Preface

There are various reasons and motivations for the point-free approach to space. We are not historians and cannot (and do not really wish to) give in this short introduction a summary of the colorful development, where the impulses from logic, special results of set-theoretical topology, partially ordered sets, algebra, and other branches of mathematics united into the point-free topology as we know it today. For an excellent overview of the early stages we can warmly recommend the Johnstone's article "*The point of pointless topology*" [36].

Here we want to emphasize the "realistic geometric" aspect in which one views a space as a structure of places of non-trivial extent rather than a set of points endowed with that or other structure. A result of a measurement is never a real number (one has a value $\pm$ a tolerance, which may be viewed as an open interval, hopefully very small, around the ideal value), or, pinpointing a position in a space (we now use the word "space" loosely) is pointing out a spot that is by no means infinitesimally small (although small enough to serve the task), not a "point" as we know it from classical geometry or topology. We can think of a point as an ideal limit of diminishing spots, and it is useful; but from our point of view it is secondary, and often not necessary (and not even advisable).

There are two big moments that initiated the development of modern theory of space that we call topology. One of them is very well defined: the Hausdorff's "*Mengenlehre*" [27] of 1914 where the idea of space is based on the concept of neighbourhood, that then opens a general perspective for concepts like closure, interior, open and closed sets. The other, much less defined in time (one can speak of a good part of the twenties), is the tendency to give more and more importance (and priority when defining the other concepts) to open sets. Gradually one started to think of a topological space as of a set endowed with a system of subsets called open sets, with very simple properties, from which one could easily derive the other concepts one might wish to have.

Now the concept of an open set fits very well to the (rather unprecise) idea of a "spot in space" mentioned above. Furthermore, the structure of open sets is simple, a complete lattice (with a special property that is easy to understand and enhances the computational potential).

Furthermore again, it has turned out that under very weak conditions (e.g., sobriety, or a separation property between T_0 and T_1 [2]) a space is already determined by the isomorphism type of its lattice of open sets (cf. [16], [64], articles that should have had more attention in their time).

Thus, the question naturally arises whether one cannot, after all, study a space as a complete lattice (with a special property — this had to be, of course, properly contemplated; it turned out that a certain distributive law does excellently, see the notion of a *frame* in 1.5 below).

The next question, however, is how to represent continuous maps. Lattice homomorphisms are not suitable, and neither are the complete lattice homomorphisms; it turned out that one can represent the continuous maps by a sort of semi-complete lattice homomorphisms (*frame homomorphisms*, preserving *all* suprema and *finite* infima, see 2.1 below).

Observing that this is the right notion was an important breakthrough in the theory that started in late fifties and early sixties (let us mention the pioneering articles [22], [47], [19]) and sent it on the track it has followed until today. (It should be noted that there were serious efforts, but not quite successful, to do topology without points much earlier, see, e.g., [43], [44], [66].)

Thus we have the following situation. Spaces are replaced by the above mentioned special complete lattices (frames); in particular, a classical space X is represented by its lattice $\Omega(X)$ of open sets. Some of the frames are not obtained from classical spaces, but the fact that we have "more spaces than before" is useful (although this was not obvious at the beginnings). The role of continuous maps is taken over by frame homomorphisms and everything is easy to work with. Moreover, for the important class of sober spaces (containing, e.g., all Hausdorff spaces, and all the finite ones) this representation is full and faithful, that is, frame homomorphisms between frames of open maps are in a natural one-to-one correspondence with the continuous maps between the spaces in question.

There is, however, an irritating drawback in this representation. Namely, it is contravariant, that is, this one-to-one correspondence associates continuous maps $X \to Y$ with frame homomorphisms $\Omega(Y) \to \Omega(Y)$. To mend it one usually turns the morphisms over formally, and considers, as the category of generalized spaces, the category $\mathsf{Frm}^{\mathrm{op}}$ dual to Frm (usually referred to as the *category of locales*). Thus we have localic morphisms $L \to M$ that are in fact frame homomorphisms $L \leftarrow M$. This helps a bit, but the intuition is, let us venture to say, not quite optimal. Instead of this elegant solution one might prefer a more down-to-earth one: a representation by suitable *maps* $L \to M$. And this can be done. Frame homomorphisms $h : M \to L$, by virtue of preserving all suprema, have uniquely defined right Galois adjoints $f : L \to M$, meet--preserving maps satisfying an extra condition (their left adjoints f^* have to preserve finite meets).

Of course, just having (special) mappings instead of formal morphisms would not count for all that much. But it turns out that such representation is often fairly apt, and thus opened perspective often surprisingly simplifies the techniques. In the present text we have chosen several topics of elementary point-free topology that illustrate the benefits of this approach. Needless to say, such "covariant technique" is not preferable to the "algebraic" one under all circumstances (and even in this text we are not unduly consequent, and sometimes slip into the contravariant thinking if it is simpler). Therefore, our topics cover only a small part of the elementary point-free topology. But there are cases in which we gain a lot. For instance, thus defined localic maps $f : L \to M$ send spectral points (that is, completely prime elements) of L to spectral points of M and make the spectrum continuous maps $\mathsf{Pt}L \to \mathsf{Pt}M$ simply restrictions of f. We are convinced that the spectrum adjunction, and the relation between classical spaces and the more general point-free ones becomes more lucid.

In particular, however, we see the striking technical advantages (and factual ones as well) when working with sublocales (generalized subspaces). They now become really *sub-spaces*, that is, precisely the subsets the embeddings of which are localic maps (and, moreover, they have a very simple intrinsic characterization). The structure of the system of sublocale is now fairly transparent (the meets are the intersections and the joins are described by a simple and expedient formula), which results, a.o., in a very short proof of the distributivity. The formula for closure is extremely plain (resulting in the characterization of dense sublocales as those containing 0, and in a three-line proof of the Isbell's density theorem). Working with images and preimages is easy (images are the set-theoretical images, and to obtain preimages the set-theoretical ones have to be only slightly modified). The analysis of separation properties (in particular of fitness, subfitness and the strong Hausdorff property) is not only technically simpler than the usual ones, but also brings new insight to some of their aspects. Also we gain some new insight into products of locales (coproducts of frames) — here, not surprisingly, our techniques are mixed.

The text does not go much beyond the topics just mentioned (and misses very important questions, even fundamental ones, like those of compactness and local compactness, of cover based uniformity and nearness, etc.). Nevertheless, it can serve as a first introduction to point-free thinking; afterwards, the reader is encouraged to open some more comprehensive text like for instance [35], or more recent but less extensive [53] and [57].

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