)\\ &= \sum_{\mu^+\in\mathbb{N}^k}\sum_{\mu\in\mathfrak{S}_k\mu^+}\widehat{\kappa}_\mu(x)\kappa_{(0^{m-k},\omega\mu^+)}(y) = \sum_{\mu\in\mathbb{N}^k}\widehat{\kappa}_\mu(x)\kappa_{(0^{m-k},\omega\mu^+)}(y). \quad (4.2)\end{aligned}$$

(Since we are dealing with two sequences of indeterminates x and y, it is convenient to write $\kappa_\alpha(x)$ and $\kappa_\alpha(y)$ instead of κ_α. Similarly for Demazure atoms.)

We now replace in the Cauchy kernel the rectangle (m^k) by the truncated staircase $\lambda = (m^{n-m+1}, m-1, \dots, n-k+1)$, with $1 \le m \le n,\ \ 1 \le k \le n$, and $n+1 \le m+k$, as shown in the green diagram below



If $n+1 = m+k$, we recover the rectangle shape (m^k). When $m = n = k$, one has the staircase partition $\lambda = (n, n-1, \dots, 2, 1)$, that is, the cells (i,j) in the NW-SE diagonal of the square diagram (n^n) and below it, and thus $(i,j) \in \lambda$ if and only if $i + j \le n+1$.

Lascoux has given the following expansion for the non-symmetric Cauchy kernel over staircases, using double crystal graphs in [30], and also in [11], based on algebraic properties of Demazure operators,

$$\prod_{\substack{i+j\le n+1\\ 1\le i,j\le n}} (1-x_iy_j)^{-1} = \sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y), \qquad (4.3)$$

where $\widehat{\kappa}$ and κ are the two families of key polynomials in x and y respectively, and ω is the longest permutation of $\mathfrak{S}_n$.

In [30], Lascoux extends (4.3) to an expansion of $\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}$, over any Ferrers shape λ, by considering $\rho(\lambda)=(t,t-1,\dots,1)$, the biggest staircase contained in λ. Take any cell in the staircase $(t+1,t,\dots,1)$ which does not belong to λ. The SW-NE diagonal passing through this cell cuts the diagram of λ/ρ, consisting of the cells in λ not in ρ, into two pieces that are called respectively the North-West part and the South-East part of λ/ρ. Fill now each cell of row $r\geq 2$ of the North-West part with the number $r-1$. Similarly, fill each cell of column $c\geq 2$ of the South-East part with the number $c-1$. Reading the columns of the North-West part, from right to left, top to bottom, and interpreting r as the simple transposition s_r, gives a reduced decomposition of a permutation $\sigma(\lambda,NW)$; similarly, reading rows, from right to left, and from top to bottom, of the South-East part, gives a permutation $\sigma(\lambda,SE)$. The example 7 in Section 1.5 is shown the $\sigma(\lambda,SE)$, where $\lambda=(m^{n-m+1},m-1,\dots,n-k+1)$, and $1\leq k\leq m\leq n$, and $n-k\leq m-1$. These two permutations depend indeed on the choice of the cell cutting the diagram λ/ρ.

Theorem 12. [[30] Theorem 7] *Let λ be a partition in $\mathbb{N}^n$, $\rho(\lambda)=(t,t-1,\dots,1)$ the maximal staircase contained in the diagram of λ, and $\sigma(\lambda,NW)$, $\sigma(\lambda,SE)$ the two permutations obtained by cutting the diagram of λ/ρ as explained above. Then*

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1}=\sum_{\mu\in\mathbb{N}^t}(\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\mu(x))(\pi_{\sigma(\lambda,SE)}\kappa_{\omega\mu}(y)). \tag{4.4}$$

For our truncated staircases λ the formula (4.4) translates to

$$\prod_{\substack{(i,j)\in\lambda\\ k\leq m}}(1-x_iy_j)^{-1}=\sum_{\mu\in\mathbb{N}^k}\widehat{\kappa}_\mu(x)(\pi_{\sigma(\lambda,SE)}\kappa_{\omega\mu}(y)); \tag{4.5}$$

$$\prod_{\substack{(i,j)\in\lambda\\ m\leq k}}(1-x_iy_j)^{-1}=\sum_{\mu\in\mathbb{N}^m}(\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\mu(x))\kappa_{\omega\mu}(y). \tag{4.6}$$

Indeed (4.6) is just (4.5), with x and y swapped, followed by a change from the basis (3.13) of Demazure characters to the basis of Demazure atoms

$$\prod_{\substack{(i,j)\in\lambda\\ m\leq k}}(1-x_iy_j)^{-1}=\prod_{\substack{(j,i)\in\overline{\lambda}\\ m\leq k}}(1-x_iy_j)^{-1}=\sum_{\mu\in\mathbb{N}^m}\widehat{\kappa}_\mu(y)\pi_{\sigma(\overline{\lambda},SE)}\kappa_{\omega\mu}(x)$$

$$\begin{aligned}
&= \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\kappa_{\omega\mu}(x) = \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)} \sum_{\beta\le\omega\mu} \widehat{\kappa}_\beta(x)\\
&= \sum_{\mu\in\mathbb{N}^m} \sum_{\substack{\beta\in\mathbb{N}^m\\ \beta\le\omega\mu}} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x) = \sum_{\beta\in\mathbb{N}^m} \sum_{\substack{\mu\in\mathbb{N}^m\\ \mu\le\omega\beta}} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x)\\
&= \sum_{\beta\in\mathbb{N}^m} \pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x) \sum_{\substack{\mu\in\mathbb{N}^m\\ \mu\le\omega\beta}} \widehat{\kappa}_\mu(y) = \sum_{\beta\in\mathbb{N}^m} \pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\beta(x)\kappa_{\omega\beta}(y).
\end{aligned}$$

Next we give a bijective proof of (4.5) and compute the Demazure character $\pi_{\sigma(\lambda,SE)}\ \kappa_{\omega\mu}(y)$ by making explicit the Young tableaux in the Demazure crystal.

4.2 Our expansions

We now use the bijection in Theorem 10 to give an expansion of the non-symmetric Cauchy kernel for the shape $\lambda = (m^{n-m+1}, m-1, \dots, n-k+1)$, where $1 \le m \le n$, $1 \le k \le n$, and $n+1 \le m+k$, which includes, in particular, the rectangle (4.1), the staircase (4.3), and implies the truncated staircases (4.5).

The generating function for the multisets of ordered pairs of positive integers $\{(a_1,b_1),\ (a_2,b_2),\ \dots,\ (a_r,b_r)\}$, $r \ge 0$, where $(a_i,b_i) \in \lambda$, that is, $a_i + b_i \le n+1$, $1 \le a_i \le k$, $1 \le b_i \le m$, $1 \le i \le r$, weighted by the contents $((\alpha, 0^{n-k});\ (\delta, 0^{n-m})) \in\mathbb{N}^k \times \mathbb{N}^m$, with α_j the number of i's such that $a_i = j$, and δ_j the number of i's such that $b_i = j$, is

$$\prod_{(i,j)\in\lambda} (1-x_iy_j)^{-1} = \sum_{\substack{\{(a_i,b_i)\}_{i=1}^r\\ r\ge 0}} x_{a_1}y_{b_1}\cdots x_{a_r}y_{b_r} = \sum_{\substack{\{(a_i,b_i)\}_{i=1}^r\\ r\ge 0}} x^\alpha y^\delta.$$

Each multiset $\{(a_1,b_1),\ (a_2,b_2),\ \dots,\ (a_r,b_r)\}$, $r \ge 0$, and, hence, each monomial $x_{a_1}\ y_{b_1}\cdots\ x_{a_r}y_{b_r}$, $r \ge 0$, is in one-to-one correspondence with the lexicographically ordered biword $\left(\begin{smallmatrix} b_r & \cdots & b_1\\ a_r & \cdots & a_1\end{smallmatrix}\right)$ in the product of alphabets $[m]\times[k]$, which is bijectively mapped by Φ into the pair (F,G) of SSAFs such that F has entries in $\{a_1,\dots,a_r\}$, G has entries in $\{b_1,\dots,b_r\}$, and their shapes $sh(F) = \mu \in \mathbb{N}^k$, and $sh(G) = \beta \in \mathbb{N}^m$, in a same $\mathfrak{S}_n$-orbit, satisfy $(\beta, 0^{n-m}) \le (0^{n-k}, \omega\mu)$ with ω the longest permutation in $\mathfrak{S}_k$. (For $r=0$, put $F=G=\emptyset$.) Thereby, $x_{a_1}y_{b_1}\cdots x_{a_r}y_{b_r} = x^F y^G$, for all $r \ge 0$. Assume

$k \leq m$. Since $(\mu, 0^{n-k})$, $(\beta, 0^{n-m})$ are in a same $\mathfrak{S}_n$-orbit, $(\mu^+, 0^{m-k}) = \beta^+ \in \mathbb{N}^m$. We may write then

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \sum_{\mu\in\mathbb{N}^k} \sum_{\substack{F,G\in SSAF_n \\ sh(G)=\beta\in\mathbb{N}^m,\ sh(F)=\mu \\ (\beta,0^{n-m})\leq(0^{n-k},\omega\mu)}} x^F y^G \tag{4.7}$$

$$= \sum_{\mu\in\mathbb{N}^k} \left(\sum_{\substack{F\in SSAF_n \\ sh(F)=\mu}} x^F \right) \left(\sum_{\substack{\beta\in\mathbb{N}^m \\ (\beta,0^{n-m})\leq(0^{n-k},\omega\mu)}} \sum_{\substack{G\in SSAF_n \\ sh(G)=\beta}} y^G \right)$$

$$= \sum_{\mu\in\mathbb{N}^k} \left(\sum_{\substack{P\in SSYT_n \\ sh(P)=\mu^+ \\ K_+(P)=key(\mu)}} x^P \right) \left(\sum_{\substack{\beta\in\mathbb{N}^m \\ (\beta,0^{n-m})\leq(0^{n-k}\omega\mu)}} \sum_{\substack{Q\in SSYT_n \\ sh(Q)=\mu^+ \\ K_+(Q)=key(\beta)}} y^Q \right)$$

$$= \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x) \sum_{\substack{Q\in\mathfrak{B}_{(0^{n-k},\omega\mu)} \\ entries \leq m}} y^Q. \tag{4.8}$$

Let $\nu := (\mu, 0^{n-k})$. Recall that $\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})} = \mathfrak{B}^{(\mu^+,0^{m-k})}$, with ω the longest permutation of $\mathfrak{S}_k$, is the crystal graph consisting of all SSYTs with shape $(\mu^+, 0^{m-k})$ and entries less or equal than m. Henceforth, one has

$$\sum_{\substack{Q\in\mathfrak{B}_{\omega\nu} \\ entries \leq m}} y^Q = \sum_{Q\in\mathfrak{B}_{\omega\nu}\cap\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}} y^Q, \tag{4.9}$$

the weight polynomial of all SSYTs in the $\mathfrak{B}_{\omega\nu}$ with entries less or equal than m, equivalently, of all SSYTs with entries $\leq m$ and shape μ^+ whose right key is bounded by $key(0^{n-k}, \omega\mu)$. It is also equivalent to consider all SSAFs such that the shape has zeros in the last $n-m$ entries, and is bounded by $\omega\nu$. Next, we determine the Demazure crystal $\mathfrak{B}_{(0^{m-k},\alpha,0^{n-m})} = \mathfrak{B}_{\omega\nu} \cap \mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}$ where $\alpha \in \mathbb{N}^k$. This shows that (4.9) is a key polynomial and describes its indexing weak composition. (See also Example 32.)

Lemma 2. *Let $\gamma \in \mathbb{N}^n$ such that $\gamma^+ = (\eta, 0^{n-m})$ is a partition of length $\leq m \leq n$. Consider the sequence of positive integers $1 \leq i_M, \dots, i_1 < n$ (not necessarily a reduced word of $\mathfrak{S}_n$) such that $\kappa_\gamma(y) = \pi_{i_M} \cdots \pi_{i_1} y^{(\eta, 0^{n-m})}$. If $j_s, \dots, j_1$ is the subsequence consisting of elements $\geq m$, it holds*

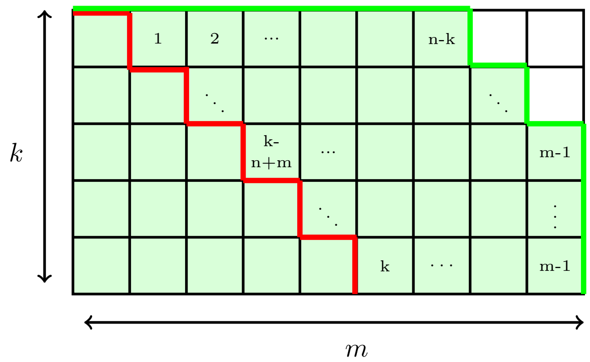
$$\sum_{\substack{Q \in \mathfrak{B}_\gamma \\ entries \leq m}} y^Q = \sum_{Q \in \mathfrak{B}_\gamma \cap \mathfrak{B}_{(\omega\eta, 0^{n-m})}} y^Q = \pi_{i_M} \cdots \tilde{\pi}_{j_s} \cdots \tilde{\pi}_{j_1} \cdots \pi_{i_1} y^{(\eta, 0^{n-m})}, \tag{4.10}$$

where the tilde means omission, and ω is the longest permutation of $\mathfrak{S}_m$.

Proof. Notice that $\mathfrak{B}_\gamma \cap \mathfrak{B}_{(\omega\eta, 0^{n-m})} = \mathfrak{B}_\gamma \cap \mathfrak{B}^\eta$. If $n = m$ or γ has the last $n - m$ entries equal to zero, then $\gamma \leq (\omega\eta, 0^{n-m})$, $\mathfrak{B}_\gamma \subseteq \mathfrak{B}^\eta$, and $1 \leq i_M, \dots, i_1 < m$. Henceforth, $\sum_{\substack{Q \in \mathfrak{B}_\gamma \\ entries \leq m}} y^Q = \sum_{Q \in \mathfrak{B}_\gamma} y^Q = \kappa_\gamma(y)$. Otherwise, $\mathfrak{B}_\gamma \cap \mathfrak{B}^\eta$ is obtained from $\mathfrak{B}_\gamma$ deleting all the vertices consisting of all SSYTs with entries $> m$, and, therefore, all i-edges incident on them (either getting in or out), in particular, those with $i \geq m$. From the combinatorial interpretation of Demazure operators π_i, (3.18), (3.20), this means we are deleting in $\pi_{i_M} \cdots \pi_{i_2} \pi_{i_1} y^{(\eta, 0^{n-m})}$ the action of the Demazure operators π_i for $i \geq m$, and, thanks to (3.14), one still has a key polynomial, precisely, (4.10). $\square$

We now calculate the indexing weak composition of the key polynomial (4.10) in the case $\eta = (\mu^+, 0^{m-k})$ and $\gamma = \omega\nu$, and, therefore, the key polynomial (4.9). For $\lambda = (m^{n-m+1}, m-1, \dots, n-k+1)$, where $1 \leq k \leq m \leq n$, and $n - k \leq m - 1$, one has the shape below where

$$\sigma(\lambda, SE) = \prod_{i=1}^{k-(n-m)-1} (s_{i+n-k-1} \dots s_i) \prod_{i=0}^{n-m} (s_{m-1} \dots s_{k-(n-m)+i})$$



Proposition 12. *Let $1 \leq k \leq m \leq n$, and $n-m+1 \leq k$. Given $\mu \in \mathbb{N}^k$, let $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ such that for each $i = k, \ldots, 1$, the entry α_i is the maximum element among the last $\min\{i, n-m+1\}$ entries of $\omega\mu$ after deleting α_j, for $i < j \leq k$. Then*

$$\begin{aligned}\sum_{\substack{Q\in\mathfrak{B}_{\omega\nu}\\ entries\leq m}} y^Q &= \sum_{Q\in\mathfrak{B}_{\omega\nu}\cap\mathfrak{B}_{(0^{m-k},\omega\mu^+,0^{n-m})}} y^Q = \sum_{Q\in\mathfrak{B}_{(0^{m-k},\alpha,0^{n-m})}} y^Q \\ &= \pi_{\sigma(\lambda,SE)}\kappa_{(\omega\mu,0^{n-k})}(y) = \kappa_{(0^{m-k},\alpha,0^{n-m})}(y). \qquad (4.11)\end{aligned}$$

In particular, when $m = n$, then $\alpha = \omega\mu$; and when $m+k = n+1$, $\alpha = \omega\mu^+$ and $\kappa_{(0^{m-k},\omega\mu^+,0^{n-m})}(y) = s_{(\mu^+,0^{m-k})}(y_1, \ldots, y_m)$ is a Schur polynomial.

Proof. Recalling the action of Demazure operators π_i on key polynomials via bubble sorting operators on their indexing weak compositions (3.14), and since $\omega\nu = (0^{n-k}, \omega\mu)$, one may write,

$$\begin{aligned}\kappa_{\omega\nu}(y) &= \prod_{i=1}^{k-(n-m)-1} (\pi_{i+n-k-1} \ldots \pi_i) \\ &\quad \bullet \prod_{i=0}^{n-m} (\pi_{m-1+i} \ldots \pi_{k-(n-m)+i})\kappa_{(\omega\mu,0^{n-k})}(y). \qquad (4.12)\end{aligned}$$

From Lemma 2, with $\eta = (\mu^+, 0^{m-k})$ and $\gamma = \omega\nu$, omitting in (4.12) the operators with indices $\geq m$, one has

$$\sum_{\substack{Q\in\mathfrak{B}_{\omega\nu}\\ entries\leq m}} y^Q = \sum_{Q\in\mathfrak{B}_{\omega\nu}\cap\mathfrak{B}(\mu^+,0^{m-k})} y^Q = \pi_{\sigma(\lambda,SE)}\kappa_{(\omega\mu,0^{n-k})}(y)$$

$$= \prod_{i=1}^{k-(n-m)-1} (\pi_{i+n-k-1}\dots\pi_i) \tag{4.13}$$

$$\bullet \prod_{i=0}^{n-m} (\pi_{m-1}\dots\pi_{k-(n-m)+i})\kappa_{(\omega\mu,0^{n-k})}(y) \tag{4.14}$$

$$= \kappa_{(0^{m-k},\alpha,0^{n-m})}(y). \tag{4.15}$$

The Demazure operators in (4.14) act as bubble sorting operators on the weak composition $(\omega\mu, 0^{n-k})$, shifting $m-k$ times to the right the last $n-m+1$ entries of $\omega\mu$, and sorting them in ascending order. Next, the operators (4.13) act similarly on the resulting vector ignoring the entry m, then ignoring the entry $m-1$, and so on. Thus the weak composition indexing the new key polynomial $\kappa_{(0^{m-k},\alpha,0^{n-m})}$ (4.15) is such that $\alpha=(\alpha_1,\dots,\alpha_k)$, where for each $i=k,\dots,1$, α_i is the maximum element of the last $\min\{i,n-m+1\}$ entries of $\omega\mu$ after deleting α_j, for $i<j\leq k$. $\square$

Therefore, for $\lambda=(m^{n-m+1},m-1,\dots,n-k+1)$, where $1\leq k\leq m\leq n$, and $n+1\leq m+k$, (4.8) can be written explicitly as

$$\prod_{\substack{(i,j)\in\lambda\\ k\leq m}} (1-x_iy_j)^{-1} = \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x)\pi_{\sigma(\lambda,SE)}\kappa_{\omega\mu}(y) = \sum_{\mu\in\mathbb{N}^k} \widehat{\kappa}_\mu(x)\kappa_{(0^{m-k},\alpha)}(y). \tag{4.16}$$

Then

$$\prod_{\substack{(i,j)\in\lambda\\ m\leq k}} (1-x_iy_j)^{-1} = \prod_{\substack{(j,i)\in\overline{\lambda}\\ m\leq k}} (1-x_iy_j)^{-1} = \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\overline{\lambda},SE)}\kappa_{\omega\mu}(x)$$

$$= \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\pi_{\sigma(\lambda,NW)}\kappa_{\omega\mu}(x) = \sum_{\mu\in\mathbb{N}^m} \widehat{\kappa}_\mu(y)\kappa_{(0^{k-m},\alpha')}(x), \tag{4.17}$$

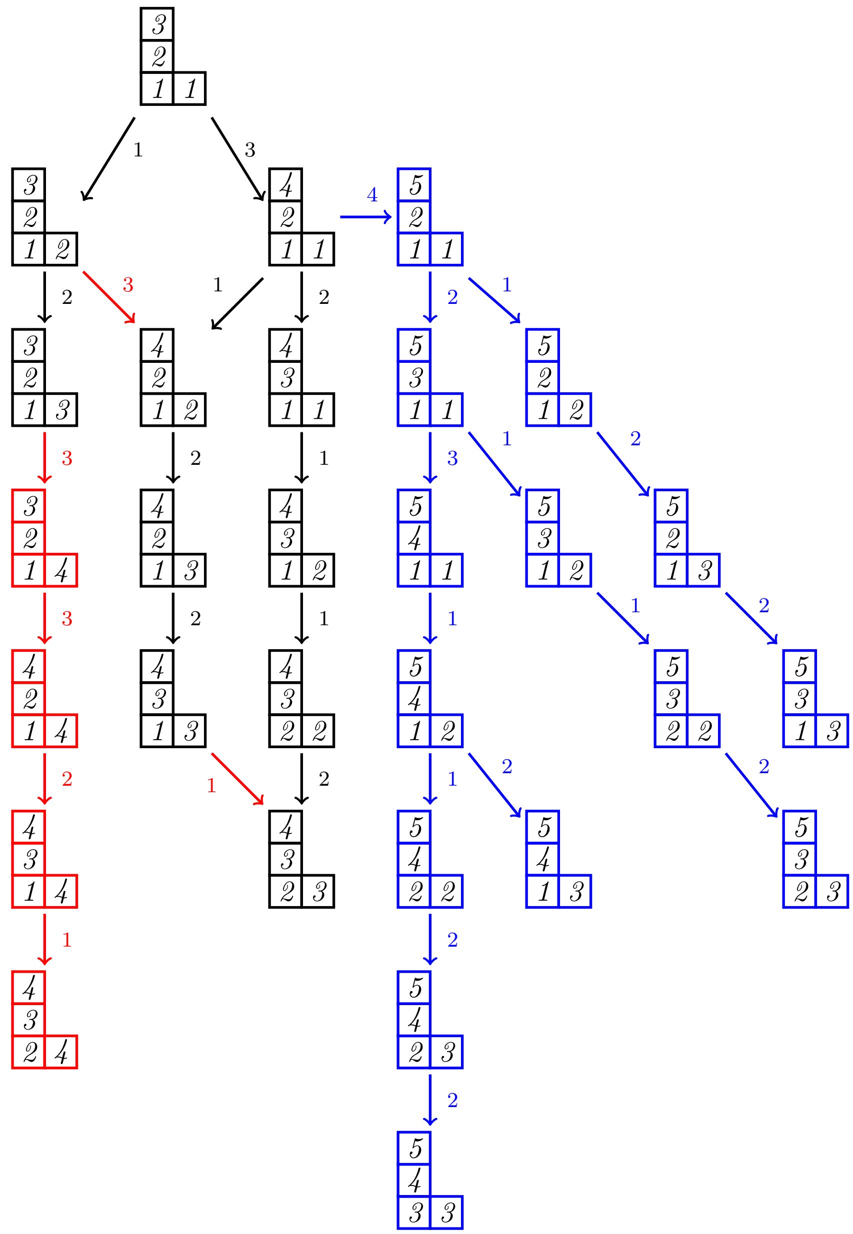
where α' is defined similarly as above, swapping k with m in Proposition 12. In identity (4.16) when $m = n$, one has for $\lambda = (n, n-1, \ldots, n-k+1)$, with $1 \leq k \leq n$,

$$\prod_{(i,j)\in\lambda} (1 - x_i y_j)^{-1} = \sum_{\substack{\mu\in\mathbb{N}^k \\ \nu=(\mu,0^{n-k})}} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y).$$

(Similarly, for $k = n$, in identity (4.17).) In particular, if $m = n = k$ ($\lambda = \overline{\lambda}$), we recover (4.3) from both previous identities. When $n + 1 = m + k$, from Proposition 12, identity (4.16) becomes (4.2), and hence we recover identity (4.1) with $k \leq m$. Similarly, (4.17) leads to (4.1) with $m \leq k$.

Example 32. *Let* $n = 5,\ k = 4 \geq m = 3$, $\mu = (1, 1, 2)$, *and* $\nu = (1, 1, 2, 0, 0)$. *The black and blue tableaux constitute the vertices of the Demazure crystal* $\mathfrak{B}_{\omega\nu} = \mathfrak{B}_{(0,0,2,1,1)} = \mathfrak{B}_{s_2 s_1 s_3 s_2 s_4 s_3 (2,1,1,0,0)}$. *One has* $\pi_2\pi_1\pi_3\pi_2\pi_3 x^{(2,1,1,0,0)} = \pi_2\pi_1\pi_2\pi_3 x^{(2,1,1,0,0)} = \kappa_{(0,1,2,1,0)}(x)$. (*The shortest element in the coset* $s_2 s_1 s_3 s_2$

$s_3 < s_2 >$ *is* $s_2 s_1\ s_2 s_3$.) *The black and the red tableaux are the vertices of the crystal* $\mathfrak{B}_{(0,\omega\mu^+,0)} = \mathfrak{B}_{(0,1,1,2,0)} = \mathfrak{B}_{s_1 s_2 s_3 s_2 s_1 \nu^+}$. *The intersection* $\mathfrak{B}_{\omega\nu} \cap \mathfrak{B}_{(0,\omega\mu^+,0)}$ *consists of the black tableaux which constitute the vertices of the Demazure crystal* $\mathfrak{B}_{(0,\alpha,0)} = \mathfrak{B}_{(0,1,2,1,0)} = \mathfrak{B}_{s_2 s_1 s_2 s_3 (2,1,1,0,0)}$, *with* α *defined in Proposition 12.* (*Note that the crystal graph has not all the edges represented. Only those of the words under consideration.*)



5. A COMBINATORIAL PROOF FOR THE LASCOUX'S CAUCHY KERNEL EXPANSION OVER FERRERS SHAPES

In this chapter we give a combinatorial proof for the Lascoux's non-symmetric Cauchy kernel expansion over an arbitrary Ferrers shapes, in the cases, of one and two non-consecutive boxes above the staircase, and an idea in the general case. The proof is given in the framework of Fomin's growth diagrams for generalized Robinson- Schensted-Knuth correspondences. The strategy is to reduce the problem of a Ferrers shape to that of a staircase, contrary to the approach of the previous section where the smallest staircase containing our shape has been considered.

5.1 An interpretation of crystal operators in terms of growth diagrams

In Section 3.3 we have defined crystal operators e_r, f_r on a column word of a SSYT. In fact they can be defined on any word over alphabet $[n]$. For details see [31]. First we recall the crystal operator e_r on a word with an example. Consider the word $u = 34263443411$, over the alphabet [7] and operator e_3. Ignoring all the entries different from 3 and 4, one gets 3434434. Applying twice the operator e_3 gives,

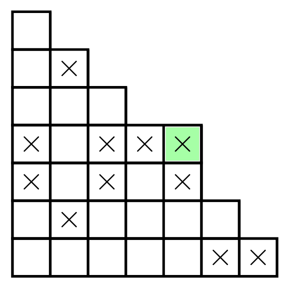
$$3434434 \xrightarrow{e_3} 343\mathbf{3}434 \xrightarrow{e_3} 343343\mathbf{3}$$

Hence $e_3^2(u) = 34263343311$. Consider now the following biword in lexicographic order where all the biletters $\begin{pmatrix} \mathrm{i} \\ \mathrm{j} \end{pmatrix}$ satisfy $i+j \leq 7+1$ except $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$

with $5+4>7+1$

$$w = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 & 7 \\ 3 & 4 & 2 & 6 & 3 & 4 & 4 & 3 & 4 & 1 & 1 \end{pmatrix}.$$

Recalling the representation of a biword in a rectangle defined in Section 1.10, we represent the biword w in the Ferrers shape $\lambda = (7,6,5,5,3,2,1)$ by putting a cross $\times$ in the cell (i,j) of λ if $\begin{pmatrix} \mathrm{j} \\ \mathrm{i} \end{pmatrix}$ is a biletter of w.



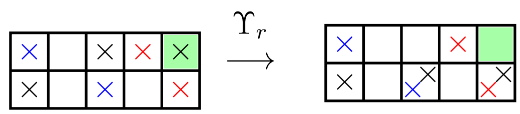
$\lambda = (7,6,5,5,3,2,1)$

Fig. 5.1: Reperesentation of a biword in a Ferrers shape

The biword w can be recovered, from this representation, by scanning the columns of the Ferrers shape λ, left to right, and bottom to top. Let $\rho(\lambda) = (7,6,\ldots,1)$ be the biggest staircase inside of λ. The green box of λ corresponds to the green biletter $\begin{pmatrix} 5 \\ 4 \end{pmatrix}$ of w where $4+5>7+1$, and, therefore, its representation is not inside the staircase $\rho(\lambda)$.

Let us consider a non-staircase Ferrers shape, $\lambda = (\lambda_1,\ldots,\lambda_r,\lambda_{r+1},\lambda_{r+2},\ldots,\lambda_n)$ where $\lambda_{r+2} < \lambda_{r+1} = \lambda_r$, $1 \le r < n$, and $\rho(\lambda)$, the biggest staircase inside of λ. Let w be a biword in the lexicographic order represented in the Ferrers shape λ. We introduce an operation Υ_r in the rows r and $r+1$ of λ, which consists of matching crosses in rows r and $r+1$, and then sliding down the unmatched crosses from row $r+1$ to row r. The objective is to put all the crosses outside of $\rho(\lambda)$ into inside. This slide of crosses translates to the action of the operator e_r, as long as it is possible, on the second row of the biword w. The operation Υ_r is the analogue of applying m times the crystal operator e_r, to the second row of w, where m is the number of unmatched

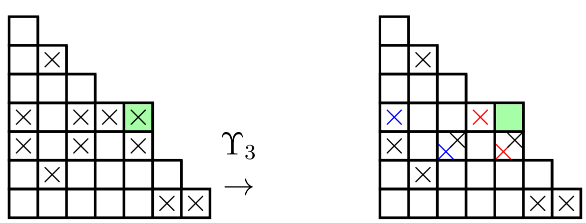
$r+1$ in the second row of w. Therefore, we also write $\Upsilon_r w$ to mean the biword obtained by applying m times the crystal operator e_r, to the second row of w, where m is the number of unmatched $r+1$ in the second row of w. We scan from right to left and match the crosses in rows r and $r+1$ that are situated in the following way: $\begin{smallmatrix}\times & \\ & \times\end{smallmatrix}$ where the cross of the row $r+1$ is the closest NW unmatched cross of the cross of the row r. Next move all the unmatched crosses of the row $r+1$ to the row r.



$$\begin{pmatrix} 1 & 1 & 3 & 3 & 4 & 5 & 5 \\ 3 & 4 & 3 & 4 & 4 & 3 & 4 \end{pmatrix} \underset{f_r^2}{\overset{\Upsilon_r}{\rightleftarrows}} \begin{pmatrix} 1 & 1 & 3 & 3 & 4 & 5 & 5 \\ 3 & 4 & 3 & 3 & 4 & 3 & 3 \end{pmatrix}$$

The action of the crystal operator f_r on the biword w is defined by the action of f_r on the second row.

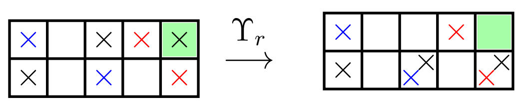
The new set of cells of λ, defined by the crosses, yields a new biword $\Upsilon_r w$, scanning λ along columns from left to right and bottom to top. The biword $\Upsilon_r w$ is obtained from the biword w by applying the crystal operator e_r as long as it is possible to the second row of the biword w.



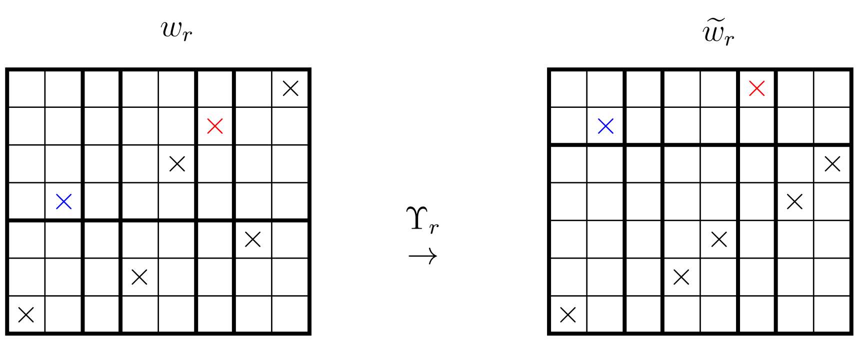
$$\begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 & 7 \\ 3 & 4 & 2 & 6 & 3 & 4 & 4 & 3 & 4 & 1 & 1 \end{pmatrix} \overset{\Upsilon_3}{\to} \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 & 7 \\ 3 & 4 & 2 & 6 & 3 & 3 & 4 & 3 & 3 & 1 & 1 \end{pmatrix}$$

Consider now the two 01-fillings of the biwords w and $\Upsilon_r w$ represented in the Ferrers shapes λ, and apply the backward local rules to them, as defined in sections 1.11 and 2.10. Notice that in the 01-filling of $\Upsilon_r w$, we match a cross of row $r+1$ with a cross to the SE, in row r, such that in these two rows there is no unmatched cross in a column between them.

These two growth diagrams have the same bottom sequences of partitions and the left sequences are different only in the partitions associated to the rows r and $r+1$. It is proved in [31] that the bottom sequence is preserved by the operations e_r and f_r, when the entries of the first row of the biword w are distinct. In the 01-filling we have standardized the first row of the biword w thus the bottom sequence is preserved, and therefore the same happens when the first row of the biword has repeated letters. Let w_r and $\widetilde{w}_r$ be the biwords that are obtained from w and $\Upsilon_r w$, after deleting all the biletters whose second rows are different from r and $r+1$. The translation of the movement of the cells in the Ferrers shape to the 01-filling is as follows: in the 01-filling of w_r, move up, without changing of columns, the matched crosses of row $r+1$, say s crosses, to the top most s rows such that they form SW chain. Then slide down the remaining unmatched crosses, from row $r+1$ to row r, without changing of columns, such that these crosses and all the crosses of row r form a SW chain. The result is the 01-filling corresponding to $\widetilde{w}_r$.



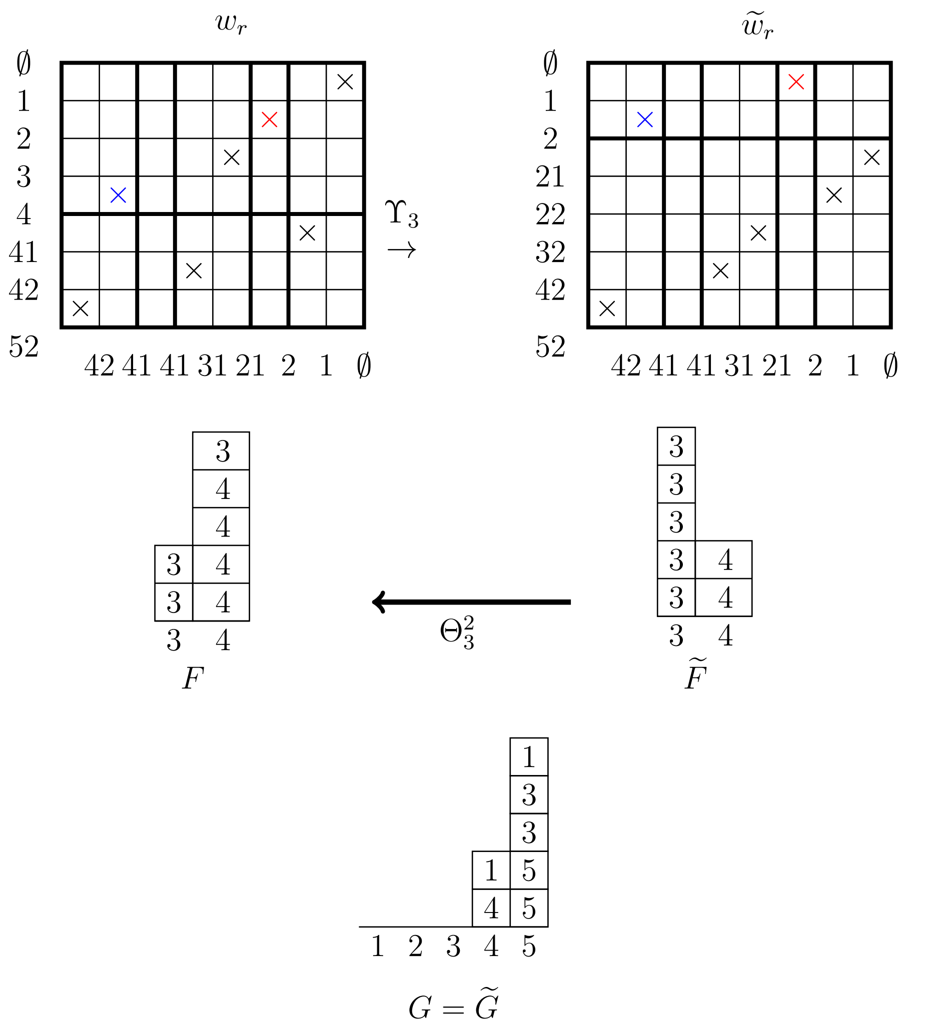
$$\begin{pmatrix} 1 & 1 & 3 & 3 & 4 & 5 & 5 \\ 3 & 4 & 3 & 4 & 4 & 3 & 4 \end{pmatrix} \underset{f_r^2}{\overset{\Upsilon_r}{\rightleftarrows}} \begin{pmatrix} 1 & 1 & 3 & 3 & 4 & 5 & 5 \\ 3 & 4 & 3 & 3 & 4 & 3 & 3 \end{pmatrix}$$



It is clear that the longest SW chain in the first k columns, from right to left, of the 01-filling of w_r and of $\widetilde{w}_r$, has length equal to the total number of crosses in rows r and rows $r+1$, of those columns, minus the number of matched crosses in row $r+1$, of those columns.

Theorem 5 implies that the bottom sequences in growth diagrams corresponding to w_r and $\widetilde{w}_r$ are the same and therefore bottom sequences of growth diagrams corresponding to w and $\widetilde{w}$ are also the same and the SSAFs corresponding to those partitions are also the same. Recall Section 2.10.

$$\begin{pmatrix} 1 & 1 & 3 & 3 & 4 & 5 & 5 \\ 3 & \color{blue}{4} & \color{blue}{3} & 4 & \color{red}{4} & \color{red}{3} & 4 \end{pmatrix} \overset{\Upsilon_3}{\rightarrow} \begin{pmatrix} 1 & 1 & 3 & 3 & 4 & 5 & 5 \\ 3 & \color{blue}{4} & \color{blue}{3} & 3 & \color{red}{4} & \color{red}{3} & 3 \end{pmatrix}$$



Recall from Section 3.3 that the operator Θ_r has the same behaviour as the operator f_r and then we have $F = \Theta_r^2 \widetilde{F}$. The next scheme shows the relation between the action of crystal operator f_r, its analogue Θ_r, the RSK and the analogue of RSK.

$$
\begin{tikzcd}
w = \begin{pmatrix} a \\ b \end{pmatrix} \arrow[r, leftrightarrow, "RSK"] & (P,Q) \arrow[r, leftrightarrow, "\Phi"] & (F,G) \\
\widetilde{w} = \begin{pmatrix} a \\ e_r(b) \end{pmatrix} \arrow[r, leftrightarrow, "RSK"] & (\widetilde{P},Q) \arrow[u, "f_r"'] \arrow[r, leftrightarrow, "\Phi"] & (\widetilde{F},G) \arrow[u, "\Theta_r"']
\end{tikzcd}
$$

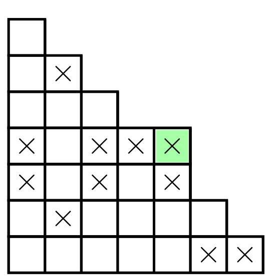
If F is SSAF, put $\Upsilon_r F$ such that $\Theta_r^m(\Upsilon_r F) = F$ where m is the number of unmatched $r+1$ in the row reading of the SSAF F. Equivalently, if $F = \Psi(P)$ with P a SSYT, then $\Upsilon_r F = \Psi(e_r^m P)$ where m is the number of unmatched $r+1$ in P.

Next theorem is therefore a consequence of our discussion.

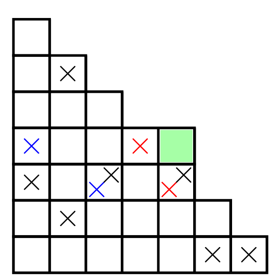
Theorem 13. *Let w be a biword in lexicographic order. If $\Phi(w) = (F,G)$ then $\Phi(\Upsilon_r w) = (\Upsilon_r F, G)$.*

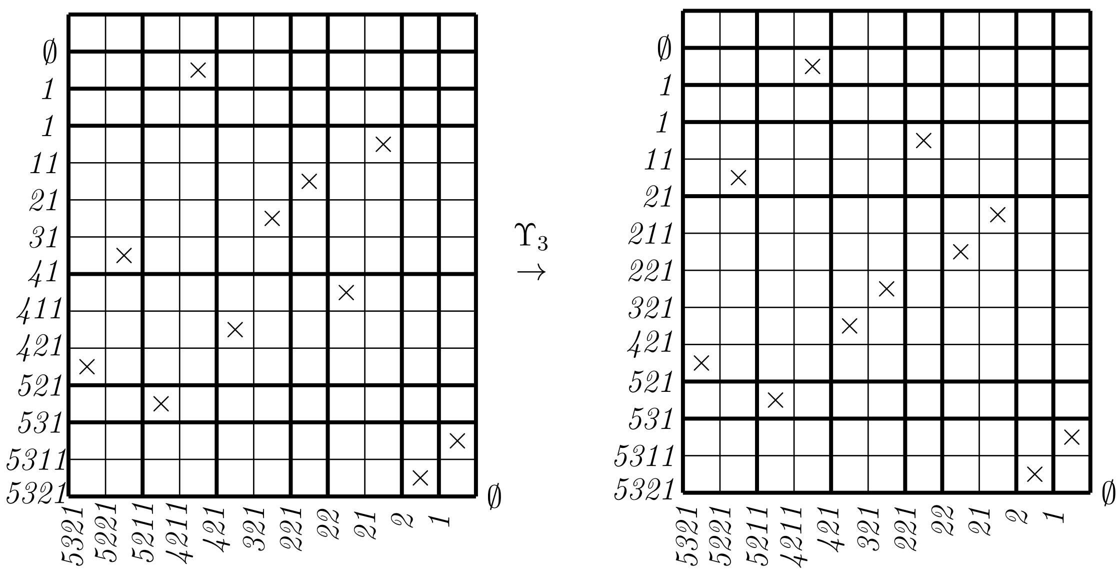
Example 33. *The procedure of passing from a biword to SSAF.*

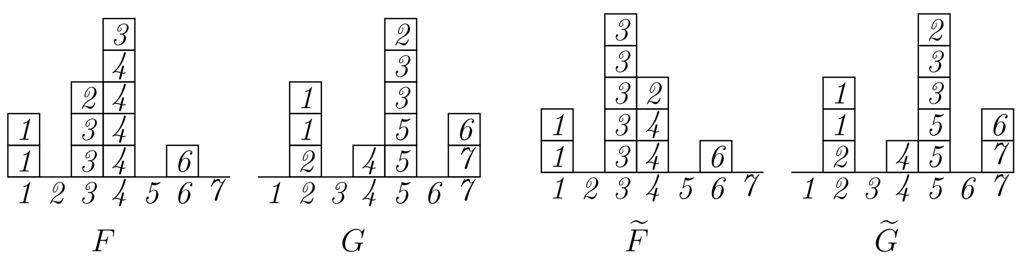
$$
\left(\begin{array}{ccccccccccc} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 & 7 \\ 3 & 4 & 2 & 6 & 3 & 4 & 4 & 3 & 4 & 1 & 1 \end{array}\right) \overset{\Upsilon_3}{\rightarrow} \left(\begin{array}{ccccccccccc} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 5 & 5 & 6 & 7 \\ 3 & 4 & 2 & 6 & 3 & 3 & 4 & 3 & 3 & 1 & 1 \end{array}\right)
$$



$\overset{\Upsilon_3}{\rightarrow}$







$sh(F) = (2, 0, 3, 5, 0, 1, 0)$ $G = \widetilde{G}$ $sh(\widetilde{F}) = (2, 0, 5, 3, 0, 1, 0)$

Theorem 14. *Let λ be a Ferrers shape where $\lambda_r = \lambda_{r+1} > \lambda_{r+2} \geq 0$, for some $r \geq 1$. Let w be a biword consisting of a multiset of cells of λ containing the cell $(r+1, \lambda_{r+1})$. Let $\Phi(w) = (F, G)$. If $sh(F) = \nu$ then $\nu_r < \nu_{r+1}$ and $sh(\Upsilon_r F) = s_r \nu$. Moreover, $\Upsilon_r w$ does not contain the biletter $\binom{\lambda_{r+1}}{r+1}$ and therefore fits the Ferrers shape λ with the cell $(r+1, \lambda_{r+1})$ deleted.*

Proof. Let $s := \lambda_{r+1}$. There is at least one cross in the position $(r+1, s)$ of the Ferrers shape λ. As $\lambda_{r+1} = \lambda_r > \lambda_{r+2}$ there is no cross in the row r to the right of it. If we consider the growth diagram corresponding to the biword w

with backward local rules, since $\lambda_r = \lambda_{r+1} > \lambda_{r+2}$ the cross corresponding to biletter $\begin{pmatrix} \text{s} \\ \text{r+1} \end{pmatrix}$ is the rightmost cross between all crosses corresponding to the rows bigger or equal than r, applying backward local rules gives that the partition corresponding to the row $r+1$, which appears in the left line of the growth diagram, has one more component than the upper partition in the left line of the growth diagram. The SSAF corresponding to that partition has a cell in the top of the basement $r+1$. If there is no new component in the partitions corresponding to the row r, which appear in the left line of the growth diagram, then in the SSAF there is no cell in the top of the basement r. Therefore if $sh(F) = \alpha$, then $\alpha_r < \alpha_{r+1}$. If there is a new component in a partition associated to the row r, which appears in the left line of the growth diagram, as the cross that creates the new component, is to the left of the cross associated to biletter $\begin{pmatrix} \text{s} \\ \text{r+1} \end{pmatrix}$, the new component can not be bigger or equal than the component associated to the biletter $\begin{pmatrix} \text{s} \\ \text{r+1} \end{pmatrix}$, and therefore $\alpha_r < \alpha_{r+1}$.

Now consider the growth diagram with backward local rules corresponding to the 01-filling of the biword $\Upsilon_r w$, as the cross corresponding to biletter $\begin{pmatrix} \text{s} \\ \text{r+1} \end{pmatrix}$ is always unmatched, then it moves down so there are just matched crosses in the row $r+1$. If the partitions corresponding to the row $r+1$ do not have the new component, as the rightmost cross between all crosses corresponding to the rows bigger or equal than r, is associated to the row r, then the partition corresponding to the row r has at least one more component than the partitions corresponding to the lines above. In the SSAF this means that there is a cell in the top of the basement r, and there is no cell in the top of the basement $r+1$. If the partitions corresponding to the row $r+1$ have the new component, as there are just matched crosses in the row $r+1$, it means that if there is chain of crosses corresponding to the row $r+1$, then there is a chain with same length corresponding to the row r. This implies that the height of the columns r and $r+1$ in SSAF will be the same, say l, in some point. After that if some component l increases to $l+1$ in some partition below the thick line row $r+1$, then in the SSAF the new cell goes to the top of the column r, because we always choose the leftmost position for new cells. As the number of crosses in the row r is bigger than the number of crosses in the row $r+1$, so the height of the column r is bigger than the height of the column $r+1$, in SSAF. Since the partitions that we get at the end of the row r in both growth diagrams should be the same, the difference between F and $\Upsilon_r F$ is in the columns r and $r+1$. So if $sh(\Upsilon_r F) = \nu$, then $sh(F) = s_r\nu$ and $\nu_r > \nu_{r+1}$. □

Example 33 illustrates Theorem 14. Transposing the Ferrers shape λ means to swap the first row and the second row of the biword w and to transpose the growth diagram of the 01-filling of w with backward local rules. Therefore the move of crosses on rows can be translated to a move of crosses on columns. As a consequence of the symmetry of the growth diagram we have the following versions of Theorem 13 and Theorem 14.

Swap the rows of w and then rearrange it in lexicographic order. This new biword is denoted by w^*. Let $\Upsilon_r^* w := \Upsilon_r w^*$.

Corollary 2. *If $\Phi(w) = (F, G)$ then $\Phi(\Upsilon_r^* w) = (F, \Upsilon_r G)$.*

Corollary 3. *Let λ be a Ferrers shape and let $\overline{\lambda} = (\lambda'_1, \lambda'_2, \ldots, \lambda'_{\lambda_1})$ be the conjugate of λ where $\lambda'_r = \lambda'_{r+1} > \lambda'_{r+2}$. Let w be a biword consisting of a* *multiset of cells of* λ *containing the cell* $(\lambda'_{r+1}, r+1)$. *Let* $\Phi(w) = (F, G)$. *If* $sh(G) = \nu$ *then* $\nu_r < \nu_{r+1}$ *and* $sh(\Upsilon_r G) = s_r\nu$. *Moreover,* $\Upsilon_r^* w$ *does not contain the biletter* $\binom{r+1}{\lambda'_{r+1}}$ *and therefore fits the Ferrers shape* λ *with the cell* $(\lambda'_{r+1}, r+1)$ *deleted.*

As a consequence of Proposition 11 and property of r-string one has,

Proposition 13. *Let* F *be a SSAF with shape* ν, *and* $\nu_r < \nu_{r+1}$, *for some* $r \geq 1$. *Then* $sh(\Upsilon_r F) = s_r\nu$.

5.2 A combinatorial proof for the Lascoux's Cauchy kernel expansion for some Ferrers shapes

We recall the Theorem 12 duo to Lascoux and bring the combinatorial proof for that. We also recall the permutations $\sigma(\lambda, NW)$ and $\sigma(\lambda, SE)$ for the Ferrers diagram λ in the figure 5.2 as below, Then $\sigma(\lambda, NW) = s_3 s_4$ and

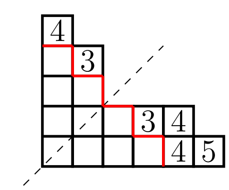


Fig. 5.2: The separation of the Ferrers shape $\lambda = 65221$ into NW and SE parts

$\sigma(\lambda, SE) = s_4 s_3 s_5 s_4$. In general these two permutations depend on the choice of the cell cutting the diagram λ/ρ, and there can be several ways to get F_λ from F_ρ.

Theorem 15. *(Theorem 12) Let* λ *be a partition,* $\rho(\lambda) = (m, \ldots, 1)$ *the maximal staircase contained in the diagram of* λ, *and* $\sigma(\lambda, NW)$, $\sigma(\lambda, SE)$

the two permutations obtained by cutting the diagram of λ/ρ as explained above. Then

$$F_\lambda(x,y) = \prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{\nu\in\mathbb{N}^m}(\pi_{\sigma(\lambda,NW)}\widehat{\kappa}_\nu(x))(\pi_{\sigma(\lambda,SE)}\kappa_{\omega\nu}(y)), \quad (5.1)$$

where the sum is over all $\nu \in \mathbb{N}^m$.

We discuss now some cases of this identity.

Case 1. Staircase partition

If the Ferrers shape is a staircase partition then $\pi_{\sigma(\lambda,NW)} = \pi_{\sigma(\lambda,SE)} = id$, and then Theorem 10 proves the identity (5.1), using the analogue of RSK.

Case 2. The boxes outside the biggest staircase are only in a NW or SE part

We give the details for one extra box and some details for two non-consecutive extra boxes in order to make clear the general case. Suppose that there is just one extra box above staircase partition in the position $(r+1, e+1)$, for $r, e \geq 0$. If r or e are zero, it means that the extra box is in the first row or in the first column of staircase partition. In this case there is just one possibility for identity (5.1). If r and e are different from zero, it means that the extra box is not in the first row or in the first column of the staircase partition. So there are two possibilities for identity (5.1). Depending on our choice of movement we can use Theorem 14 or Corollary 3 to find a proof for

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{\nu\in\mathbb{N}^n}\pi_r\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y) \quad (5.2)$$

or

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(x)\pi_e\kappa_{\omega\nu}(y) \quad (5.3)$$

We consider equation (5.2). Write

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{\substack{c\geq 0\\(i_t,j_t)\in\lambda\setminus(r+1,e+1)}} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}$$

$$+\sum_{d>0}\sum_{\substack{c\geq 0\\(i_t,j_t)\in\lambda\setminus(r+1,e+1)}} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r+1}^d y_{e+1}^d.$$

To each monomial $x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}$ we associate the pair (F,G). As the bi-word corresponding to these monomials are in the staircase partition, using Theorem 10 one gets, $sh(G)\leq \omega sh(F)$. From now on all the (i_t,j_t) belong to the maximum staircase partition contained in the partition λ. To avoid cumbersome notation we shall write $x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c} := \begin{pmatrix} j_1 & \cdots & j_c \\ i_1 & \dots & i_c \end{pmatrix}$. Using Theorem 14 one gets,

$$\begin{aligned}
\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} &= \sum_{c\geq 0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c} + \sum_{c\geq 0}\sum_{d>0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r+1}^d y_{e+1}^d \\
:= \ &\sum_{c\geq 0}\begin{pmatrix} j_1 & \cdots & j_c \\ i_1 & \dots & i_c \end{pmatrix} + \sum_{c\geq 0}\sum_{d>0}\begin{pmatrix} j_1 & \cdots & (e+1)^d & \dots & j_c \\ i_1 & \dots & (r+1)^d & \dots & i_c \end{pmatrix} \\
&= \sum_{\substack{\nu\in\mathbb{N}^n \\ sh(F)=\nu \\ sh(G)\leq\omega\nu}} x^F y^G + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r<\nu_{r+1}}} \sum_{\substack{(F,G)\in SSAF \\ sh(F)=\nu \\ sh(\Upsilon_r F)=s_r\nu \\ sh(G)\leq\omega s_r\nu \\ sh(G)\not\leq\omega\nu}} x^F y^G
\end{aligned} \tag{5.4}$$

Using the fact that if $\nu_r<\nu_{r+1}$ then $\omega\nu<\omega(s_r\nu)$, one gets,

$$\sum_{\substack{\nu\in\mathbb{N}^n \\ sh(F)=\nu \\ \nu_r<\nu_{r+1} \\ sh(G)\leq\omega\nu<\omega(s_r\nu)}} x^F y^G = \sum_{\substack{\nu\in\mathbb{N}^n \\ \theta_r>\theta_{r+1} \\ sh(F)=s_r\theta \\ sh(G)\leq\omega(s_r\theta)<\omega\theta}} x^F y^G.$$

Hence,

$$\sum_{\substack{\nu\in\mathbb{N}^n \\ sh(F)=\nu \\ sh(G)\leq\omega\nu}} x^F y^G = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r\geq\nu_{r+1} \\ sh(F)=\nu \\ sh(G)\leq\omega\nu}} x^F y^G + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r>\nu_{r+1} \\ sh(F)=s_r\nu \\ sh(G)\leq\omega(s_r\nu)<\omega\nu}} x^F y^G,$$

and therefore

$$(5.4) = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r\geq\nu_{r+1} \\ sh(F)=\nu \\ sh(G)\leq\omega\nu}} x^F y^G + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r>\nu_{r+1} \\ sh(F)=s_r\nu \\ sh(G)\leq\omega(s_r\nu)<\omega\nu}} x^F y^G + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r>\nu_{r+1} \\ sh(F)=s_r\nu \\ sh(G)\leq\omega\nu \\ sh(G)\not\leq\omega(s_r\nu)}} x^F y^G$$

$$= \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r\geq\nu_{r+1} \\ sh(F)=\nu \\ sh(G)\leq\omega\nu}} x^F y^G + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r>\nu_{r+1} \\ sh(F)=s_r\nu \\ sh(G)\leq\omega\nu}} x^F y^G.$$

Note that equation (3.17) implies that

$$\sum_{\nu\in\mathbb{N}^n} \pi_r\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y) = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r\geq\nu_{r+1}}} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y) + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r>\nu_{r+1}}} \widehat{\kappa}_{s_r\nu}(x)\kappa_{\omega\nu}(y). \tag{5.5}$$

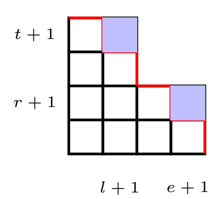
Therefore

$$(5.4) \quad = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r\geq\nu_{r+1} \\ sh(F)=\nu \\ sh(G)\leq\omega\nu}} x^F y^G + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r>\nu_{r+1} \\ sh(F)=s_r\nu \\ sh(G)\leq\omega\nu}} x^F y^G = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r\geq\nu_{r+1}}} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)$$

$$+ \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r>\nu_{r+1}}} \widehat{\kappa}_{s_r\nu}(x)\kappa_{\omega\nu}(y) = \sum_{\nu\in\mathbb{N}^n} \pi_r\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)$$

The equation (5.3) can be obtained by either conjugating the Ferrers shape λ or using Corollary 3 instead of Theorem 14 and change from the basis (3.13) of Demazure characters to the basis of Demazure atoms.

Now suppose that there are two extra boxes above staircase partition in the positions $(r+1, e+1)$ and $(t+1, l+1)$, where $r < t$ and $l < e$ as below:



Using two times Theorem 14 one gets,

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{c\geq 0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c} + \sum_{c\geq 0}\sum_{d>0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r+1}^d y_{e+1}^d$$

$$+ \sum_{c\geq 0}\sum_{d'>0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{t+1}^{d'}y_{l+1}^{d'} + \sum_{c\geq 0}\sum_{d,d'>0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r+1}^d y_{e+1}^d x_{t+1}^{d'}y_{l+1}^{d'}$$

$$
\begin{aligned}
:= & \sum_{c\geq 0}\begin{pmatrix} j_1 & \dots & j_c \\ i_1 & \dots & i_c \end{pmatrix} + \sum_{c\geq 0}\sum_{d>0}\begin{pmatrix} j_1 & \dots & (e+1)^d & \dots & j_c \\ i_1 & \dots & (r+1)^d & \dots & i_c \end{pmatrix} \\
& + \sum_{c\geq 0}\sum_{d>0}\begin{pmatrix} j_1 & \dots & (l+1)^{d'} & \dots & j_c \\ i_1 & \dots & (t+1)^{d'} & \dots & i_c \end{pmatrix} \\
& + \sum_{c\geq 0}\sum_{d,d'>0}\begin{pmatrix} j_1 & \dots & (l+1)^{d'} & \dots & (e+1)^d & \dots & j_c \\ i_1 & \dots & (t+1)^{d'} & \dots & (r+1)^d & \dots & i_c \end{pmatrix} \\
= & \sum_{\substack{\nu\in\mathbb{N}^n \\ (F,G)\in SSAF \\ sh(F)=\nu \\ sh(G)\leq\omega\nu}} x^F y^G + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r<\nu_{r+1}}} \sum_{\substack{(F,G)\in SSAF \\ sh(F)=\nu \\ sh(\Upsilon_r F)=s_r\nu \\ sh(G)\leq\omega s_r\nu \\ sh(G)\nleq\omega\nu}} x^F y^G \\
& + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t<\nu_{t+1}}} \sum_{\substack{(F,G)\in SSAF \\ sh(F)=\nu \\ sh(\Upsilon_t F)=s_t\nu \\ sh(G)\leq\omega s_t\nu \\ sh(G)\nleq\omega\nu}} x^F y^G \\
& + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t<\nu_{t+1} \\ \nu_r<\nu_{r+1}}} \sum_{\substack{(F,G)\in SSAF \\ sh(F)=\nu \\ sh(\Upsilon_r\Upsilon_t F)=s_r s_t\nu \\ sh(G)\leq\omega s_r s_t\nu \\ sh(G)\nleq\omega s_t\nu \\ sh(G)\nleq\omega s_r\nu}} x^F y^G \\
= & \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r\geq\nu_{r+1} \\ \nu_t\geq\nu_{t+1}}} \sum_{\substack{(F,G)\in SSAF \\ sh(F)=\nu \\ sh(G)\leq\omega\nu}} x^F y^G + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_r>\nu_{r+1} \\ \nu_t\geq\nu_{t+1}}} \sum_{\substack{(F,G)\in SSAF \\ sh(F)=s_r\nu \\ sh(G)\leq\omega\nu}} x^F y^G \\
& + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t>\nu_{t+1} \\ \nu_r\geq\nu_{r+1}}} \sum_{\substack{(F,G)\in SSAF \\ sh(F)=s_t\nu \\ sh(G)\leq\omega\nu}} x^F y^G + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t>\nu_{t+1} \\ \nu_r>\nu_{r+1}}} \sum_{\substack{(F,G)\in SSAF \\ sh(F)=s_r s_t\nu \\ sh(G)\leq\omega\nu}} x^F y^G
\end{aligned}
$$

$$= \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t\geq\nu_{t+1} \\ \nu_r\geq\nu_{r+1}}} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y) + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t\geq\nu_{t+1} \\ \nu_r>\nu_{r+1}}} \widehat{\kappa}_{s_r\nu}(x)\kappa_{\omega\nu}(y) + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t>\nu_{t+1} \\ \nu_r\geq\nu_{r+1}}} \widehat{\kappa}_{s_t\nu}(x)\kappa_{\omega\nu}(y)$$

$$+ \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t>\nu_{t+1} \\ \nu_r>\nu_{r+1}}} \widehat{\kappa}_{s_rs_t\nu}(x)\kappa_{\omega\nu}(y) = \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t>\nu_{t+1} \\ \nu_r>\nu_{r+1}}} (\widehat{\kappa}_\nu(x) + \widehat{\kappa}_{s_r\nu}(x))\kappa_{\omega\nu}(y)$$

$$+ \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t>\nu_{t+1} \\ \nu_r=\nu_{r+1}}} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y) + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t>\nu_{t+1} \\ \nu_r>\nu_{r+1}}} (\widehat{\kappa}_{s_t\nu}(x) + \widehat{\kappa}_{s_rs_t\nu}(x))\kappa_{\omega\nu}(y)$$

$$+ \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t>\nu_{t+1} \\ \nu_r=\nu_{r+1}}} \widehat{\kappa}_{s_t\nu}(x)\kappa_{\omega\nu}(y) + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t=\nu_{t+1} \\ \nu_r>\nu_{r+1}}} (\widehat{\kappa}_\nu(x) + \widehat{\kappa}_{s_r\nu}(x))\kappa_{\omega\nu}(y) + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t=\nu_{t+1} \\ \nu_r=\nu_{r+1}}} \widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)$$

$$= \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t>\nu_{t+1}}} \pi_r\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y) + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t>\nu_{t+1}}} \pi_r\widehat{\kappa}_{s_t\nu}(x)\kappa_{\omega\nu}(y) + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t=\nu_{t+1}}} \pi_r\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)$$

$$= \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t>\nu_{t+1}}} \pi_r(\widehat{\kappa}_\nu(x) + \widehat{\kappa}_{s_t\nu}(x))\kappa_{\omega\nu}(y) + \sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_t=\nu_{t+1}}} \pi_r\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)$$

$$= \sum_{\nu\in\mathbb{N}^n} \pi_r\pi_t\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)$$

Notice that if two extra boxes are consecutive then in order to apply Theorem 14 we should first consider the upper extra box.

The next equation can be obtained by either conjugating the Ferrers shape λ or using Corollary 3 instead of Theorem 14 and change from the basis (3.13) of Demazure characters to the basis of Demazure atoms.

$$\prod_{(i,j)\in\lambda} (1 - x_iy_j)^{-1} = \sum_{\nu\in\mathbb{N}^n} \widehat{\kappa}_\nu(x)\pi_l\pi_e\kappa_{\omega\nu}(y).$$

Before going to the next step let us calculate the action of π_i on the $\widehat{\kappa}_\nu(x)$, by using several times the equation (5.5).

$$\sum_{\nu\in\mathbb{N}^n} \pi_{r_k}\dots\pi_{r_2}\pi_{r_1}\widehat{\kappa}_\nu(x) =$$

$$= \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_i}\geq\nu_{r_i+1}\\ 1\leq i\leq k}} \widehat{\kappa}_\nu(x) + \sum_{1\leq i\leq k} \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_i}>\nu_{r_i+1}\\ \nu_{r_j}\geq\nu_{r_j+1}\quad j<i\\ (s_{r_i}\nu)_{r_t}\geq(s_{r_i}\nu)_{r_t+1}\quad i<t\leq k}} \widehat{\kappa}_{s_{r_i}\nu}(x)$$

$$+ \sum_{1\leq i<j\leq k} \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_i}>\nu_{r_i+1}\\ (s_{r_i}\nu)_{r_j}>(s_{r_i}\nu)_{r_j+1}\\ \nu_{r_p}\geq\nu_{r_p+1}\quad p<i\\ (s_{r_i}\nu)_{r_t}\geq(s_{r_i}\nu)_{r_t+1}\quad i<t<j\\ (s_{r_j}s_{r_i}\nu)_{r_l}\geq(s_{r_j}s_{r_i}\nu)_{r_l+1}\quad j<l\leq k}} \widehat{\kappa}_{s_{r_j}s_{r_i}\nu}(x) + \dots +$$

$$\sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_1}>\nu_{r_1}+1\\ \vdots\\ (s_{r_{k-1}}\dots s_{r_1}\nu)_{r_k}>(s_{r_{k-1}}\dots s_{r_1}\nu)_{r_k}+1}} \widehat{\kappa}_{s_{r_k}\dots s_{r_1}\nu}(x)$$

In general if there are k extra boxes in positions $(r_1+1, e_1+1), \dots (r_k+1, e_k+1)$, such that we can consider them in the NW part of Ferrers shape λ, i.e there is an empty box above staircase that is below them, or the first row has no box of the biggest staircase to its right. We proceed as follows: we read, column wise, extra boxes from left to right and bottom to top, to guarantee that we are able to use Theorem 14. It means that (r_1+1, e_1+1) is the lowest extra box in the leftmost column contains the extra boxes. See Figure 5.3.

This reading implies that each time we create the new box above staircase, in the new Ferrers shape we have $\lambda_s = \lambda_{s+1} > \lambda_{s+2}$, where $s+1$ is the row the new box just created. Using several times Theorem 14, one has,

$$\prod_{(i,j)\in\lambda} (1-x_iy_j)^{-1} = \sum_{c\geq 0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}$$

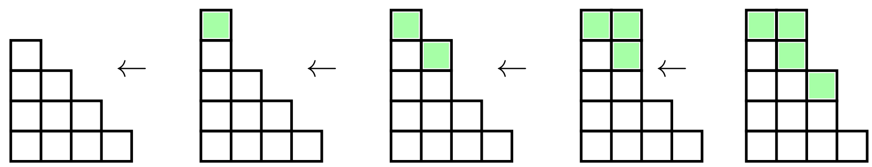


Fig. 5.3: the order of appearing the extra boxes

$$+ \quad \sum_{1\le f\le k}\sum_{c\ge 0}\sum_{d_f>0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r_f+1}^{d_f}y_{e_f+1}^{d_f}$$

$$+ \quad \sum_{1\le f<h\le k}\sum_{c\ge 0}\sum_{d_f,d_h>0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r_f+1}^{d_f}y_{e_f+1}^{d_f}x_{r_h+1}^{d_h}y_{e_h+1}^{d_h}$$

$$+ \quad \cdots + \sum_{c\ge 0}\sum_{d_1,\ldots,d_k>0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x_{r_1+1}^{d_1}y_{e_1+1}^{d_1}\ldots x_{r_k+1}^{d_k}y_{e_k+1}^{d_k}$$

$$= \quad \sum_{c\ge 0}\begin{pmatrix} j_1 & \cdots & j_c \\ i_1 & \ldots & i_c \end{pmatrix} + \sum_{1\le f\le k}\sum_{c\ge 0}\sum_{d_f>0}\begin{pmatrix} j_1 & \cdots & (e_f+1)^{d_f} & \ldots & j_c \\ i_1 & \ldots & (r_f+1)^{d_f} & \ldots & i_c \end{pmatrix}$$

$$+ \quad \sum_{1\le f<h\le k}\sum_{c\ge 0}\sum_{d_f,d_h>0}\begin{pmatrix} j_1 & \ldots & (e_h+1)^{d_h} & \ldots & (e_f+1)^{d_f} & \ldots & j_c \\ i_1 & \ldots & (r_h+1)^{d_h} & \ldots & (r_f+1)^{d_f} & \ldots & i_c \end{pmatrix}$$

$$+ \quad \cdots + \sum_{c\ge 0}\sum_{d_1,\ldots,d_k>0}\begin{pmatrix} j_1\ldots & (e_k+1)^{d_k}\ldots & (e_1+1)^{d_1}\ldots & j_c \\ i_1\ldots & (r_k+1)^{d_k}\ldots & (r_1+1)^{d_1}\ldots & i_c \end{pmatrix}$$

$$= \sum_{\nu\in\mathbb{N}^n}\sum_{\substack{(F,G)\in SSAF \\ sh(F)=\nu \\ sh(G)\le\omega\nu}} x^F y^G + \sum_{1\le i\le r}\sum_{\substack{\nu\in\mathbb{N}^n \\ \nu_{r_i}<\nu_{r_i+1}}}\sum_{\substack{(F,G)\in SSAF \\ sh(F)=\nu \\ sh(\Upsilon_{r_i}F)=s_{r_i}\nu \\ sh(G)\le\omega s_{r_i}\nu \\ sh(G)\not\le\omega\nu}} x^F y^G$$

$$+\cdots+\sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_k}<\nu_{r_k+1},\\ (s_{r_k}\nu)_{r_{k-1}}<(s_{r_k}\nu)_{r_{k-1}+1}\\ \vdots\\ (s_{r_2}\ldots s_{r_k}\nu)_{r_1}<(s_{r_2}\ldots s_{r_k}\nu)_{r_1+1}}}\ \sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(\Upsilon_{r_1}\ldots\Upsilon_{r_k}F)=s_{r_1}\ldots s_{r_k}\nu\\ sh(G)\leq\omega s_{r_1}\ldots s_{r_k}\nu\\ sh(G)\nleq\omega s_{r_1}\ldots\widehat{s}_{r_i}\ldots s_{r_k}\nu\ \ 1\leq i\leq k}} x^F y^G$$

$$=\sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_i}\geq\nu_{r_i+1}\\ 1\leq i\leq k}}\ \sum_{\substack{(F,G)\in SSAF\\ sh(F)=\nu\\ sh(G)\leq\omega\nu}} x^F y^G+\sum_{1\leq i\leq k}\ \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_i}>\nu_{r_i+1}\\ \nu_{r_j}\geq\nu_{r_j+1}\quad j<i\\ (s_{r_i}\nu)_{r_t}\geq(s_{r_i}\nu)_{r_t+1}\quad i<t\leq k}}\ \sum_{\substack{(F,G)\in SSAF\\ sh(F)=s_{r_i}\nu\\ sh(G)\leq\omega\nu}} x^F y^G$$

$$+\cdots+\sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_1}>\nu_{r_1}+1\\ \vdots\\ (s_{r_{k-1}}\ldots s_{r_1}\nu)_{r_k}>(s_{r_{k-1}}\ldots s_{r_1}\nu)_{r_k}+1}}\ \sum_{\substack{(F,G)\in SSAF\\ sh(F)=s_{r_k}\ldots s_{r_1}\nu\\ sh(G)\leq\omega\nu}} x^F y^G$$

$$=\sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_i}\geq\nu_{r_i+1}\\ 1\leq i\leq k}}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)+\sum_{1\leq i\leq k}\ \sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_i}>\nu_{r_i+1}\\ \nu_{r_j}\geq\nu_{r_j+1}\quad j<i\\ (s_{r_i}\nu)_{r_t}\geq(s_{r_i}\nu)_{r_t+1}\quad i<t\leq k}}\widehat{\kappa}_{s_{r_i}\nu}(x)\kappa_{\omega\nu}(y)$$

$$+\cdots+\sum_{\substack{\nu\in\mathbb{N}^n\\ \nu_{r_1}>\nu_{r_1}+1\\ \vdots\\ (s_{r_{k-1}}\ldots s_{r_1}\nu)_{r_k}>(s_{r_{k-1}}\ldots s_{r_1}\nu)_{r_k}+1}}\widehat{\kappa}_{s_{r_k}\ldots s_{r_1}\nu}(x)\kappa_{\omega\nu}(y)$$

$$=\sum_{\nu\in\mathbb{N}^n}\pi_{r_k}\ldots\pi_{r_1}\widehat{\kappa}_\nu(x)\kappa_{\omega\nu}(y)$$

If we conjugate the Ferrers shape then the extra boxes are in the SE part of Ferrers shape. It means that there is an empty box above staircase which is above all of the extra boxes. We read, row wise, the extra boxes from bottom to top and left to right, $(r'_1+1,e'_1+1),\ldots(r'_k+1,e'_k+1)$. It means that (r'_1+1,e'_1+1) is the leftmost extra box in the lowest row containing the extra boxes. This reading implies that each time we create the new box above staircase, in the new Ferrers shape we have, $\lambda'_s = \lambda'_{s+1} > \lambda'_{s+1}$, where $s+1$ is the column that new box is created. The next equation can be obtained by either conjugating the Ferrers shape λ or using Corollary 2 instead of Theorem 13 and change from the basis (3.13) of Demazure characters to the basis of Demazure atoms.

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{\nu\in\mathbb{N}^n}\widehat{\kappa}_\nu(x)\pi_{e'_k}\dots\pi_{e'_1}\kappa_{\omega\nu}(y)$$

Note that the strict truncated staircase studied in the Chapter 4 is included in this approach by considering the biggest staircase inside the Ferrers shape. Hear we work with the biggest staircase contained in the Ferrers shape and try to reduce the extra boxes into staircase but in the Chapter 4 we work with the smallest staircase that contains the Ferrers shape.

Case 3. The boxes outside the biggest staircase are in NW and SE parts

Now for the arbitrary Ferrers shapes, depending on our choice of row movements or column movements, we can mix Theorem 14 and Corollary 3 several times to arrive to the staircase partition. Suppose extra boxes in positions $(r_1+1, e_1+1), \dots (r_k+1, e_k+1)$ are those with row movement and extra boxes in positions $(r_{k+1}+1, e_{k+1}+1), \dots (r_{k'}+1, e_{k'}+1)$ are those with column movement. Then using k times Theorem 14 and k' times Corollary 3 we obtain,

$$\prod_{(i,j)\in\lambda}(1-x_iy_j)^{-1} = \sum_{c\geq 0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}$$

$$+ \quad \sum_{1\leq f\leq k'}\sum_{c\geq 0}\sum_{d_f>0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x^{d_f}_{r_f+1}y^{d_f}_{e_f+1}$$

$$+ \quad \sum_{1\leq f<h\leq k'}\sum_{c\geq 0}\sum_{d_f,d_h>0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c}x^{d_f}_{r_f+1}y^{d_f}_{e_f+1}x^{d_h}_{r_h+1}y^{d_h}_{e_h+1}$$

$$+ \quad \cdots + \sum_{c\geq 0} \sum_{d_1,\ldots,d'_k>0} x_{i_1}y_{j_1}\cdots x_{i_c}y_{j_c} x_{r_1+1}^{d_1} y_{e_1+1}^{d_1} \ldots x_{r_{k'}+1}^{d_{k'}} y_{e'_k+1}^{d_{k'}}$$

$$= \sum_{c\geq 0} \begin{pmatrix} j_1 & \ldots & j_c \\ i_1 & \ldots & i_c \end{pmatrix} + \sum_{1\leq f\leq k'} \sum_{c\geq 0} \sum_{d_f>0} \begin{pmatrix} j_1 & \ldots & (e_f+1)^{d_f} & \ldots & j_c \\ i_1 & \ldots & (r_f+1)^{d_f} & \ldots & i_c \end{pmatrix}$$

$$+ \quad \sum_{1\leq f<h\leq k'} \sum_{c\geq 0} \sum_{d_f,d_h>0} \begin{pmatrix} j_1 & \ldots & (e_h+1)^{d_h} & \ldots & (e_f+1)^{d_f} & \ldots & j_c \\ i_1 & \ldots & (r_h+1)^{d_h} & \ldots & (r_f+1)^{d_f} & \ldots & i_c \end{pmatrix}$$

$$+ \quad \cdots + \sum_{c\geq 0} \sum_{d_1,\ldots,d'_k>0} \begin{pmatrix} j_1 \ldots & (e'_k+1)^{d_{k'}} \ldots & (e_1+1)^{d_1} \ldots & j_c \\ i_1 \ldots & (r'_k+1)^{d_{k'}} \ldots & (r_1+1)^{d_1} \ldots & i_c \end{pmatrix}$$

$$= \sum_{\nu\in\mathbb{N}^n} \pi_{r_k} \ldots \pi_{r_1} \widehat{\kappa}_\nu(x) \sum_{\beta\leq\omega\nu} \pi_{e_{k'}} \ldots \pi_{e_{k+1}} \widehat{\kappa}_\beta(y)$$

$$= \sum_{\nu\in\mathbb{N}^n} \pi_{r_k} \ldots \pi_{r_1} \widehat{\kappa}_\nu(x) \pi_{e_{k'}} \ldots \pi_{e_{k+1}} \sum_{\beta\leq\omega\nu} \widehat{\kappa}_\beta(y)$$

$$= \sum_{\nu\in\mathbb{N}^n} \pi_{r_k} \ldots \pi_{r_1} \widehat{\kappa}_\nu(x) \pi_{e_{k'}} \ldots \pi_{e_{k+1}} \kappa_{\omega\nu}(y)$$

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