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★**Frames and locales.**
Topology without points.
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The book covers, in a comprehensive and self-contained way, the development of point-free topology in the last 30 years, since P. T. Johnstone published his book [*Stone spaces*, Cambridge Stud. Adv. Math., 3, Cambridge Univ. Press, Cambridge, 1982; MR0698074 (85f:54002)].

It can be also considered as a natural continuation of the authors' previous monograph [*Locales treated mostly in a covariant way*, Textos Mat. Sér. B, 41, Univ. Coimbra, Coimbra, 2008; MR2459570 (2010d:06012)]. The main originality of this approach to point-free topology consists of the fact that the authors consider morphisms in the category Loc of locales as localic maps, i.e. maps between locales which have left adjoints that preserve all meets. Consequently localic maps are functions in the usual sense, and not simply arrows obtained by reversing frame homomorphisms. It turns out in many occasions that this change helps one to understand different notions and constructions. However, the authors do not act in a dogmatic way and on some occasions, when lucidity calls for that and the contravariant approach brings more clarity, the authors change to the frame perspective.

The text starts with a nicely written introduction. It is at the same time a motivating introduction for readers for whom point-free topology is an unknown topic and a concise history of the subject, of interest also for involved specialists.

In the first two chapters the authors motivate the reader to go further in the study of the subject by delimiting the relationship between both point-free and classical topology. The basic concepts are introduced; in particular sobriety and the T_D-axiom are naturally presented together with their basic properties. The category of locales and localic maps is then introduced. The new idea is in the choice of the morphisms: the authors define a localic map between locales L and M to be a mapping $f\colon L \to M$ such that the left adjoint $f^*\colon M \to L$ preserves finite meets, including the top. The second chapter ends with the spectrum adjunction and some criteria for spatiality.

The third chapter is devoted to a first study of sublocales. They are introduced as follows: $S \subseteq L$ is a *sublocale* if it is closed under all meets and for every $s \in S$ and every $x \in L$, $x \to s \in S$, where $\to$ denotes the Heyting implication. Equivalent notions like frame congruences and nuclei are also considered, but the main concepts like open and closed sublocales and open and closed localic maps are developed in terms of sublocales. This is one of the many places where the use of localic maps shows its advantages, in particular in the treatment of images and preimages of sublocales.

In Chapter IV some categorical facts of the categories Loc and Frm are studied. Particular attention is paid to the construction of products of locales (coproducts of frames). In this respect it deserves to be mentioned that the authors change to the contravariant approach since it would be unnecessarily complicated to try to introduce products of locales in a covariant way. The comparison of products of locales and those of spaces deserves particular attention, as does the detailed discussion on epimorphisms.

Chapter V is devoted to the study of the separation axioms. Special attention is paid to the (almost) counterparts of the T_1 axiom in the point-free setting: fitness and subfitness. The rest of the usual separation axioms are then studied in the natural order: Hausdorffness (including both approaches due to Isbell and to Dowker and Strauss), regularity, complete regularity and normality.

In Chapter VI the authors continue the study of sublocales, in particular the complemented, spatial and induced ones. At the end of the section characterizations of locales with no non-spatial sublocales and spaces with no non-induced sublocales are provided.

Chapter VII is devoted to compactness and local compactness. The cover definition of compactness being basically point-free, most of the basic facts are parallels of the classical ones.

In Chapters VIII–XII frames are enriched with some additional structure: nearness and uniformity in Chapters VIII, X and XII and (dia)metric in Chapter XI. After the introduction of the basic concepts on uniformities and completeness, in Chapter IX the authors introduce paracompact locales. The reason to insert paracompactness here is the point-free result which states that a locale is paracompact iff it admits a complete uniformity.

The point-free theory of reals is treated in Chapter XIV. The chapter starts by introducing continuous real-valued functions along the lines of [B. Banaschewski, *The real numbers in pointfree topology*, Textos Mat. Sér. B, 12, Univ. Coimbra, Coimbra, 1997; MR1621835 (99c:54017)]. Then they go further and arbitrary real-valued functions are also considered (in the sense of [J. Gutiérrez García, T. Kubiak and J. Picado, J. Pure Appl. Algebra **213** (2009), no. 6, 1064–1074; MR2498797 (2010a:06020)]). The chapter ends with a list of comments on related topics which shows that the subject of real functions on a locale is too vast to be covered in the present book.

The last chapter discusses group structures in the point-free context. It includes (among other results) the Closed Subgroup Theorem, which states that each subgroup of a localic group is always closed.

The book ends with two short appendices containing those facts about partially ordered sets and categories needed in the book.

In conclusion, this is a very good book; it is nicely written and is highly recommended for anyone wishing to gain an overview of point-free topology.

Reviewed by *Javier Gutiérrez García*