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How to deal with the ring of (continuous) real functions in terms of scales[1]

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Abstract

There are different approaches in the literature to the study of (continuous) real functions in terms of scales.
Our first purpose with this survey-type paper is to provide motivation for the study of scales as a kind of generalization of the notion of Dedekind cut.
Secondly, we make explicit the well-known relationship between real functions and scales and we show how one can deal with the algebraic and lattice operations of the ring of real functions purely in terms of scales.
Finally we consider two particular situations: (1) if the domain is endowed with a topology we characterize the scales that generate upper and lower semicontinuous and also continuous functions and/or (2) if the domain is enriched with a partial order we characterize the scales that generate functions preserving the partial order and the order embeddings.

1. Introduction

Let us denote by $C(X, \mathcal{O}X)$ the ring of continuous real functions on a topological space $(X, \mathcal{O}X)$[2] and by $F(X)$ the collection of *all* real functions on X.

We would like to start by discussing the following question:

Question. *What is more general, the study of the rings $C(X, \mathcal{O}X)$ or that of the rings $F(X)$?*

A first obvious answer immediately comes to our mind:

- For a given topological space $(X, \mathcal{O}X)$, the family $F(X)$ is much bigger than $C(X, \mathcal{O}X)$. Hence the study of the rings of real functions is more general than the study of the rings of continuous real functions.

But looking at this question from a different perspective we could argue as follows:

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[2]If there is no need to specify the topology $\mathcal{O}X$ on X, we will simply write $C(X)$, as usual.

- For each set X we have that $F(X) = C(X, \mathfrak{D}(X))$ (where $\mathfrak{D}(X)$ denotes the discrete topology on X), i.e. the real functions on X are precisely the continuous real functions on $(X, \mathfrak{D}(X))$. Hence the study of all $F(X)$ is the study of all $C(X, \mathcal{O}X)$ for discrete topological spaces, a particular case of the study of all $C(X, \mathcal{O}X)$.

We can conclude then that the study of *all* rings of the form $C(X, \mathcal{O}X)$ (see [8]) is equivalent to the study of *all* rings of the form $F(X)$. However, for a *fixed* topological space $(X, \mathcal{O}X)$, the study of $F(X)$ is clearly more general than that of $C(X, \mathcal{O}X)$.

The reason to start this introduction with the question above is that it is directly related with the issue of dealing with real functions in terms of scales that we want to address in this paper. Depending of the focus of the study, that of $C(X)$ or that of $F(X)$, different notions of scale can be found in the literature.

The origin of the notion of scale goes back to the work of P. Urysohn [17] and it is based on his approach to the construction of a *continuous* function on a topological space from a given family of open sets.

On the other hand, it was probably M.H. Stone [16] who initiated the study of an *arbitrary* (not necessarily continuous) real function by considering what he called the *spectral family* of the function.

Note that in both approaches the families involved can be considered to be either decreasing or increasing. In this paper we will deal only with decreasing families, but we point out that each statement here could be also rephrased in increasing terms.

For people mainly interested in $C(X)$ a scale is a family of open sets U_d of a given topological space X indexed by a countable and dense subset D (e.g. the dyadic numbers or the set $\mathbb{Q}$ of rationals) of a suitable part of the reals (e.g. $[0, 1]$ or the whole $\mathbb{R}$) and such that

$$(1) \text{ if } d < d', \text{ then } \overline{U_{d'}} \subseteq U_d,^{3} \quad (2) \textstyle\bigcup_{d \in D} U_d = X \text{ and } \bigcap_{d \in D} U_d = \varnothing.$$

Then the real function defined by $f(x) = \sup\{d \in D \mid x \in U_d\}$ for each $x \in X$, is continuous. Of course, arbitrary real functions appear when the topology $\mathcal{O}X$ is discrete; then any subset is open and closed and condition (1) simply reads as: if $d < d'$, then $U_{d'} \subseteq U_d$.

On the other hand, when the main focus of interest is $F(X)$, a scale must be a family of arbitrary subsets S_d of a set X indexed by D (as before) and such that

$$(1) \text{ if } d < d', \text{ then } S_{d'} \subseteq S_d, \quad (2) \textstyle\bigcup_{d \in D} S_d = X \text{ and } \bigcap_{d \in D} S_d = \varnothing.$$

Now the f given by $f(x) = \sup\{d \in D \mid x \in S_d\}$ for each $x \in X$, is a real function (not necessarily continuous). If the set X is endowed with a topology, then additional conditions on the scale can be added in order to ensure upper or lower semicontinuity or even continuity. In the same vein, we may be interested in endowing X with a partial order and characterize those functions which preserve the partial order. This can be also done in a similar way, by adding some additional conditions to the corresponding scales.

In this work we will follow the latter approach, i.e. we will focus our attention on scales of arbitrary subsets generating arbitrary real functions and then we will study particular types of scales generating continuous functions. We will see also how one can deal similarly with order-preserving functions.

The paper is organized as follows. In Section 2 we provide some motivation for the study of real functions in terms of scales, based on the construction of the real numbers

[3] Or $\overline{U_d} \subseteq U_{d'}$ in case one prefers to work with increasing scales.

in terms of Dedekind cuts. In Section 3 we make explicit the well-known relationship between real functions and scales. In Section 4 we show how one can deal with the usual algebraic operations in terms of scales, without constructing the corresponding real functions. In Section 5 we consider scales on a topological space and characterize those generating lower and upper semicontinous real functions. Finally, in Section 6 we briefly study the representability of preorders in terms of scales.

2. Motivation for the study of real functions in terms of scales

The purpose of this work is to try to show how one can deal with the ring of real functions without using the real numbers at all. This will be achieved by using the notion of a *scale*. In order to motivate its definition we start by recalling some well-known facts about the construction of the real numbers via Dedekind cuts.

2.1. Yet another look at Dedekind cuts.

As it is well-known, the purpose of Dedekind (see [7]) with the introduction of the notion of *cut* was to provide a logical foundation for the real number system. Dedekind's motivation is the fundamental observation that a real number r is completely determined by the rationals strictly smaller than r and those strictly larger than r; he originally defined a cut (A, B) as a *partition* of the rationals into two non-empty classes where every member of one of the classes is smaller than every member in the other.[4] It is important to recall his remark in [7]:

> *Every rational number produces one cut or, strictly speaking, two cuts, which, however, we shall not look as essentially different.*

In other words, there are two cuts associated to each $q \in \mathbb{Q}$, namely,

$$\big((\leftarrow, q], \mathbb{Q} \setminus (\leftarrow, q]\big) \qquad \text{and} \qquad \big((\leftarrow, q), \mathbb{Q} \setminus (\leftarrow, q)\big),$$

where $(\leftarrow, q] = \{p \in \mathbb{Q} \mid p \leq q\}$ and $(\leftarrow, q) = \{p \in \mathbb{Q} \mid p < q\}$.

In fact, (assuming excluded middle) we may take the lower part A as the representative of any given cut (A, B) since the upper part of the cut B is completely determined by A.[5] Hence one can consider the following equivalent description of the real numbers:

Dedekind's construction of the reals. A real number is a *Dedekind cut*, i.e. a subset $r \subseteq \mathbb{Q}$ such that

(D1) r is a down-set, i.e. if $p < q$ in $\mathbb{Q}$ and $q \in r$, then $p \in r$;
(D2) $\varnothing \neq r \neq \mathbb{Q}$;
(D3) r contains no greatest element, i.e. if $q \in r$, then there is some $p \in r$ such that $q < p$.

We denote the set of real numbers by $\mathbb{R}$ and define a total ordering on the set $\mathbb{R}$ as $r \leq s \equiv r \subseteq s$. We also write $r < s$ to denote the negation of $s \subseteq r$, that is $r < s \equiv r \subsetneq s$.

Any subset $S \subseteq \mathbb{R}$ which has an upper bound in $\mathbb{R}$ has a least upper bound $\bigvee S$ in $\mathbb{R}$ and $\bigvee S = \bigcup\{r : r \in S\}$.

A real number r is said to be *irrational* if $\mathbb{Q} \setminus r$ contains no least element.

Condition (D3) in the definition above just serves to eliminate subsets of the form $(\leftarrow, q]$ for a given $q \in \mathbb{Q}$ since it determines the same real number as $(\leftarrow, q)$. This allows us to embed the rational numbers into the reals by identifying the rational number $q \in \mathbb{Q}$ with

[4] We will not recall here the precise formulation, it can be found in [7].

[5] By doing this we may think intuitively of a real number as being represented by the set of all smaller rational numbers. Of course, everything could be equivalently stated in a dual way by considering Dedekind cuts as the upper part B if we think of a real number as being represented by the set of all greater rational numbers.

the subset $(\leftarrow, q) \in \mathbb{R}$. In particular the restriction of the total order in $\mathbb{R}$ coincides with the usual order in $\mathbb{Q}$. Also, for each $q \in \mathbb{Q}$ and each r irrational real number we have that

$$q \leq r \text{ in } \mathbb{R} \iff (\leftarrow, q) \subseteq r \iff q \in r \iff (\leftarrow, q) \subsetneq r \iff q < r \text{ in } \mathbb{R}.$$

Remark 1. Note that one can define the *extended real numbers* in a similar way by considering those subsets of $\mathbb{Q}$ satisfying only conditions (D1) and (D3). Under this definition we have now two additional cuts, namely $\varnothing$ and $\mathbb{Q}$ which determine the extended real numbers usually denoted as $-\infty$ and $+\infty$, respectively.[6]

Let us consider now the family of subsets $A \subseteq \mathbb{Q}$ satisfying only conditions (D1)–(D2), and call them *indefinite Dedekind cuts*.[7] In other words, we will take into consideration now both subsets $(\leftarrow, q)$ and $(\leftarrow, q]$ for each $q \in \mathbb{Q}$.

After identifying each subset $A \subseteq \mathbb{Q}$ with its characteristic function $\chi_A : \mathbb{Q} \to \mathbf{2}$ into the two-element lattice $\mathbf{2} = \{0, 1\}$ (given by $\chi_A(q) = 1$ iff $q \in A$) one has, equivalently:

Definition 2. An *indefinite Dedekind cut* is a function $\mathcal{S} : \mathbb{Q} \to \mathbf{2}$ such that
(D1) $\mathcal{S}$ is decreasing, i.e. $\mathcal{S}(q) \leq \mathcal{S}(p)$ whenever $p < q$,
(D2) $\bigvee_{q \in \mathbb{Q}} \mathcal{S}(q) = 1$ and $\bigwedge_{q \in \mathbb{Q}} \mathcal{S}(q) = 0$.

Remark 3. A Dedekind cut in the previous sense is an indefinite Dedekind cut if it is right continuous, i.e. if it satisfies the additional condition
(D3) $\mathcal{S}(q) = \bigvee_{p > q} \mathcal{S}(p)$ for each $q \in \mathbb{Q}$.

2.2. **From indefinite Dedekind cuts to scales.** We can now try to extend the previous notion by considering an arbitrary frame L instead of the two element lattice $\mathbf{2}$.

Recall that a *frame* is a complete lattice L in which $a \wedge \bigvee B = \bigvee \{a \wedge b : b \in B\}$ for all $a \in L$ and $B \subseteq L$. The universal bounds are denoted by 0 and 1. The most familiar examples of frames are
(a) the two element lattice $\mathbf{2}$ (and, more generally, any complete chain),
(b) the topology $\mathcal{O}X$ of a topological space $(X, \mathcal{O}X)$, and
(c) the complete Boolean algebras.

Being a Heyting algebra, each frame L has the implication $\to$ satisfying $a \wedge b \leq c$ iff $a \leq b \to c$. The *pseudocomplement* of an $a \in L$ is

$$a^* = a \to 0 = \bigvee \{b \in L : a \wedge b = 0\}.$$

Given $a, b \in L$, we denote by $\prec$ the relation defined by

$$a \prec b \quad \text{iff} \quad a^* \vee b = 1.$$

In particular, when $L = \mathcal{O}X$ for some topological space X, one has $U^* = \mathrm{Int}\,(X \setminus U)$ and $U \prec V$ iff $\mathrm{Cl}\, U \subseteq V$ for each $U, V \in \mathcal{O}X$. Also, in a Boolean algebra, the pseudocomplement is a complement and $a \prec b$ iff $a \leq b$.

One arrives now to the notion of an (extended) scale on a frame:[8]

Definition 4. ([16, 2, 10, 3]) Let L be a frame. An *extended scale* on L is a family $(s_q \mid q \in \mathbb{Q})$[9] of elements in L satisfying

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[6] There are actually two slightly different notions that both go by the name *extended real number*: one in which $+\infty$ and $-\infty$ are identified, and one in which they are not. We are dealing here with the latter. The former notion forms a quotient space of the latter.

[7] The name *indefinite Dedekind cut* is motivated from the notation used in [6].

[8] Note that the terminology *scale* used here differs from its use in [15] where it refers to maps to L from the unit interval of $\mathbb{Q}$ and not all of $\mathbb{Q}$. In [2] the term *descending trail* is used instead.

[9] From now on we will identify a function $s : \mathbb{Q} \to L$ with $(s_q \equiv s(q) \mid q \in \mathbb{Q})$.

(S1) $s_q \prec s_p$ whenever $p < q$;
It is a *scale* if it additionally satisfies
(S2) $\bigvee_{q\in\mathbb{Q}} s_q = 1 = \bigvee_{q\in\mathbb{Q}} s_q^*$.

Now given a topological space $(X, \mathcal{O}X)$, we can particularize the previous notion in two different ways:

For $L = \mathcal{O}X$, a *scale* on $\mathcal{O}X$ (or a *scale of open sets*) is a family $(U_q \mid q \in \mathbb{Q})$ of open sets such that
(S1) $\mathrm{Cl}\, U_q \subseteq U_p$ whenever $p < q$;
(S2) $\bigcup_{q\in\mathbb{Q}} U_q = X$ and $\bigcap_{q\in\mathbb{Q}} U_q = \varnothing$.[10]

However, in this work we will deal with scales on $L = \mathfrak{D}(X)$:

Definition 5. Let X be a set. A family $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ of subsets of X is said to be a *scale* on X if it is a scale on $\mathfrak{D}(X)$, i.e. if it satisfies
(S1) $S_q \subseteq S_p$ whenever $p < q$;
(S2) $\bigcup_{q\in\mathbb{Q}} S_q = X$ and $\bigcap_{q\in\mathbb{Q}} S_p = \varnothing$.
We shall denote by $\mathrm{Scale}(X)$ the collection of all scales over X.

Remark 6. Another extension of the notion of scale has been considered in [9] (see also [4] and [**?**]) in order to deal with functions with values in a completely distributive lattice with a a countable join-dense subset consisting of non-supercompact elements. Several parts in what follows could be stated in this more general ssetting, but we will restrict ourselves to the real-valued case.

3. Scales and real functions

In this section we will analyze in detail the relationship between scales and real functions on a given set.

We would like to emphasize again that a similar analysis could be done for scales of open subsets. Also, note that when dealing with scales, one can always use either decreasing or increasing scales.

3.1. Some binary relations in Scale(X).

We will consider three different binary relations between scales defined on a given set, which will be denoted as $\leq$, $\preceq$ and $\sim$:

Given $\mathcal{S}, \mathcal{T} \in \mathrm{Scale}(X)$, we write:

$$\begin{array}{lcll} \mathcal{S} \leq \mathcal{T} & \iff & S_q \subseteq T_q & \text{for each } q \in \mathbb{Q} \\ \mathcal{S} \preceq \mathcal{T} & \iff & S_q \subseteq T_p & \text{for each } p < q \in \mathbb{Q} \end{array}$$

Clearly enough we have that $\mathcal{S} \leq \mathcal{T}$ implies that $\mathcal{S} \preceq \mathcal{T}$. (Indeed, let $\mathcal{S} \leq \mathcal{T}$ and $p < q \in \mathbb{Q}$, then $S_q \subseteq T_q \subseteq T_p$.)

It is easy to check that both relations are reflexive and transitive and $\leq$ is additionally antisymmetric, in other words, $\leq$ is a partial order while $\preceq$ is only a preorder.

Now we can use the preorder $\preceq$ on $\mathrm{Scale}(X)$ to define an equivalence relation $\sim$ on $\mathrm{Scale}(X)$ such that

$$\mathcal{S} \sim \mathcal{T} \iff \mathcal{S} \preceq \mathcal{T} \text{ and } \mathcal{T} \preceq \mathcal{S} \iff S_q \cup T_q \subseteq S_p \cap T_p \quad \text{for each } p < q \in \mathbb{Q}.$$

This relation, determines a partial order on the quotient set $\mathrm{Scale}(X)/\sim$ (the set of all equivalence classes of $\sim$): given $[\mathcal{S}], [\mathcal{T}] \in \mathrm{Scale}(X)/\sim$,

$$[\mathcal{S}] \preceq [\mathcal{T}] \iff \mathcal{S} \preceq \mathcal{T}.$$

[10] Note that $\bigvee_{q\in\mathbb{Q}} U_q^* = \bigcup_{q\in\mathbb{Q}} \mathrm{Int}\,(X \setminus U_q) = X \setminus \left(\bigcap_{q\in\mathbb{Q}} \mathrm{Cl}\, U_q\right) = X \setminus \left(\bigcap_{q\in\mathbb{Q}} U_q\right)$.

By the construction of $\sim$, this definition is independent of the chosen representatives and the corresponding relation is indeed well-defined. It is also easy to check that this yields a partially ordered set $\big(\mathrm{Scale}(X)/\sim, \preceq\big)$.

3.2. **The real function generated by a scale.** We shallll start now by establishing the relation between scales and real functions.

Notation 7. Given $f : X \to \mathbb{R}$ and $q \in \mathbb{Q}$, we write $[f \geq q] = \{x \in X \mid q \leq f(x)\}$ and $[f > q] = \{x \in X \mid q < f(x)\}$.

Proposition 8. *Let X be a set and $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ a scale on X. Then $f_{\mathcal{S}}(x) = \bigvee\{q \in \mathbb{Q} \mid x \in S_q\}$ determines a unique function $f_{\mathcal{S}} : X \to \mathbb{R}$ such that $[f_{\mathcal{S}} > q] \subseteq S_q \subseteq [f_{\mathcal{S}} \geq q]$ for each $q \in \mathbb{Q}$.*

In view of the previous result, we can now introduce the following:

Definition 9. Let $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ be a scale in X. The function $f_{\mathcal{S}} : X \to \mathbb{R}$ defined by

$$f_{\mathcal{S}}(x) = \bigvee\{q \in \mathbb{Q} \mid x \in S_q\}$$

for each $x \in X$, is said to be *the real function generated* by $\mathcal{S}$.

We immediately have:

Proposition 10. *Let $\mathcal{S}$ and $\mathcal{T}$ be two scales on X generating real functions $f_{\mathcal{S}}$ and $f_{\mathcal{T}}$, respectively. Then $\mathcal{S} \preceq \mathcal{T}$ if and only if $f_{\mathcal{S}} \leq f_{\mathcal{T}}$; consequently, $\mathcal{S} \sim \mathcal{T}$ if and only if $f_{\mathcal{S}} = f_{\mathcal{T}}$.*

3.3. **Scales generating a given real function.** It follows immediately from the preceding proposition that different scales may generate the same real function. Our intention now is to study the set of all scales generating a given real function, or, equivalently, the equivalence class of a given scale.

We start by proving the following auxiliary result:

Lemma 11. *Let X be a set, $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ a scale on X and*

$$\mathcal{S}^{min} \equiv \big(S_q^{min} = \textstyle\bigcup_{p>q} S_p \mid q \in \mathbb{Q}\big) \quad \textit{and} \quad \mathcal{S}^{max} \equiv \big(S_q^{max} = \textstyle\bigcap_{p<q} S_p \mid q \in \mathbb{Q}\big).$$

Then:

(1) $\mathcal{S}^{min}$ and $\mathcal{S}^{max}$ are scales on X.
(2) $\mathcal{S}^{min} \leq \mathcal{S} \leq \mathcal{S}^{max}$. and $\mathcal{S}^{min} \sim \mathcal{S} \sim \mathcal{S}^{max}$.
(3) If $\mathcal{T} \sim \mathcal{S}$, then $\mathcal{S}^{min} \leq \mathcal{T} \leq \mathcal{S}^{max}$.
(4) If $\mathcal{T} \sim \mathcal{S}$, then $\mathcal{T}^{min} = \mathcal{S}^{min}$ and $\mathcal{T}^{max} = \mathcal{S}^{max}$.
(5) $\mathcal{S}^{min} = \{[f_{\mathcal{S}} > q] \mid q \in \mathbb{Q}\}$ and $\mathcal{S}^{max} = \{[f_{\mathcal{S}} \geq q] \mid q \in \mathbb{Q}\}$.

Now we can characterize the equivalence class of a given scale as an interval in the partially ordered set $(\mathrm{Scale}(X), \leq)$:

Proposition 12. *Let X be a set and $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ a scale on X. Then*

$$[\mathcal{S}] = \left\{\mathcal{T} \mid \mathcal{S}^{min} \leq \mathcal{T} \leq \mathcal{S}^{max}\right\}.$$

Finally, we can characterize the scales generating a given real function:

Proposition 13. *Let X be a set and $f : X \to \mathbb{R}$ a real function. Then*

(1) $\mathcal{S}_f^{min} = \{[f > q] \mid q \in \mathbb{Q}\}$ and $\mathcal{S}_f^{max} = \{[f \geq q] \mid q \in \mathbb{Q}\}$ are scales generating f.
(2) If $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ is a scale on X that generates f, then $\mathcal{S}^{min} = \mathcal{S}_f^{min}$ and $\mathcal{S}^{max} = \mathcal{S}_f^{max}$.

(3) $\mathcal{S} = (S_q \mid q \in \mathbb{Q})$ *is a scale on* X *that generates* f *if and only if* $\mathcal{S}_f^{min} \leq \mathcal{S} \leq \mathcal{S}_f^{max}$.
(4) *The collection of all scales on* X *that generate* f *is precisely the class* $\left[\mathcal{S}_f^{min}\right] = \left[\mathcal{S}_f^{max}\right]$.

3.4. **Correspondence between real functions and equivalence classes of scales.** We can now establish the desired correspondence:

Proposition 14. *Let* X *be a set. There exists an order isomorphism between the partially ordered sets* $(F(X), \leq)$ *of real functions on* X *and* $(\mathrm{Scale}(X)/\sim, \preceq)$.

In fact, this correspondence is more than an order isomorphism. As we will see in what follows it can be used to express the algebraic operations between real functions purely in terms of scales. Furthermore, when the space is enriched with some additional structure (e.g. a topology or a preorder) the real functions preserving the structure ((semi)continuous functions or increasing functions, respectively) can be characterized by mean of scales.

4. Algebraic operations on $\mathrm{Scale}(X)$

In this section we will try to show how one can deal with the usual algebraic operations in terms of scales, without constructing the corresponding real functions.

4.1. **Constant scale and characteristic scale of a set.**

• Let $r \in \mathbb{R}$ and $\mathcal{S}^{\mathbf{r}} = (S_q^{\mathbf{r}} \mid q \in \mathbb{Q})$ be defined by

$$S_q^{\mathbf{r}} = X \text{ if } q < r \quad \text{and} \quad S_q^{\mathbf{r}} = \varnothing \text{ if } r \leq q.$$

Clearly, $\mathcal{S}^{\mathbf{r}}$ is a scale on X and it will be called the *constant scale* with value r.
In case $r \in \mathbb{Q}$, we have that $[\mathcal{S}^{\mathbf{r}}] = \{\mathcal{S}^{\mathbf{r},min}, \mathcal{S}^{\mathbf{r},max}\}$, where $\mathcal{S}^{\mathbf{r},min} = \mathcal{S}^{\mathbf{r}}$ and $S_q^{\mathbf{r},max} = X$ if $q \leq r$ and $S_t^{\mathbf{r},max} = \varnothing$ otherwise.
On the other hand, if r is irrational, then $\mathcal{S}^{\mathbf{r},min} = \mathcal{S}^{\mathbf{r},max} = \mathcal{S}^{\mathbf{r}}$ and so $[\mathcal{S}^{\mathbf{r}}] = \{\mathcal{S}^{\mathbf{r}}\}$.

• Let $A \subseteq X$ and $\mathcal{S}^A = (S_q^A \mid q \in \mathbb{Q}) \subseteq X$ be defined by

$$S_q^A = X \text{ if } q < 0, \quad S_q^A = A \text{ if } 0 \leq q < 1 \quad \text{and} \quad S_q^A = \varnothing \text{ if } q \geq 1.$$

Once again, $\mathcal{S}^A$ is a scale on X and it will be called the *characteristic scale* of A.
In this case $\left[\mathcal{S}^A\right]$ is order isomorphic to the 4 element Boolean algebra and $\mathcal{S}^{A,min} = \mathcal{S}^A$ while $S_q^{A,max} = X$ if $q \leq 0$, $S_q^{A,max} = A$ if $0 < q \leq 1$ and $S_q^{A,max} = \varnothing$ if $q \geq 1$.

4.2. **Opposite scale.** Given a scale $\mathcal{S}$ on X, define

$$-\mathcal{S} = (X \setminus S_{-q} \mid q \in \mathbb{Q}).$$

(1) $-\mathcal{S}$ is a scale on X;
(2) If $\mathcal{S} \preceq \mathcal{T}$ then $-\mathcal{T} \preceq -\mathcal{S}$ and hence, if $\mathcal{S} \sim \mathcal{T}$ then $-\mathcal{T} \sim -\mathcal{S}$;
(3) $[-\mathcal{S}] = \{-\mathcal{T} \mid \mathcal{T} \in [\mathcal{S}]\}$;
(4) $(-\mathcal{S})^{min} = -(\mathcal{S}^{max})$ and $(-\mathcal{S})^{max} = -(\mathcal{S}^{min})$;
(5) $-\mathcal{S}^{\mathbf{r}} \sim \mathcal{S}^{-\mathbf{r}}$ for each $r \in \mathbb{R}$.

4.3. **Finite joins and meets.** Given two scales $\mathcal{S}$ and $\mathcal{T}$ on X, we write

$$\mathcal{S} \vee \mathcal{T} = (S_q \cup T_q \mid q \in \mathbb{Q}) \quad \text{and} \quad \mathcal{S} \wedge \mathcal{T} = (S_q \cap T_q \mid q \in \mathbb{Q}).$$

(1) $\mathcal{S} \vee \mathcal{T} = \mathcal{T} \vee \mathcal{S}$ is a scale on X;
(2) If $\mathcal{S} \preceq \mathcal{S}'$ and $\mathcal{T} \preceq \mathcal{T}'$ then $\mathcal{S} \vee \mathcal{T} \preceq \mathcal{S}' \vee \mathcal{T}'$ and hence, if $\mathcal{S} \sim \mathcal{S}'$ and $\mathcal{T} \sim \mathcal{T}'$ then $\mathcal{S} \vee \mathcal{T} \sim \mathcal{S}' \vee \mathcal{T}'$;
(3) $(\mathcal{S} \vee \mathcal{T})^{min} = \mathcal{S}^{min} \vee \mathcal{T}^{min}$ and $(\mathcal{S} \vee \mathcal{T})^{max} = \mathcal{S}^{max} \vee \mathcal{T}^{max}$;
(4) $\mathcal{S} \wedge \mathcal{T} = -\big((-\mathcal{S}) \vee (-\mathcal{T})\big) = \mathcal{T} \wedge \mathcal{S}$ is a scale on X;
(5) If $\mathcal{S} \preceq \mathcal{S}'$ and $\mathcal{T} \preceq \mathcal{T}'$ then $\mathcal{S} \wedge \mathcal{T} \preceq \mathcal{S}' \wedge \mathcal{T}'$ and hence, if $\mathcal{S} \sim \mathcal{S}'$ and $\mathcal{T} \sim \mathcal{T}'$ then $\mathcal{S} \wedge \mathcal{T} \sim \mathcal{S}' \wedge \mathcal{T}'$;
(6) $(\mathcal{S} \wedge \mathcal{T})^{min} = \mathcal{S}^{min} \wedge \mathcal{T}^{min}$ and $(\mathcal{S} \wedge \mathcal{T})^{max} = \mathcal{S}^{max} \wedge \mathcal{T}^{max}$;
(7) $\mathcal{S} \prec \mathcal{T}$ if and only if $\mathcal{S} \vee \mathcal{T} \sim \mathcal{T}$ if and only if $\mathcal{S} \wedge \mathcal{T} \sim \mathcal{S}$.

4.4. **Arbitrary joins and meets.** As expected, given an arbitrary family of scales on X we cannot always ensure the existence of its join and/or meet in $\mathrm{Scale}(X)$. More precisely, given a family of scales $\{\mathcal{S}^i\}_{i\in I}$ on X, we define

$$\textstyle\bigvee_{i\in I}\mathcal{S}^i = \big(\bigcup_{i\in I} S^i_q \mid q \in \mathbb{Q}\big) \quad \text{and} \quad \bigwedge_{i\in I}\mathcal{S}^i = \big(\bigcap_{i\in I} S^i_q \mid q \in \mathbb{Q}\big).$$

If $\bigcap_{q\in\mathbb{Q}}\bigcup_{i\in I} S^i_q = \varnothing$, then we have that:

(1) $\bigvee_{i\in I}\mathcal{S}^i$ is a scale on X;
(2) $f_{\bigvee_{i\in I}\mathcal{S}^i} = \bigvee_{i\in I} f_{\mathcal{S}^i}$;
(3) $\big(\bigvee_{i\in I}\mathcal{S}^i\big)^{min} = \bigcup_{i\in I}\big(\mathcal{S}^i\big)^{min}$.

Dually, if $\bigcup_{q\in\mathbb{Q}}\bigcap_{i\in I} S^i_q = X$ we have that:

(4) $\bigwedge_{i\in I}\mathcal{S}^i = -\big(\bigvee_{i\in I} - \mathcal{S}^i\big)$ is a scale on X;
(5) $f_{\bigwedge_{i\in I}\mathcal{S}^i} = \bigwedge_{i\in I} f_{\mathcal{S}^i}$;
(6) $\big(\bigwedge_{i\in I}\mathcal{S}^i\big)^{max} = \bigcap_{i\in I}\big(\mathcal{S}^i\big)^{max}$.

In particular, if there is a scale $\mathcal{T}$ on X such that $\mathcal{S}^i \preceq \mathcal{T}$ for each $i \in I$, then $\bigcap_{q\in\mathbb{Q}}\bigcup_{i\in I} S^i_q \subseteq \bigcap_{q\in\mathbb{Q}}\bigcup_{i\in I} T_q = \varnothing$ and so $\bigvee_{i\in I}\mathcal{S}^i$ is a scale on X and $\bigvee_{i\in I}\mathcal{S}^i \preceq \mathcal{T}$. Similarly, if $\mathcal{T} \preceq \mathcal{S}^i$ for each $i \in I$, then $\bigwedge_{i\in I}\mathcal{S}^i$ is a scale on X and $\mathcal{T} \preceq \bigwedge_{i\in I}\mathcal{S}^i$.

4.5. **Product with a scalar.** Given $r \in \mathbb{R}$ such that $r > 0$ and a scale $\mathcal{S}$ on X, we define

$$r \cdot \mathcal{S} = \Big(\textstyle\bigcup_{p<r} S_{\frac{q}{p}} \mid q \in \mathbb{Q}\Big).$$

We have that:

(1) $r \cdot \mathcal{S}$ is a scale on X;
(2) If $\mathcal{S} \preceq \mathcal{T}$ then $r \cdot \mathcal{S} \preceq r \cdot \mathcal{T}$ and hence, if $\mathcal{S} \sim \mathcal{T}$ then $r \cdot \mathcal{S} \sim r \cdot \mathcal{T}$;
(3) $[r \cdot \mathcal{S}] = \{r \cdot \mathcal{T} \mid \mathcal{T} \in [\mathcal{S}]\}$;
(4) $(r \cdot \mathcal{S})^{min} = r \cdot (\mathcal{S})^{min}$ and $(r \cdot \mathcal{S})^{max} \sim r \cdot (\mathcal{S})^{max}$;
(5) $1 \cdot \mathcal{S} \sim \mathcal{S}$;
(6) $r \cdot \mathcal{S}^{\mathbf{s}} = \mathcal{S}^{\mathbf{rs}}$ for each $s \in \mathbb{R}$;
(7) $-\big(r \cdot \mathcal{S}\big) \sim r \cdot \big(-\mathcal{S}\big)$;
(8) $r \cdot \big(\mathcal{S} \vee \mathcal{T}\big) \sim \big(r \cdot \mathcal{S}\big) \vee \big(r \cdot \mathcal{T}\big)$ and $r \cdot \big(\mathcal{S} \wedge \mathcal{T}\big) \sim \big(r \cdot \mathcal{S}\big) \wedge \big(r \cdot \mathcal{T}\big)$.

Further, we define

$$r \cdot \mathcal{S} = -\big((-r) \cdot \mathcal{S}\big) \ \text{ if } r < 0 \quad \text{and} \quad 0 \cdot \mathcal{T} = \mathcal{S}^{\mathbf{0}}.$$

4.6. **Sum and difference.** Given two scales $\mathcal{S}$ and $\mathcal{T}$ on X, we define

$$\mathcal{S}+\mathcal{T}=\Big(\bigcup_{p\in\mathbb{Q}} S_p\cap T_{q-p}\mid q\in\mathbb{Q}\Big)\quad\text{and}\quad \mathcal{S}-\mathcal{T}=\Big(\bigcup_{p\in\mathbb{Q}} S_p\setminus T_{p-q}\mid q\in\mathbb{Q}\Big).$$

We have that:

(1) $\mathcal{S}+\mathcal{T}=\mathcal{T}+\mathcal{S}$ is a scale on X;
(2) If $\mathcal{S}\preceq\mathcal{S}'$ and $\mathcal{T}\preceq\mathcal{T}'$ then $\mathcal{S}+\mathcal{T}\preceq\mathcal{S}'+\mathcal{T}'$ and hence, if $\mathcal{S}\sim\mathcal{S}'$ and $\mathcal{T}\sim\mathcal{T}'$ then $\mathcal{S}+\mathcal{T}\sim\mathcal{S}'+\mathcal{T}'$;
(3) $\mathcal{S}^{\mathbf{0}}+\mathcal{S}\sim\mathcal{S}$, i.e. the constant scale with value 0 is the neutral element w.r.t the sum;
(4) $\mathcal{S}^{\mathbf{r}}+\mathcal{S}^{\mathbf{s}}\sim\mathcal{S}^{\mathbf{r+s}}$ for each $r,s\in\mathbb{R}$;
(5) $-\big(\mathcal{S}+\mathcal{T}\big)\sim\big(-\mathcal{S}\big)+\big(-\mathcal{T}\big)$;
(6) $r\cdot\big(\mathcal{S}+\mathcal{T}\big)\sim\big(r\cdot\mathcal{S}\big)+\big(r\cdot\mathcal{T}\big)$ for each $r\in\mathbb{R}$;
(7) $\mathcal{S}-\mathcal{T}\sim\mathcal{T}+\big(-\mathcal{S}\big)$.

4.7. **Product.** Given two scales $\mathcal{S}$ and $\mathcal{T}$ on X such that $\mathcal{S}^{\mathbf{0}}\preceq\mathcal{S},\mathcal{T}$, we define

$$\mathcal{S}\cdot\mathcal{T}=\Big(\bigcup_{0<p} S_p\cap T_{\frac{q}{p}}\mid q\in\mathbb{Q}\Big).$$

Then $\mathcal{S}\cdot\mathcal{T}$ is a scale on X.

More generally, given a scale $\mathcal{S}$ on X let

$$\mathcal{S}^{+}=\mathcal{S}\vee\mathcal{S}^{\mathbf{0}}\quad\text{and}\quad\mathcal{S}^{-}=(-\mathcal{S})\vee\mathcal{S}^{\mathbf{0}}$$

(Notice that $\mathcal{S}\sim\mathcal{S}^{+}-\mathcal{S}^{-}$.) Given two arbitrary scales $\mathcal{S}$ and $\mathcal{T}$ on X, we define

$$\mathcal{S}\cdot\mathcal{T}=\big((\mathcal{S}^{+}\cdot\mathcal{T}^{+})-(\mathcal{S}^{+}\cdot\mathcal{T}^{-})\big)-\big((\mathcal{S}^{-}\cdot\mathcal{T}^{+})+(\mathcal{S}^{-}\cdot\mathcal{T}^{-})\big).$$

We have that:

(1) $\mathcal{S}\cdot\mathcal{T}\sim\mathcal{T}\cdot\mathcal{S}$ is a scale on X;
(2) If $\mathcal{S}\sim\mathcal{S}'$ and $\mathcal{T}\sim\mathcal{T}'$ then $\mathcal{S}\cdot\mathcal{T}\sim\mathcal{S}'\cdot\mathcal{T}'$;
(3) $\mathcal{S}^{\mathbf{1}}\cdot\mathcal{S}\sim\mathcal{S}$, i.e. the constant scale with value 1 is the neutral element w.r.t the product;
(4) $r\cdot\mathcal{S}=\mathcal{S}^{\mathbf{r}}\cdot\mathcal{S}$ for each $r\in\mathbb{R}$;
(5) $-\big(\mathcal{S}\cdot\mathcal{T}\big)\sim\big(-\mathcal{S}\big)\cdot\mathcal{T}\sim\mathcal{S}\cdot\big(-\mathcal{T}\big)$;
(6) $\mathcal{S}\cdot\big(\mathcal{T}+\mathcal{T}'\big)=\big(\mathcal{S}\cdot\mathcal{T}\big)+\big(\mathcal{S}\cdot\mathcal{T}'\big)$.

5. Semicontinuous real functions and scales

In what follows the space X will be endowed with a topology $\mathcal{O}X$ and we will try to see how to deal with semicontinuous real functions in terms of scales.

Let $(X,\mathcal{O}X)$ be a topological space. A function $f:X\to\mathbb{R}$ is *lower* (resp. *upper*) *semicontinuous* if and only if $[f>q]\in\mathcal{O}X$ (resp. $[f<q]\in\mathcal{O}X$) for each $q\in\mathbb{Q}$. The collections of all lower (resp. upper) semicontinuous real functions on X will be denoted by $\mathrm{LSC}(X)$ (resp. $\mathrm{USC}(X)$). Elements of $\mathrm{C}(X)=\mathrm{LSC}(X)\cap\mathrm{USC}(X)$ are called *continuous*.

As mentioned in the Introduction, in this work we focus our attention on scales of arbitrary subsets generating arbitrary real functions and then we study particular types of scales generating continuous (and semicontinuous) functions. We introduce now the following terminology:

Definition 15. Let $(X,\mathcal{O}X)$ be a topological space. A scale $\mathcal{S}$ on X is said to be:

(1) *lower semicontinuous* if $S_q\subseteq\operatorname{Int}S_p$ whenever $p<q\in\mathbb{Q}$.
(2) *upper semicontinuous* if $\operatorname{Cl}S_q\subseteq S_p$ whenever $p<q\in\mathbb{Q}$.
(3) *continuous* if $\operatorname{Cl}S_q\subseteq\operatorname{Int}S_p$ whenever $p<q\in\mathbb{Q}$.

Remarks 16. (1) If $S_q\in\mathcal{O}X$ for each $q\in\mathbb{Q}$, i.e if $\mathcal{S}$ is a scale of open subsets of X, then it is automatically lower semicontinuous and it is continuous if $\operatorname{Cl}S_q\subseteq S_p$ whenever $p<q\in\mathbb{Q}$.

Consequently a continuous scale of open subsets of X is precisely a scale on $\mathcal{O}X$ in the sense of Definition 4.

(2) Any scale on X is continuous when $\mathcal{O}X$ is the discrete topology on X. On the other hand, the only continuous scales when $\mathcal{O}X$ is the indiscrete topology on X are the constant ones.

Now we have the following result which motivates the notation introduced.

Proposition 17. *Let $\mathcal{S}$ be a scale on $(X, \mathcal{O}X)$ and $f_{\mathcal{S}}$ the real function generated by $\mathcal{S}$:*
(1) *$\mathcal{S}$ is lower semicontinuous if and only if $f_{\mathcal{S}} \in \mathrm{LSC}(X)$;*
(2) *$\mathcal{S}$ is upper semicontinuous if and only if $f_{\mathcal{S}} \in \mathrm{USC}(X)$;*
(3) *$\mathcal{S}$ is continuous if and only if $f_{\mathcal{S}} \in \mathrm{C}(X)$;*

Since our intention is to work purely in terms of scales, we need still some further characterizations:

Proposition 18. *For a scale $\mathcal{S}$ on $(X, \mathcal{O}X)$ the following are equivalent:*
(1) *$\mathcal{S}$ is lower semicontinuous;*
(2) *There exists a scale of open subsets $\mathcal{T}$ such that $\mathcal{T} \sim \mathcal{S}$;*
(3) *$\mathcal{S}^{min}$ is a scale of open subsets, i.e. $\bigcup_{q>p} S_q$ is open for each $p \in \mathbb{Q}$.*

Clearly enough, $\mathcal{S}$ is upper semicontinuous if and only if $-\mathcal{S}$ is lower semicontinuous. Hence we have:

Corollary 19. *For a scale $\mathcal{S}$ on $(X, \mathcal{O}X)$ the following are equivalent:*
(1) *$\mathcal{S}$ is upper semicontinuous;*
(2) *There exists a scale of closed subsets $\mathcal{T}$ such that $\mathcal{T} \sim \mathcal{S}$;*
(3) *$\mathcal{S}^{max}$ is a scale of closed subsets, i.e. $\bigcap_{q<p} S_q$ is closed for each $p \in \mathbb{Q}$.*

Corollary 20. *For a scale on $\mathcal{S}$ on $(X, \mathcal{O}X)$ the following are equivalent:*
(1) *$\mathcal{S}$ is continuous;*
(2) *There exist a scale $\mathcal{T}$ of open subsets and a scale $\mathcal{T}'$ of closed subsets satisfying $\mathcal{T} \sim \mathcal{T}' \sim \mathcal{S}$;*
(3) *$\mathcal{S}^{min}$ is a scale of open subsets and $\mathcal{S}^{max}$ is a scale of closed subsets.*

Now we use the descriptions of the algebraic operations obtained in the previous section together with these characterization to obtain the following:

Proposition 21. *Let $\mathcal{S}, \mathcal{T}$ and $\mathcal{S}^i$ $(i \in I)$ be scales on $(X, \mathcal{O}X)$ and $r \in \mathbb{R}^+$. Then:*
(1) *$\mathcal{S}^{\mathbf{r}}$ is continuous;*
(2) *If $\mathcal{S}$ is lower (resp. upper) semicontinuous, then $-\mathcal{S}$ is upper (resp. lower) semicontinuous;*
(3) *If $\mathcal{S}$ and $\mathcal{T}$ are lower (resp. upper) semicontinuous, then so are $\mathcal{S} \vee \mathcal{T}$ and $\mathcal{S} \wedge \mathcal{T}$;*
(4) *If all $\mathcal{S}^i$ are lower semicontinuous and $\bigvee_{i \in I} \mathcal{S}^i$ is a scale, then it is lower semicontinuous;*
(5) *If all $\mathcal{S}^i$ are upper semicontinuous and $\bigwedge_{i \in I} \mathcal{S}^i$ is a scale, then it is lower semicontinuous;*
(6) *If $\mathcal{S}$ is lower (resp. upper) semicontinuous, then so is $r \cdot \mathcal{S}$;*
(7) *If $\mathcal{S}$ and $\mathcal{T}$ are lower (resp. upper) semicontinuous, then so is $\mathcal{S} + \mathcal{T}$;*
(8) *If $\mathcal{S}$ and $\mathcal{T}$ are lower (resp. upper) semicontinuous and $\mathcal{S}^{\mathbf{0}} \preceq \mathcal{S}, \mathcal{T}$, then so is $\mathcal{S} \cdot \mathcal{T}$.*

Of course, the previous results are well-known properties when we think in terms of real functions. But we want to stress here that the interest of this approach (in terms of scales) is that it can be easily generalized to the pointfree setting, as it has been recently done in [12].

6. Representability of preorders through scales

Finally, in this section the topological space $(X, \mathcal{O}X)$ will be additionally endowed with a preorder $\mathcal{R}$ (a reflexive and transitive relation on X). The pair $(X, \mathcal{R})$ will be referred to as a preordered set and the triple $(X, \mathcal{O}X, \mathcal{R})$ consisting of a topological space $(X, \mathcal{O}X)$ endowed with a preorder $\mathcal{R}$ will be referred to as a topological preordered space. The *asymmetric part* $\mathcal{P}$ of $\mathcal{R}$ is defined for each $x, y \in X$ as $x\mathcal{P}y$ if and only if $x\mathcal{R}y$ and not $y\mathcal{R}x$.

In this section we will try to see how to deal with real functions defined on a topological preordered space $(X, \mathcal{O}X, \mathcal{R})$ which preserve the preorder $\mathcal{R}$ as well as its asymmetric part $\mathcal{P}$, in terms of scales.

A subset A of $(X, \mathcal{R})$ is said to be *increasing* if $x\mathcal{R}y$ together with $x \in A$ imply $y \in A$. For a subset A of X we write $i(A) = \{y \in X \mid \exists x \in A \text{ such that } x\mathcal{R}y\}$ to denote the smallest increasing subset of X containing A.

A function $f : (X, \mathcal{R}) \to (\mathbb{R}, \leq)$ is *increasing* if $f(x) \leq f(y)$ whenever $x\mathcal{R}y$, *stricly increasing* if $f(x) < f(y)$ whenever $x\mathcal{P}y$ aand it is a *preorder embedding* in case $f(x) \leq f(y)$ if and only if $x\mathcal{R}y$. A preorder $\mathcal{R}$ on X is said to be *representable* if there exists a preorder embedding (also called "*utility function*") $f : (X, \mathcal{R}) \to (\mathbb{R}, \leq)$. We introduce now the following terminology:

Definition 22. Let $(X, \mathcal{R})$ be a preordered set. A scale $\mathcal{S}$ on X is said to be:

(1) *increasing* if $i(S_q) \subseteq S_p$ whenever $p < q \in \mathbb{Q}$;
(2) *strictly increasing* if for each $x, y \in X$ with $x\mathcal{P}y$ there exist $p < q \in \mathbb{Q}$ such that $x \in S_p$ and $y \notin S_q$;
(3) *preorder embedding* in case it is both increasing and strictly increasing.

Remarks 23. (1) If S_q is increasing for each $q \in \mathbb{Q}$, i.e if $\mathcal{S}$ is a scale of increasing subsets of X, then $\mathcal{S}$ is automatically a increasing scale.

(2) The notion of continuous preorder embedding scale is closely related with that of linear separable system in a preordered topological space ([13, 14, 5]), i.e. a family $\mathcal{F}$ of open decreasing subsets of X which is linearly ordered by set inclusion and such that there exist sets $E_1, E_2 \in \mathcal{F}$ such that $E_1 \subseteq E_2$ and for all sets $E_1, E_2 \in \mathcal{F}$ such that $\operatorname{Cl} E_1 \subseteq E_2$ there exists some set $E_3 \in \mathcal{F}$ such that $\operatorname{Cl} E_1 \subseteq E_3 \subseteq \operatorname{Cl} E_3 \subseteq E_2$.

The following result that justifies the notation introduced (cf. [1, Theorem 2.2]).

Proposition 24. *Let $\mathcal{S}$ be a scale on $(X, \mathcal{R})$ and $f_{\mathcal{S}}$ the real function generated by $\mathcal{S}$. Then:*

(1) *$\mathcal{S}$ is increasing if and only if $f_{\mathcal{S}}$ is increasing;*
(2) *$\mathcal{S}$ is strictly increasing if and only if $f_{\mathcal{S}}$ is strictly increasing;*
(3) *$\mathcal{S}$ is a preorder embedding if and only if $f_{\mathcal{S}}$ is a utility function;*

Finally we provide a sample result which shows how the concept of a scale furnishes interesting results on the existence of (continuous) utility representations:

Theorem 25. [1, Theorem 2.5] *Let $(X, \mathcal{O}X, \mathcal{R})$ be a preordered topological space. The following conditions are equivalent:*

(1) *There exists a (continuous) preorder embedding scale.*
(2) *There exists a (continuous) utility function $u : (X, \mathcal{O}X, \mathcal{R}) \to (\mathbb{R}, \leq)$.*

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