Notes on point-free topology

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Abstract Point-free topology is the study of the category of locales and localic maps and its dual category of frames and frame homomorphisms. These notes cover the topics presented by the first author in his course on *Frames and Locales* at the *Summer School in Algebra and Topology*. We give an overview of the basic ideas and motivation for point-free topology, explaining the similarities and dissimilarities with the classical setting and stressing some of the new features.

Introduction

Topology can be cultivated as the natural geometry of places ("spots") and their interrelations. One does not have to think of them as sets of points: they can be thought as entities in their own right (similarly as lines in classical geometry are not sets of incident points). Such (point-free) approach to general geometry appeared in topology already in the late thirties and forties, started to be systematically cultivated in the last decades of previous century, and flourishes since. It has turned out that by forgetting about points one does not lose really important information (unless, of course, when needing a topology as technical means for solving a problem in a

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concrete — typically otherwise structured — set, which is another matter). On the contrary, one gains important new insights into the general structure of space and obtains results that are in some respects better than the classical counterparts, or such that have no classical counterparts at all. Also, the point-free approach opens the door to applications in theoretical computer science.

In this text we would like to introduce the reader to point-free thinking, to illustrate the reasoning (for this, we present some proofs hoping to persuade the reader that the techniques are in fact fairly friendly) and present some results to illustrate the merits of the approach. In Prologue we expound point-free topology as a natural synthetic general geometry and briefly outline the history. Then we discuss the necessary order-theoretic background, the relation to classical spaces, and some categorical aspects. Next we introduce the reader to some concrete facts about point-free spaces (locales), and finally we present a few facts showing their advantages and merits.

1 Prologue

1.1 General (set-theoretic) topology is a generalized geometry. Note, however, that it fundamentally differs from the classical geometry: it lacks a synthetic variant preceding the current analytic one. Let us explain.

In the classical (synthetic Euclidean) geometry we work with entities like points, lines, or planes, entities in their own right. One studies their interrelations, in particular the incidence, which should not be confused with the set-theoretic $\in$. A point can be incident (or non-incident) with a line. A line p, however, is **not** identified with the set of all the points incident with p, and similarly in stereometry, the relation between lines and planes is not a set-theoretic inclusion.

Only much later, in the *analytic* version, one starts with a beforehand given set of elements (say, pairs of numbers); points, lines, etc., are defined as specific types of subsets (the fact that points are represented as one-element subsets is more or less accidental and not important for what we want to emphasize).

General topology, like many modern structures, comes right away in the analytic form: one starts with a set, and the structure is given by assigning specific roles to some of the subsets.

1.2 The following is not an account of what historically happened. Let us just pretend designing a *synthetic generalized geometry* from scratch. We will do it modelling the intuition of the behavior of "pieces of space" (we will call them *spots*) as we think about them in the "space around us".

1.2.1 Order First, the system L of such spots is naturally ordered:

a spot can be an extension of another one

(we do not have in mind a set-theoretic inclusion — a spot is not a set of elements — just think of a spot being larger than another one).

1.2.2 Glueing spots together Next, given a system A of spots we can think of conglutinating (merging, pasting) them together and obtaining a new one. It will be the smallest spot larger than all the $a \in A$ and hence, formally, this amounts to assuming that the ordered L is a complete lattice, and that the systems A combine to their suprema (joins) $\bigvee A$.

1.2.3 Meeting conglutinated spots It is natural to assume that

> *a spot b meets the result of pasting the $a \in A$ together, the conglutination $\bigvee A$, only if it meets some of the constituents $a \in A$.*

Formally,

$$(\textstyle\bigvee A) \wedge b \neq 0 \quad \text{only if} \quad a \wedge b \neq 0 \text{ for some } a \in A. \tag{meet}$$

It is easy to see that this means precisely that our complete lattice admits pseudo-complements. Thus, our (we hope admittedly natural) assumptions lead us to the conclusion that

> *a general synthetic geometry can be viewed as a complete pseudocomple-mented lattice.*

1.2.4 One more assumption Finally, let us agree that at least some naturally defined sublattices of L are synthetic geometries as well. Confining ourselves to the up-sets $\uparrow u = \{x \mid u \leq x\} \subseteq L$ we obtain the condition (meet) strengthened to

$$\forall u \in L \ ((\textstyle\bigvee A) \wedge b \not\leq u \quad \text{only if} \quad a \wedge b \not\leq u \text{ for some } a \in A) \tag{MEET}$$

and this is easily seen to be equivalent to the *frame distribution law*

$$\forall A \subseteq L, \ \forall b \in L, \ (\textstyle\bigvee A) \wedge b = \bigvee\{a \wedge b \mid a \in A\}. \tag{frm}$$

Complete lattices satisfying (frm) are called *frames*; taking into account that this is equivalent to the Heyting structure (see 2.6 below) and that it is a more expedient condition than the mere existence of pseudocomplements, we can view frames as representatives of fairly general synthetic geometries. And there will be a strong corroboration of this view in the next subsections.

1.3 A more realistic account of the events The development of topology did not follow the reckoning outlined in the subsection 1.2. General topology has been created as an *analytic* theory, *preceding a synthetic one*; but the synthetic successor turns out to be precisely what we have obtained above. Moreover it leads to a suitable definition of mappings between generalized geometries, which we have in 1.2 not even started to contemplate.

Modern topology originated in the pioneering Hausdorff's article [27] published in 1914. The intuition behind the introduced general concept of a space is based on the natural distinction between a set surrounding a point as opposed to a set just including it: think of a ship in a lake surrounded by water from all the sides as opposed to a landed one touching a pier. The structure of a space, carried by a

set, is constituted by determining the *neighborhoods* $U \ni x$ as special subsets of X containing x, satisfying very natural assumptions.

Soon (already in the twenties) an equivalent alternative approach emerged based on the notion of an *open set* (in the original concept setting: a set that is a *neighborhood of all its points*; and on the other hand, if we start with the notion of an open set we can define a neighborhood of x as a $U \ni x$ such that there is an open V with $x \in V \subseteq U$). Comparing it with the neighborhood intuition it might look somewhat less transparent to start with (trading an obvious intuition for technical advantages which are indisputable), but nowadays we know better: in fact it is the intuition of a synthetic version of topology as outlined above: an open set, a set without boundary, is a good model of a spot, and the system of open sets constitutes a complete sublattice $\Omega(X)$ of the powerset of X closed under arbitrary unions and finite meets, and hence satisfying the frame distribution law. Thus we have here an example (in fact a typical one) of a frame, a general geometry presented in subsection 1.2.

1.3.1 Although one did not necessarily have in mind developing synthetic topology, the ideas of harnessing lattice theory in topology (via the lattices of open resp. closed sets) appeared already in the late thirties and in the forties. The Stone duality ([59]) replacing (very special) topological spaces and continuous maps by Boolean algebras and homomorphisms is deservedly a most cited example. But one should not forget the outstanding Wallman's article [61] where the lattice technique allowed for an ingenious compactification (and not only that: the author is consequent in the lattice technique even to the point of using a specific point-free separation axiom).

In the fifties the attempts to develop a variant of topology without points became more and more frequent. It is not our intention to present here a detailed history. The reader can find a short account in the introduction to our monograph [50], but we can particularly recommend the excellent Johnstone's "The point of pointless topology" [34] and "Elements of the history of locale theory" [38]. Here let us just state that the basic concepts started to settle in late fifties [14, 18, 48] and continued in works of Banaschewski, who had been regarding to topology from a lattice-theoretic point of view since 1953 ([3]), Dowker, Isbell, Johnstone, Joyal and Tierney, that the first stage of the theory culminated in the monograph [33], and that the theory flourishes.

1.4 Frame homomorphisms We have spoken of spaces, either classical or point-free, as objects. The most important concept of classical topology, however, is continuity. What is the counterpart of continuous maps in the point-free context? The characteristic property of continuous maps is that they preserve openness by preimage. Thus, with a continuous map $f \colon X \to Y$ there is associated a mapping $\Omega(f) \colon \Omega(Y) \to \Omega(X)$ sending each U to $\Omega(f)(U) = f^{-1}[U]$. Since preimage preserves unions and intersections and since *all* unions of open sets are open and *finite* intersections of open sets are open as well, we see that $\Omega(f)$ preserves arbitrary joins and finite meets. This has been adopted for the definition of a *frame homomorphism*: it is a mapping $h \colon L \to M$ between frames preserving all joins and finite meets.[1]

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[1] In fact this property is characteristic for the representation of continuous maps: if the spaces are *sober*, which is a very general condition (see 3.4 below), each frame homomorphism $h \colon \Omega(Y) \to \Omega(X)$ is an $\Omega(f)$ for some continuous f [55].

1.4.1 Note The general geometry (leading, ultimately, to frames) was based on the idea of spots that can merge and the relation of intersecting. Thus, while the former really has to do with joins, the latter involves only the question whether $x \wedge y = 0$ or $x \wedge y \neq 0$. Thus we can naturally ask whether the appropriate maps between geometries should not be those that

(1) preserve all joins, and
(2) satisfy the implication $x \wedge y = 0 \implies h(x) \wedge h(y) = 0$.

It turns out that in a large (and important) class of frames such maps are frame homomorphisms anyway (see [13]). But in full generality these conditions are weaker.

2 Background: posets

2.1 When dealing with posets we will use the standard notation. If necessary we use different symbols for different orders, but if there is no danger of confusion we write simply $\leq$ (like in saying that "$f : (X, \leq) \to (Y, \leq)$ is monotone if $x \leq y$ implies that $f(x) \leq f(y)$" even if there is in fact on Y a relation different from that on X — similarly like we do not hesitate to use the same arrow symbol "$\to$" when indicating morphisms in two distinct categories).

We write

$$\downarrow A \ \text{ for } \ \{x \mid \exists\, a \in A, x \leq a\} \quad \text{and} \quad \uparrow A \ \text{ for } \ \{x \mid \exists\, a \in A, x \geq a\}$$

and abbreviate $\downarrow\{a\} = \downarrow a$, $\uparrow\{a\} = \uparrow a$. An element a is an *upper* (resp. *lower*) *bound* of $A \subseteq (X, \leq)$ if $A \subseteq \downarrow a$ (resp. $A \subseteq \uparrow a$) and the least upper (resp. largest lower) bound, if it exists, is called the *supremum* or *join* (resp. *infimum* or *meet*) of A and denoted by $\bigvee A$ resp. $\bigwedge A$. We also use symbols $a \vee b$ for $\bigvee\{a, b\}$, $a_1 \vee \cdots \vee a_n$, $\bigvee_{i \in I} a_i$, and, similarly, with $\bigwedge$ or $\wedge$, in the obvious sense.

The least resp. largest element of $(X, \leq)$, that is, $\bigvee \emptyset$ resp. $\bigwedge \emptyset$, if it exists, will be denoted by 0 resp. 1.

The poset obtained by reversing the order, that is $(X, \leq')$ with $x \leq' y \equiv y \leq x$ is called the dual of $(X, \leq)$ and denoted as $(X, \leq)^{\mathrm{op}}$. We may also write $\leq^{\mathrm{op}}$ for thus defined $\leq'$.

2.1.1 A poset in which all the subsets have infima and suprema are called *complete lattices*. If all finite sets have suprema and infima we speak of *bounded lattices*, in case of *non-void* finite sets simply of *lattices*.

2.2 Adjunction We say that monotone maps

$$f : (X, \leq) \to (Y, \leq) \quad \text{and} \quad g : (Y, \leq) \to (X, \leq)$$

are (*Galois*) *adjoint* [19], *f to the left and g to the right*, and write $f \dashv g$, if

$$f(x) \leq y \iff x \leq g(y). \tag{adj1}$$

It is standard and (very) easy to see that this is equivalent to assuming that

$$fg \leq \mathrm{id}_Y \quad \text{and} \quad \mathrm{id}_X \leq gf. \tag{adj2}$$

Note that from these inequalities it readily follows that

$$fgf = f \quad \text{and} \quad gfg = g. \tag{adj3}$$

2.2.1 Facts (a) *Generally, left adjoints preserve all existing suprema and right adjoints preserve all existing infima.*
(b) *On the other hand, if X, Y are complete lattices then each $f \colon X \to Y$ preserving all suprema is a left adjoint (i.e. it has an adjoint on the right), and each $g \colon Y \to X$ preserving all infima is a right adjoint.*

Proof (a) Let f be a left adjoint and $s = \bigvee A$ in X. Then obviously $f(s)$ is an upper bound of $f[A]$. Now let b be a general upper bound of $f[A]$, that is, let $f(a) \leq b$ for all $a \in A$. Then for all $a \in A$, $a \leq g(b)$, hence $g(b)$ is an upper bound of A, hence $s \leq g(b)$, and finally $f(s) \leq b$.
(b) Let f preserve all joins. Set $g(y) = \bigvee \{z \mid f(z) \leq y\}$. If $f(x) \leq y$ then trivially $x \leq g(y)$. On the other hand, if $x \leq g(y) = \bigvee \{z \mid f(z) \leq y\}$ then $f(x) \leq f(\bigvee \{z \mid f(z) \leq y\}) = \bigvee \{f(z) \mid f(z) \leq y\} \leq y$. $\qquad \square$

2.3 For category minded readers: posets as special categories A category is said to be *thin* if for any two objects A, B there is at most one morphism $A \to B$. A poset $(X, \leq)$ (more generally, a preordered set) is a thin category in which a morphism $x \to y$ is the statement that $x \leq y$, if it holds true: reflexivity of $\leq$ provides the identity morphisms and transitivity provides the composition of morphisms.

Note that, on the other hand

every thin category is a preordered partially ordered class,

(which differs from a partially ordered set by possibly being carried by a proper class, and by $a \leq b$ and $b \leq a$ for isomorphic distinct objects a, b).

Monotone maps $f \colon (X, \leq) \to (Y, \leq)$ are in this perspective precisely the functors between such categories.

Now the adjunction from 2.2 is a special case of the adjunction of functors $L \colon \mathcal{A} \to \mathcal{B}$ and $R \colon \mathcal{B} \to \mathcal{A}$ in categories. Recall (adj1) and compare

$$f(a) \leq b \text{ iff } a \leq g(b) \quad \text{with} \quad \mathcal{B}(L(A), B) \cong \mathcal{A}(A, R(B));$$

further compare (adj2) with the adjunction unit and co-unit

$$\lambda \colon LR \to \mathrm{Id}_{\mathcal{B}} \quad \text{and} \quad \rho \colon \mathrm{Id}_{\mathcal{A}} \to RL,$$

and (adj3) with

$$\left(L \xrightarrow{\;L\rho\;} LRL \xrightarrow{\;\lambda_L\;} L\right) = \mathrm{id}_L \quad \text{and} \quad \left(R \xrightarrow{\;\rho_R\;} RLR \xrightarrow{\;R\lambda\;} R\right) = \mathrm{id}_R$$

(in the general case the latter has to be assumed, in the thin case it comes for free and becomes the former).

Further realize that the (partial order) upper or lower bounds are precisely the categorical upper or lower bounds in the thin case, and infima resp. suprema coincide with products and coproducts:
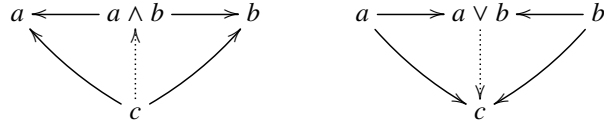
Thus, the first fact in 2.2.1 is a special case of preservation of colimits and limits by left and right adjoint functors. Since equalizers (resp. coequalizers) exist trivially in thin categories, the *existence of limits* (resp. *colimits*) means *existence of products* (resp. *coproducts*), that is, infima (resp. suprema). Hence, complete lattices are the complete (and cocomplete) such categories.

The correspondence of the second fact in 2.2.1 with the theorems on the existence of adjoint functors is not quite so straightforward. One has to keep in mind that in general categories we do not have a proper counterpart to the concept of a complete lattice: in complete and cocomplete categories we consider small (set) diagrams in large (class carried) categories (and large diagrams would not make sense) while in complete lattices we admit (and need) "diagrams of the same size". But there are theorems that present adjoint functors under reasonable circumstances (namely, the Freyd's "*Solution Set*" condition [43]).

To finish this short excursion to categorial reasoning: we have seen posets and Galois adjunction as a special case of adjunction in categories. It is, however, sometimes also profitable to look at the situation the other way, namely as primarily a phenomenon in posets extended to categories where instead of one arrow between nodes one has labelled ones, with structured labelling.

2.4 Some special posets A *meet* resp. *join semilattice* has $a \wedge b$ resp. $a \vee b$ for any a, b (and consequently all non-empty finite meets resp. joins); if it is obvious from the context whether the meets or joins are meant, one speaks simply of a *semilattice*.

We have already introduced lattices, bounded lattices and complete lattices in 2.1.1. Further, a lattice is *distributive* if we have

$$(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c) \quad \text{which is equivalent to} \quad (a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$$

(the equivalence may be slightly surprising; it is important to realize that it amounts to the fact that the dual of a distributive lattice is also distributive).

2.5 Pseudocomplements, supplements and complements We might consider a more general situation, but for our purposes everything in the remaining part of this section will happen in bounded lattices L. A *pseudocomplement* (resp. *supplement*) of an element $a \in L$ is an element b such that

$$a \wedge x = 0 \ \text{ iff } \ x \leq b \quad (\text{resp.} \quad a \vee x = 1 \ \text{ iff } \ x \geq b).$$

None of them has to exist, but if it does it is obviously uniquely determined. If it exists we usually denote it by a^* resp. $a^\#$.

A *complement* of a is an element b such that $a \wedge b = 0$ and $a \vee b = 1$. It does not have to exist and in general it is not even uniquely determined. But in a distributive lattice there is at most one and if it exists it is simultaneously a pseudocomplement and a supplement. One then speaks of a *complemented* element and the complement is usually denoted a^* (if there is no danger of confusion, otherwise another symbol, e.g. a^c, is used).

2.6 Heyting algebras A bounded lattice L is called a *Heyting algebra* if there is a binary operation $x \to y$ (the *Heyting operation*) such that for all a, b, c in L,

$$a \wedge b \leq c \quad \text{iff} \quad a \leq b \to c. \tag{Hey}$$

Recall 2.2 and realize that (Hey) says precisely that for every b

> the mapping $b \to (-) = (x \mapsto b \to x) \colon L \to L$ is a right Galois adjoint of $(-) \wedge b = (x \mapsto x \wedge b) \colon L \to L$

and hence

(H1) *the operation $\to$, if it exists, is uniquely determined; thus, being Heyting is in fact a condition on the meet in L,*

(H2) *in a Heyting algebra one has $(\bigvee A) \wedge b = \bigvee_{a \in A}(a \wedge b)$ for any $A \subseteq L$ such that $\bigvee A$ exists, and $b \to \bigwedge A = \bigwedge_{a \in A}(b \to a)$ for any $A \subseteq L$ such that $\bigwedge A$ exists,*

(H3) *and if L is complete then the distributivity rule*

$$(\bigvee A) \wedge b = \bigvee_{a \in A} (a \wedge b)$$

is a necessary and sufficient condition for the existence of a Heyting operation on L.

2.6.1 Notes (a) Unlike the plain distributivity $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$ the distributivity from (H3) is not carried over to the dual. See 3.2.2 below.
(b) It follows immediately from (Hey) that $a \leq b \to c$ iff $b \leq a \to c$. This is a contravariant Galois adjunction that yields moreover the rule

(H4) $(\bigvee A) \to b = \bigwedge_{a \in A} (a \to b)$.

(c) Dually one defines a coHeyting algebra as a bounded lattice with a binary operation (*coHeyting operation*)

$$c \smallsetminus b$$

(called the *difference*) such that

$$a \vee b \geq c \quad \text{iff} \quad a \geq c \smallsetminus b.$$

The importance of this concept will be apparent in 5.4 below.

2.6.2 A few Heyting rules In the sequel we will often need to compute with the Heyting operation. Here are some formulas that are immediate consequences of (Hey).

(1) $a \leq b \to a$ (since $a \wedge b \leq a$).
(2) $1 \to a = a$ (since $x \leq a$ iff $x \wedge 1 \leq a$, that is, $x \leq 1 \to a$).
(3) $a \to b = 1$ iff $a \leq b$ (since $1 \leq a \to b$ iff $1 \wedge a \leq b$).
(4) $a \wedge (a \to b) \leq b$ (since $a \to b \leq a \to b$, the well known "*modus ponens*" rule).
(5) $a \wedge (a \to b) = a \wedge b$ ($\leq$ by (4) and $\geq$ by (1)).

And here are three further useful ones, perhaps slightly less trivial, but still very simple.

(6) $a \to (b \to c) = (a \wedge b) \to c = b \to (a \to c)$
    (we have $x \leq a \to (b \to c)$ iff $x \wedge a \leq b \to c$ iff $x \wedge a \wedge b \to c$ iff $x \leq (a \wedge b) \to c$).
(7) $a \to b = a \to c$ iff $a \wedge b = a \wedge c$
    ($\Rightarrow$: By (5) and (4), $a \wedge b = a \wedge (a \to b) = a \wedge (a \to c) = a \wedge c$. $\Leftarrow$: By (3) and (H2), $a \to b = (a \to a) \wedge (a \to b) = a \to (a \wedge b) = a \to (a \wedge c) = a \to c$.
(8) $x = (x \vee a) \wedge (a \to x)$
    (by (4) and (1), $(x \vee a) \wedge (a \to x) = (a \wedge (a \to x)) \vee (x \wedge (a \to x)) \leq x$; on the other hand, by (1), $x \leq (x \vee a) \wedge (a \to x)$).

2.6.3 Pseudocomplement rules In a Heyting algebra one obviously has a pseudo-complement, namely $a^* = a \to 0$, with the following properties:

(1) $a \leq b \Rightarrow b^* \leq a^*$.
(2) $a \leq a^{**}$ and $a^{***} = a^*$.
(3) $(\bigvee A)^* = \bigwedge_{a \in A} a^*$ for any $A \subseteq L$ such that $\bigvee A$ exists (*De Morgan law*). Caution: the dual law for $\bigwedge A$ does not hold in general.

(Note that, in $\Omega(X)$, U^* is the interior of $X \smallsetminus U$.)

Dually, in a coHeyting algebra one has the supplement $a^{\#} = 1 \smallsetminus a$.

2.7 Boolean algebras On the other hand we easily prove

2.7.1 Observation Let b have a complement b^c in a distributive lattice L. Then

$$a \wedge b \leq c \text{ iff } a \leq b^c \vee c, \quad \text{and} \quad a \vee b \geq c \text{ iff } a \geq c \wedge b^c.$$

Thus, in a Heyting algebra we have for any complemented element b, $b \to c = b^* \vee c$ and in a coHeyting algebra we have for any complemented element b, $c \smallsetminus b = c \wedge b^{\#}$.

Note All the assumptions are essential, though. In particular the formulas $b \to c = b^* \vee c$ resp. $c \smallsetminus b = c \wedge b^{\#}$ hold for complemented elements only; the Heyting resp. coHeyting operation cannot be thus reduced to pseudocomplementing resp. supplementing in no other case.

2.7.2 A *Boolean algebra* is a distributive lattice in which every element is comple-mented. From 2.7.1 we immediately obtain

Corollary *A Boolean algebra is both a Heyting and a coHeyting algebra.*

3 Frames and spaces

3.1 A *frame* (resp. *coframe*) is a complete lattice L satisfying the distributivity law

$$(\bigvee A) \wedge b = \bigvee \{a \wedge b \mid a \in A\} \tag{frm}$$

$$(\text{resp. } (\bigwedge A) \vee b = \bigwedge \{a \vee b \mid a \in A\}) \tag{cofrm}$$

for all $A \subseteq L$ and $b \in L$. A *frame homomorphism* $h\colon L \to M$ between two frames is a mapping preserving all joins and all finite meets. The resulting category will be denoted by

Frm.

Similarly we have *coframe homomorphisms* between coframes preserving all meets and finite joins.

3.2 Spaces and frames. The functor Ω A typical frame is the lattice $\Omega(X)$ of all open subsets of a topological space. Furthermore, if $f\colon X \to Y$ is a continuous mapping we have a frame homomorphism $\Omega(f)\colon \Omega(Y) \to \Omega(X)$ defined by $\Omega(f)(U) = f^{-1}[U]$. Thus we obtain a *contravariant* functor

$$\Omega\colon \mathbf{Top} \to \mathbf{Frm},$$

a basic link between classical spaces and what will turn out to be the generalized ones (see already in 3.5.2 below, then in 3.9, and again and again).

3.2.1 Let us agree that our spaces will be, from now on, always T_0: the frames $\Omega(X)$ will be central in our approach to spaces and it will make no sense to discuss classical points that cannot be distinguished by open sets. In particular we will, without further particular mentioning, use the following

Observation *Let $f, g\colon X \to Y$ be distinct continuous maps, and let Y be T_0. Then $\Omega(f) \neq \Omega(g)$.*

(Indeed, if $f(x) \neq g(x)$ consider a U such that, say, $g(x) \notin U \ni f(x)$. Then $x \in \Omega(f)(U) \smallsetminus \Omega(g)(U)$.)

3.2.2 Notes (a) Unlike plain (finite) distributivity, the frame distributivity typically does not carry over to the dual, that is, a frame is seldom simultaneously a coframe. Take e.g. any T_1-space X with a non-isolated point x and an open set $W \ni x$. Set $V = W \smallsetminus \{x\}$ and $\mathcal{U} = \{U \in \Omega(X) \mid x \in U\}$. Then $\bigwedge \mathcal{U} = \text{int} \bigcap \mathcal{U} = \emptyset$ and hence $(\bigwedge \mathcal{U}) \cup V = V$ while $\bigwedge \{U \cup V \mid U \in \mathcal{U}\} = W \neq V$.

(b) As the example shows, coframes will seldom come as models of (generalized) spaces. They will play, however, a fundamental role in the study of the structure of generalized subspaces.

3.3 The Heyting structure Recall 2.6 (and 2.2). The distributivity rule (frm) makes a frame a Heyting algebra and computing with the Heyting operation will be extensively used; similarly we will use computing with the difference in coframes. But we have to keep in mind that the category **Frm** is not that of (complete) Heyting algebras: frame homomorphisms generally do not respect the Heyting operation.

3.4 Prime elements and sobriety Recall that an element $p < 1$ in a distributive lattice is *prime* if

$$a \wedge b = p \quad \text{implies that either} \quad a = p \ \text{ or } \ b = p$$

(compare with primeness of numbers); equivalently, $a \wedge b \leq p$ only if $a \leq p$ or $b \leq p$ (readily deduced replacing $a \wedge b \leq p$ by $(a \vee p) \wedge (b \vee p) = (a \wedge b) \vee p = p$).

Typical prime elements in $\Omega(X)$ are the open sets $X \smallsetminus \overline{\{x\}}$. A T_0-space is *sober* [50] if there are no other primes in $\Omega(X)$.

3.4.1 Notes (a) Sobriety is a very common property of topological spaces (see [23, 33]). For instance, Scott spaces are *mostly* sober (cf. [32]), and *every* Hausdorff space is sober:

Suppose a prime $P \in \Omega(X)$ lacks two distinct points x and y. Separate them by disjoint $U \ni x$ and $V \ni y$ and consider the intersection $P = (P \cup U) \cap P \cup V)$ where P contains none of $P \cup U, P \cup V$.

On the other hand, sobriety is incomparable with the axiom T_1.
(b) Because of the relation with the Hausdorff axiom, sobriety is sometimes viewed as one of the so called separation axioms. But as it was rightly pointed out by Marcel Erné, it is, rather, a completion condition akin to the completion in metric or more generally uniform spaces. As we will see in the following proposition, it amounts to the assumption that filters that have the natural property of a neighborhood system have "a point in the center" that is, are really neighborhood systems.

Recall that a filter F in a distributive lattice is *prime* if $a \vee b \in F$ implies that $a \in F$ or $b \in F$, and *completely prime* if $\bigvee_{i \in I} a_i \in F$ implies that some $a_i \in F$ for any system $\{a_i \mid i \in I\}$. A typical completely prime filter in $\Omega(X)$ is the system $\mathcal{U}(x) = \{U \mid x \in U\}$ of all neighborhoods of a point x.

3.4.2 Proposition *A (T_0-)space is sober iff each completely prime filter in $\Omega(X)$ is $\mathcal{U}(x)$ for some $x \in X$.*

Proof $\Rightarrow$: Let X be sober and let $\mathcal{F} \subseteq \Omega(X)$ be a completely prime filter. Set $V_0 = \bigcup \{V \in \Omega(X) \mid V \notin \mathcal{F}\}$. By completeness, $V_0 \notin \mathcal{F}$, hence it is the largest such open set, and

$$U \in \mathcal{F} \quad \text{iff} \quad U \nsubseteq V_0. \tag{$*$}$$

Now if $U_1 \cap U_2 \subseteq V_0$ then both of the U_i cannot be in $\mathcal{F}$ (since it is a filter); thus, for some i, $U_i \subseteq V_0$ and V_0 is prime, and by sobriety $V_0 = X \smallsetminus \overline{\{x_0\}}$ for some x_0. Thus by ($*$), $U \in \mathcal{F}$ iff $U \nsubseteq X \smallsetminus \overline{\{x_0\}}$ which holds precisely when $x_0 \in U$, that is, $U \in \mathcal{U}(x_0)$.

$\Leftarrow$: Let V_0 be prime. Set $\mathcal{F} = \{U \in \Omega(X) \mid U \nsubseteq V_0\}$. Obviously $\mathcal{F}$ is a complete prime filter, hence $\mathcal{U}(x_0)$ for some x_0, so that $U \nsubseteq V_0$ iff $x_0 \in U$, that is, $U \subseteq V_0$ iff $U \subseteq X \smallsetminus \overline{\{x_0\}}$ and we conclude that $V_0 = X \smallsetminus \overline{\{x_0\}}$. $\qquad \square$

3.5 Theorem *Let Y be sober and let $h : \Omega(Y) \to \Omega(X)$ be a frame homomorphism. Then there is precisely one continuous map $f : X \to Y$ such that $h = \Omega(f)$.*

Proof Obviously a preimage $h^{-1}[\mathcal{F}]$ of a completely prime filter is a completely prime filter. Take an $x \in X$ and consider $h^{-1}[\mathcal{U}(x)]$. By sobriety and 3.4.2 it is $\mathcal{U}(y)$ for some $y \in Y$. Choose such y and denote it by $f(x)$. Thus, $h^{-1}[\mathcal{U}(x)] = \mathcal{U}(f(x))$, that is,

$$h(U) \ni x \;\; \text{iff} \;\; U \ni f(x), \text{ that is, iff } \; x \in f^{-1}[U],$$

hence thus defined f is continuous and $h = \Omega(f)$. Uniqueness of f follows immediately from T_0 property: if $f(x) \neq g(x)$ choose a U such that, say, $f(x) \in U \not\ni g(x)$ showing that $\Omega(f)(U) \neq \Omega(g)(U)$.   □

Thus the restriction $\Omega \colon \mathbf{Sob} \to \mathbf{Frm}$ of Ω is a full embedding.

3.5.1 Corollary *A sober space X can be homeomorphically reconstructed from the frame $\Omega(X)$ as the set*

$$\{h \colon \Omega(X) \to \mathbf{2} = \{0,1\} \mid h \text{ is a frame homomorphism}\}$$

endowed by the topology consisting of the $\widetilde{U} = \{h \mid h(U) = 1\}$ with $U \in \Omega(X)$.

(Indeed, consider the one-point space $P = \{*\}$. Then $\Omega(P) = \{\emptyset, P\} \cong \{0,1\}$ and we can consider the elements $x \in X$ represented by the continuous maps f_x with $f_x(*) = x$. Those are then by 3.5 in a one-to-one correspondence with the $h_x = \Omega(f_x)$, and $h_x(U) = 1$ iff $x = f_x(*) \in U$.)

3.5.2 Locales — so far formally Consider, so far just formally, the dual of the category of frames. It is called the category of locales and it will be studied later in a more transparent and useful concrete form. For the purposes of this section, however, it will be simply

$$\mathbf{Loc} = \mathbf{Frm}^{\mathrm{op}}$$

with frame homomorphisms understood in opposite direction for morphisms. Then we have a covariant functor

$$\Omega \colon \mathbf{Top} \to \mathbf{Loc}$$

and Theorem 3.5 can be interpreted as that this functor embeds sober spaces, a substantial part of the category of spaces, into **Loc** as a full subcategory. This justifies viewing frame theory, the "point-free topology", as an extension of (at least a substantial part) of the classical one. This point of view will be corroborated and confirmed in the sequel; in this section we only wish to demonstrate the basic linkage between the two.

3.6 Points and spectra The role of the sobriety in 3.5 and 3.5.1 was in the one-to-one correspondences, not in detecting (classical) points in the lattice $\Omega(X)$: any point x in any space X is represented by the map $(* \mapsto x) \colon P \to X$. This leads to the definition of a *point* in a frame L as a map of locales $\Omega(P) \to L$, that is, a frame homomorphism $L \to \Omega(P) = \mathbf{2} = \{0,1\}$ (cf. Clementino [16]).

The following representations of points will come handy.

3.6.1 Proposition (1) *Points h in L are in a one-to-one correspondence with the completely prime filters F in L given by $h \mapsto F_h = \{x \mid h(x) = 1\}$ and $F \mapsto h_F$ with $h_F(x) = 1$ iff $x \in F$.*

(2) *Points h in L are in a one-to-one correspondence with the prime elements p of L given by $h \mapsto p_h = \bigvee\{x \mid h(x) = 0\}$ and $p \mapsto h_p$ with $h_p(x) = 1$ iff $x \nleq p$.*

Proof It is a matter of straightforward checking. $\square$

3.6.2 Spectra In the following construction we will represent points as completely prime filters (briefly, *cp-filters*). This has technical advantages but it is also fairly intuitive: think of points represented by their systems of neighborhoods. The morphisms in **Loc** will be (so far) represented as frame homomorphisms, one has only to be careful with the interchanged domain and codomain.

   The spectrum of a frame L is the topological space

$$\Sigma(L) = \Big(\{F \mid F \text{ cp-filter in } L\}, \{\Sigma_a \mid a \in L\}\Big)$$

where $\Sigma_a = \{F \mid a \in F\}$. Note that

$$\Sigma_0 = \emptyset, \ \ \Sigma_1 = \Sigma(L), \ \ \Sigma_{a \wedge b} = \Sigma_a \cap \Sigma_b \ \text{ and } \ \Sigma_{\bigvee a_i} = \bigcup \Sigma_{a_i} \qquad (\Sigma 1)$$

so that $\{\Sigma_a \mid a \in L\}$ is really a topology. For each frame homomorphism $h\colon M \to L$ ($L \to M$ in **Loc**) set $\Sigma(h)(F) = h^{-1}[F]$. We have

$$(\Sigma(h))^{-1}[\Sigma_a] = \{F \mid a \in h^{-1}[F]\} = \{F \mid h(a) \in F\} = \Sigma_{h(a)}; \qquad (\Sigma 2)$$

hence, each $\Sigma(h)$ is continuous and we have obtained a functor

$$\Sigma\colon \textbf{Loc} \to \textbf{Top}.$$

Observation *Each $\Sigma(L)$ is a sober space.* Indeed:

   If $F \nsubseteq G$, with $a \in F$, $a \notin G$ then $G \notin \Sigma_a \ni F$. Thus $\Sigma(L)$ is T_0. Let Σ_a be a prime in $\Omega\Sigma(L)$. Set $p = \bigvee\{b \in L \mid \Sigma_b \subseteq \Sigma_a\}$. In particular, $\Sigma_p = \Sigma_a$. If $x \wedge y \leq p$ then $\Sigma_x \cap \Sigma_y \subseteq \Sigma_p = \Sigma_a$ and hence, say, $\Sigma_x \subseteq \Sigma_p$ so that $x \leq p$. Thus, p is a prime in L. Now note that $F \in \overline{\{G\}}$ iff $F \subseteq G$. Consider the $F = F_{h_p}$ from 3.6.1. We have $G \notin \overline{\{F\}}$ iff $G \nsubseteq F$ iff $h_p(c) = 0$ for some $c \in G$ iff $c \leq p$ for some $c \in G$ iff $G \in \Sigma_p = \Sigma_a$. Hence, $\Sigma_a = \Sigma(L) \smallsetminus \overline{\{F\}}$.

Theorem Σ *is a right adjoint to* Ω.

Proof Consider the mappings

$$\sigma_L\colon L \to \Omega\Sigma(L) \ (\Omega\Sigma(L) \to L \text{ in } \textbf{Loc}) \quad \text{and} \quad \rho_X\colon X \to \Sigma\Omega(X)$$

given by $\sigma_L(a) = \Sigma_a$ and $\rho_X(x) = \mathcal{U}(x) = \{U \mid x \in U\}$. We have already seen in ($\Sigma 1$) that σ_L is a homomorphism, and since

$$\rho_X^{-1}[\Sigma_U] = \{x \mid \mathcal{U}(x) \in \Sigma_U\} = \{x \mid \mathcal{U}(x) \ni U\} = \{x \mid x \in U\} = U, \qquad (\Sigma 3)$$

ρ_X is continuous.

Next, they constitute natural transformations $\sigma \colon \Omega\Sigma \to \mathrm{Id}$ (viewed as in **Loc**) and $\rho \colon \mathrm{Id} \to \Sigma\Omega$: indeed,

$$\Omega\Sigma(h)(\sigma_L(a)) = (\Sigma(h))^{-1}[\Sigma_a] = \Sigma_{h(a)} = \sigma_M(h(a))$$

(recall ($\Sigma 2$)), and $\Sigma\Omega(f)(\rho_X(x)) = \Omega(f)^{-1}[\mathcal{U}(x)] = \{U \mid f^{-1}[U] \in \mathcal{U}(x)\} = \{U \mid x \in f^{-1}[U]\} = \{U \mid f(x) \in U\} = \rho_Y(f(x))$.

Finally, we have to check that the composites

$$\Sigma(L) \xrightarrow{\ \rho_{\Sigma(L)}\ } \Sigma\Omega\Sigma(L) \xrightarrow{\ \Sigma\sigma_L\ } \Sigma(L) \tag{$\Sigma 4$}$$

and

$$\Omega(X) \xrightarrow{\ \sigma_{\Omega(X)}\ } \Omega\Sigma\Omega(X) \xrightarrow{\ \Omega(\rho_X)\ } \Omega(X) \tag{$\Sigma 5$}$$

result in identities. We have

$$\Sigma\sigma_L(\rho_{\Sigma(L)}(F)) = \sigma_L^{-1}[\mathcal{U}(F)] = \sigma_L^{-1}[\{\Sigma_a \mid F \in \Sigma_a\}] =$$
$$= \{x \mid \Sigma_x \ni F\} = \{x \mid x \in F\} = F$$

and

$$\Omega(\rho_X)(\sigma_{\Omega(X)}(U)) = \rho_X^{-1}[\Sigma_U] = \{x \mid \mathcal{U}(x) \in \Sigma_U\} =$$
$$= \{x \mid U \in \mathcal{U}(x)\} = \{x \mid x \in U\} = U. \ \square$$

3.7 Spatial frames A frame L is said to be *spatial* if it is isomorphic to $\Omega(X)$ for some space X. The adjointness counit σ of the spectrum offers an expedient criterion of spatiality. We have that

    a frame L is spatial iff σ_L is one-to-one

(which is the same as saying that it is an isomorphism). Indeed, by the definition of the space $\Sigma(L)$, σ_L is always onto, hence if the condition holds we have, trivially, $L \cong \Omega\Sigma(L)$. On the other hand in the identity in ($\Sigma 4$) above, $\sigma_{\Omega(X)}$ is a coretract and hence an isomorphism. Now if there is an isomorphism $\phi \colon L \to \Omega(X)$ we obtain from the transformation commutativity an isomorphism

$$\sigma_L = (\Omega\Sigma(\phi))^{-1} \cdot \sigma_{\Omega(X)} \cdot \phi.$$

Note that in view of 3.6.1 this condition can be reformulated as saying that for any two $a, b \in L$ with $a \not\leq b$ there is a prime p such that $b \leq p$ and $a \not\leq p$, and hence

$$\text{for every } a \in L, \ a = \bigwedge\{p \mid p \text{ prime}, \ a \leq p\}. \tag{spatial}$$

3.8 Sober reflection The unit ρ constitutes a reflection of **Top** to **Sob**. We have that

    a space X is sober iff ρ_X is a homeomorphism.

Indeed, ρ_X is invertible by 3.5 and by ($\Sigma 3$), $\rho_X[U] = \rho_X[\rho_X^{-1}(\Sigma_U)] = \Sigma_U$, so that an invertible ρ_X is an open map. The converse follows from Observation 3.6.2.

3.9 Classical and generalized (point-free) spaces Now we are ready for a rough
outline of the relation of the point-free and classical spaces.
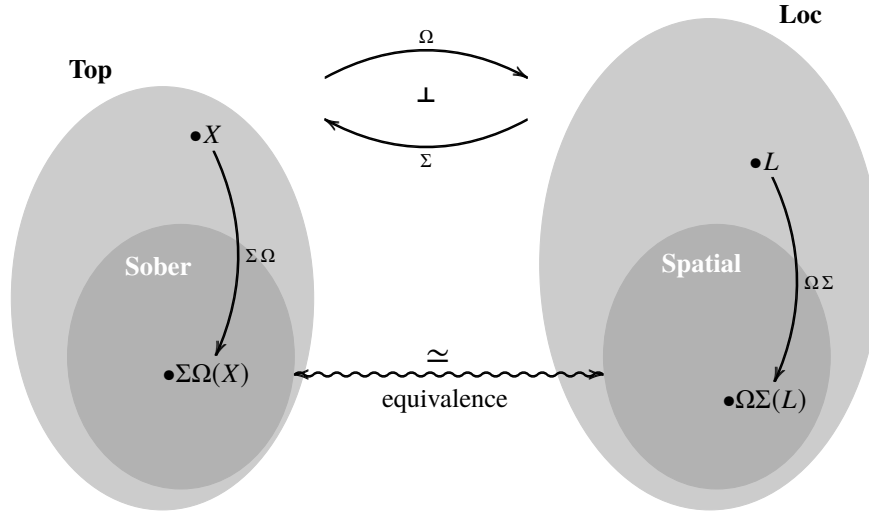
"sobrification" of a space   *"spatialization" of a locale*

On the one hand the point-free theory extends the classical one (or, at least a
substantial part of it; precisely, the subcategory of sober spaces). On the other hand,
the scope of the generalized theory is much larger; we will present two simple
examples shortly. The reader may of course ask whether such an extension is useful.
It turns out that it is, as will be hopefully apparent from the following sections.

3.9.1 Two easy examples First, consider a complete Boolean algebra B without
atoms (e.g., the lattice of all *regular open sets*[2] of any Euclidean space). There are
no primes at all: indeed, let p be one. Since it is not an atom, there is an x with
$p < x < 1$. We have $x^* \wedge x = 0 \leq p$, hence $x^* = 0$ and $x = 1$, a contradiction.

Next, take such a B again and consider $L = \{(x, y) \in B \times B \mid x \leq y\}$. Since all the
$(0, x)$ are in L we obtain by the same reasoning as above that a prime in L has to be
of the form $(q, 1)$ and hence there are no primes to separate distinct $(0, a)$ and $(0, b)$.

The latter example seems to be very similar to the former, but it is in fact much
more interesting. While Boolean frames are something like a generalization of
discrete spaces (albeit constituting a much more colorful class), the frames L of this
example are geometrically rather peculiar (do not forget that a subframe is, due to
the contravariance, geometrically more like a quotient space, not like a subspace):
interpreted as spaces they are behaving like Hausdorff ones, while on the other hand
they are not even subfit (see 7.6.2 below) which is a property weaker than T_1!

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[2] That is, the open sets $U = \operatorname{int} \overline{U}$, "open sets without lesions", the open sets one thinks about first.

4 Categorical remarks

4.1 Semilattices and a free functor Under a *semilattice* we here understand a *meet semilattice with 0 and 1*, and *semilattice homomorphisms* preserve $\wedge, 0$ and 1.

4.1.1 Note that in the category of semilattices (similarly like in abelian groups), the cartesian product with the injections and projections as in the following diagram

constitutes a biproduct (note that $p_1 \iota_1 \wedge p_2 \iota_2 = \text{id}$ and check the coproduct and product properties).

4.1.2 A free construction For a semilattice L consider the down-set lattice

$$\mathfrak{D}(L) = \{U \subseteq L \mid \emptyset \neq U = {\downarrow}U\}$$

ordered by inclusion. Further, define $\lambda = \lambda_L \colon L \to \mathfrak{D}(L)$ by setting $\lambda(a) = {\downarrow}a$. Obviously,

$$\mathfrak{D}(L) \text{ is a frame and } \lambda \text{ is a semilattice homomorphism}$$

(since we take only the non-empty down-sets, the zero of $\mathfrak{D}(L)$ is $\{0\}$; all the other joins are the unions).

Proposition *Let M be a frame and let $h \colon L \to M$ be a semilattice homomorphism. Then there is precisely one frame homomorphism $\widetilde{h} \colon \mathfrak{D}(L) \to M$ such that the diagram*

commutes. This $\widetilde{h}$ is given by the formula $\widetilde{h}(U) = \bigvee\{h(a) \mid a \in U\}$.

Proof Since for a down-set U, $U = \bigcup\{{\downarrow}a \mid a \in U\} = \bigcup\{\lambda(a) \mid a \in U\}$ we have for a (possible) frame homomorphism $\widetilde{h}$ commuting as desired $\widetilde{h}(U) = \bigvee\{\widetilde{h}\lambda(a) \mid a \in U\} = \bigvee\{h(a) \mid a \in U\}$; hence the formula and the uniqueness. Obviously, this formula gives a mapping $\widetilde{h} \colon \mathfrak{D}(L) \to M$ that preserves all joins, and $\widetilde{h}(L) = 1$. Finally, we have

$$\widetilde{h}(U) \wedge \widetilde{h}(V) = \bigvee\{h(a) \mid a \in U\} \wedge \bigvee\{h(b) \mid b \in V\} =$$
$$= \bigvee\{h(a) \wedge h(b) \mid a \in U, b \in V\} = \bigvee\{h(a \wedge b) \mid a \in U, b \in V\} \le$$
$$\le \bigvee\{h(c) \mid c \in U \cap V\} = \widetilde{h}(U \cap V) \le \widetilde{h}(U) \wedge \widetilde{h}(V),$$

so $\widetilde{h}$ is indeed a frame homomorphism. $\square$

4.2 Free objects in Frm For a set X define

$$F(X) = \{A \subseteq X \mid A \text{ finite}\}$$

ordered by $\le \; = \; \supseteq$ so that we have the meet $A \wedge B = A \cup B$. Denote by β_X the mapping

$$\beta_X = (x \mapsto \{x\}) \colon X \to F(X).$$

Then we have for each meet-semilattice S with 1 and each mapping $f \colon X \to S$ precisely one meet-semilattice homomorphism $\overline{f} \colon F(X) \to S$ such that the diagram

commutes and $\overline{f}(\emptyset) = 1$, namely the homomorphism defined by $\overline{f}(A) = \bigwedge_{x \in A} f(x)$.

The free frame over a set can be now obtained combining F and $\mathfrak{D}$, that is, as $\mathfrak{D}F(X)$. This provides a functor $\mathfrak{D}F \colon \mathbf{Set} \to \mathbf{Frm}$ right adjoint to the forgetful functor $\mathbf{Frm} \to \mathbf{Set}$.

4.3 Algebraic aspects of Frm The category $\mathbf{Frm}$ is clearly *equationally presentable* i.e. its objects are described by a (proper class of) operations, namely

- **0-ary**: $0, 1 \colon L^0 \to L$,
- **binary**: $L^2 \to L$, $(a, b) \mapsto a \wedge b$,
- **κ-ary (any cardinal κ)**: $L^\kappa \to L$, $(a_i)_\kappa \mapsto \bigvee_\kappa a_i$,

and equations

- $(L, \wedge, 1)$ is an idempotent commutative monoid,
- with a zero element satisfying the absorption law $a \wedge 0 = 0 = 0 \wedge a$,
- $\bigvee_0 a_i = 0$, $a_j \wedge \bigvee_\kappa a_i = a_j$, $a \wedge \bigvee_\kappa a_i = \bigvee_\kappa (a \wedge a_i)$.

Then, by general results of category theory (see Manes [46], Chapter 1, or Johnstone [33]), it follows that

4.3.1 Proposition Frm *has all* (*small*) *limits* (*i.e., it is a complete category*) *and they are constructed exactly as in the category* **Set** *of sets* (*i.e., the forgetful functor* **Frm** $\to$ **Set** *preserves them*).

Combining this with the fact that it has free objects over **Set**, **Frm** is a *monadic category over* **Set** ([46]). This means that, in particular,

4.3.2 Proposition (1) **Frm** *has all* (*small*) *colimits* (*i.e., it is a cocomplete category*).

(2) *The monomorphisms in* **Frm** *are exactly the injective homomorphisms.*

(3) *Epimorphisms in* **Frm** *need not be surjective but the regular epimorphisms are precisely the surjective homomorphisms.*

(4) *Every morphism in* **Frm** *can be factored* (*uniquely up to isomorphism*) *as a regular epimorphism followed by a monomorphism.*

(5) *Quotients are described by congruences.*

4.3.3 A consequence: presentations by generators and relations The fact that one has free frames and quotient frames implies, in particular, that, like in traditional categories of algebras, we may present frames by generators and relations: take the quotient of the free frame on the given set of generators modulo the congruence generated by the pairs (u, v) for the given relations $u = v$.

For example, the point-free space of reals is introduced as the *frame of reals* $\mathfrak{L}(\mathbb{R})$ ([39, 6]) generated by all ordered pairs (p, q) $(p, q \in \mathbb{Q})$, subject to the relations

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee \{(r, s) \mid p < r < s < q\}$,

(R4) $\bigvee_{p, q \in \mathbb{Q}} (p, q) = 1$.

This provides a development of the "theory of function rings $C(X)$" in frames and locales ([6]) and the treatment of more general point-free real functions (see e.g. [26]).

As another example, the *product* of two locales L and M (see 4.5 below) is the frame generated by all pairs $a \otimes b$, $a \in L$, $b \in M$, subject to the relations

(P1) $1 \otimes 1 = 1$,

(P2) $a \otimes 0 = 0 \otimes b = 0$,

(P3) $(a \otimes b) \wedge (a' \otimes b') = (a \wedge a') \otimes (b \wedge b')$,

(P4) $\bigvee_{i \in I} (a_i \otimes b) = (\bigvee_{i \in I} a_i) \otimes b$, $\bigvee_{i \in I} (a \otimes b_i) = a \otimes (\bigvee_{i \in I} b_i)$.

For further examples, see e.g. [25, 41].

4.4 Taking quotients In general, extending a binary relation to a congruence (and subsequent factorizing) in an algebra can be a hard task. In frames, however, it is surprisingly easy.

4.4.1 Saturation Let $R \subseteq L \times L$ be an arbitrary relation on a frame L. An element $s \in L$ is said to be *R-saturated* (briefly, *saturated*) if

$$aRb \quad \Rightarrow \quad a \rightarrow s = b \rightarrow s.$$

The set of all R-saturated elements of L is a frame: since $b \rightarrow (-)$ is a right adjoint, it is closed under meets, and by 2.6.2(6) we have for any x and aRb, $a \rightarrow (x \rightarrow s) = x \rightarrow (a \rightarrow s) = x \rightarrow (b \rightarrow s) = b \rightarrow (x \rightarrow s)$, hence it is also closed under the Heyting operation and therefore it is a complete Heyting algebra,

hence a frame[3] (with the same meets and the same Heyting operation as in L but not necessarily the same joins). It will be denoted by L/R.

We will show that L/R is the quotient of L by the congruence generated by R, and more.

4.4.2 The associated nucleus For any $a \in L$ set

$$v(a) = v_R(a) = \bigwedge \{s \in L/R \mid a \leq s\}.$$

We have

Proposition (1) *For every $a \in L$ and $s \in L/R$, $a \to s = v(a) \to s$.*
(2) *v is a nucleus, that is, it is monotone, $a \leq v(a)$, $vv(a) = v(a)$ and $v(a \wedge b) = v(a) \wedge v(b)$.*

Proof (1) For any x we have trivially $x \leq a \to s$ iff $a \leq x \to s$ and since this last is in L/R, this is the same as $v(a) \leq x \to s$, and this is equivalent with $x \leq v(a) \to s$.
(2) The first three formulas are trivial, and also trivially $v(a \wedge b) \leq v(a) \wedge v(b)$. Now since $a \wedge b \leq v(a \wedge b)$, we have, by (1), $a \leq b \to v(a \wedge b) = v(b) \to v(a \wedge b)$, hence $v(a) \leq v(b) \to v(a \wedge b)$ and finally $v(a) \wedge v(b) \leq v(a \wedge b)$. □

4.4.3 Proposition *v understood as a mapping $L \to L/R$ is an onto frame homomorphism. We have for aRb, $v(a) = v(b)$; moreover, if a frame homomorphism $h \colon L \to M$ is such that $h(a) = h(b)$ for all aRb, then there is an $\overline{h} \colon L/R \to M$ such that $\overline{h} \cdot v = h$.*
Furthermore, $\overline{h}(s) = h(s)$ for all $s \in L/R$.

Proof v preserves finite meets by 4.4.2(2). The joins $\bigsqcup$ in L/R are given by $\bigsqcup_i s_i = v(\bigvee_i s_i)$ (if $t \in L/R$ and $t \geq s_i$ for all i then $t \geq \bigvee_i s_i$ and $t = v(t) \geq v(\bigvee_i s_i)$) and hence $v(\bigvee_i a_i) \leq v(\bigvee_i v(a_i)) = \bigsqcup_i v(a_i) \leq v(\bigvee_i a_i)$. Hence v is a frame homomorphism.

Next, if aRb then $1 = a \to v(a) = b \to v(a)$ and hence $b \leq v(a)$, and $v(b) \leq v(a)$; equality by symmetry.

Finally, let $h \colon L \to M$ be such that $h(a) = h(b)$ for aRb. Set $\sigma(a) = \bigvee \{x \mid h(x) \leq h(a)\}$. Then obviously

$$a \leq \sigma(a), \quad \text{and} \quad h\sigma(a) \leq h(a) \text{ and hence } h\sigma = h. \tag{$*$}$$

Hence we have $x \leq \sigma(a)$ iff $h(x) \leq h(a)$ ('$\Rightarrow$' by $(*)$ and '$\Leftarrow$' by the definition of σ) so that for any uRv we have for any x, $x \leq u \to \sigma(a)$ iff $x \wedge u \leq \sigma(a)$ iff $h(x \wedge v) = h(x \wedge u) \leq h(a)$ iff $x \wedge v \leq \sigma(a)$ iff $x \leq v \to \sigma(a)$. Thus, $\sigma(a)$ is saturated, hence $a \leq v(a) \leq \sigma(a)$ and we have

$$h(a) \leq hv(a) \leq h\sigma(a) = h(a)$$

so that $h(a) = hv(a)$ and the statement follows. □

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[3] This proof shows indeed more: L/R is a *sublocale* of L, see 5.4 below.

4.4.4 Proposition *Let there be a join-basis $C \subseteq L$ such that for all $c \in C$ and aRb we have $(a \wedge c)R(b \wedge c)$. Then s is R-saturated iff for all aRb, $a \leq s$ iff $b \leq s$.*

Proof If the statement holds we have for every $c \in C$ and aRb, $c \wedge a \leq s$ iff $c \wedge b \leq s$, that is, $c \leq a \to s$ iff $c \leq b \to s$, and $a \to s = b \to s$. On the other hand, if s is saturated then in particular $a \to s = 1$ iff $b \to s = 1$. $\qquad\qquad\square$

4.5 Product in Loc (coproduct in Frm) concretely We will present a construction of the coproduct in the category of frames [33, 51]. It will be done for just two factors; the idea of the general case is precisely the same, only one has to use a more complicated notation which makes the presentation less transparent. The reader may do the general construction as a simple exercise taking instead of the $\iota_i \colon L_i \to L_1 \times L_2$ below in the role of the coproduct in the category of semilattices the general coproduct in that category, namely

$$\iota_i \colon L_i \to \coprod_{j \in J} L_j = \{(a_j)_{j \in J} \in \prod_{j \in J} L_j \mid \text{for all but finitely many } j \in J, a_j = 1\}$$

where $\iota_i(a) = (x_j)_{j \in J}$ with $x_i = a$ and $x_j = 1$ otherwise.

On the frame $\mathfrak{D}(L_1 \times L_2)$ define a relation R by setting

$$R = \left\{ \left(\bigcup_{i \in I} \mathord{\downarrow}(a_i, b), \mathord{\downarrow}(\bigvee_{i \in I} a_i, b) \right) \mid a_i \in L_1, b \in L_2 \right\} \cup$$
$$\cup \left\{ \left(\bigcup_{i \in I} \mathord{\downarrow}(a, b_i), \mathord{\downarrow}(a, \bigvee_{i \in I} b_i) \right) \mid a \in L_1, b_i \in L_2 \right\}.$$

Note that

- the void index set is not excluded, hence we have

$$\{(0,0)\} R \mathord{\downarrow}(0, b) \quad \text{and} \quad \{(0,0)\} R \mathord{\downarrow}(a, 0)$$

  for all $a \in L_1$ and $b \in L_2$;
- it is easy to check that the R-saturated $U \in \mathfrak{D}(L_1 \times L_2)$ are precisely those that

$$\text{for any } (a_i, b) \in U, i \in I, \text{ also } (\bigvee_{i \in I} a_i, b) \in U,$$

$$\text{and, for any } (a, b_i) \in U, i \in I, \text{ also } (a, \bigvee_{i \in I} b_i) \in U$$

  (the relation satisfies the conditions of 4.4.4, hence we can use the simplified saturation formula).

Theorem *The maps*

$$\bar{\iota}_i \colon \nu_R \cdot \lambda_{L_1 \times L_2} \cdot \iota_i \colon L_i \to L_1 \oplus L_2 = \mathfrak{D}(L_1 \times L_2)/R \quad (i = 1, 2)$$

are frame homomorphisms and constitute a coproduct in **Frm**.

Proof Let $h_i \colon L_i \to M$ be frame homomorphisms. Consider first the semilattice homomorphism $h' \colon L_1 \times L_2 \to M$ obtained for the h_i understood as semilattice homomorphisms (recall 4.1) and, using 4.1.2, lift it to a frame homomorphism $g = \widetilde{h'} \colon \mathfrak{D}(L_1 \times L_2) \to M$. Consider the following diagram.

The $\lambda \iota_i$ are semilattice homomorphisms that generally do not need to preserve the joins in L_i. The nucleus homomorphism ν_R, however, obviously provides the necessary equalities, and since it preserves meets, we obtain frame homomorphisms $\nu_R \lambda \iota_i$. Using the formula for $\widetilde{h'}$ from 4.1.2 (and taking into account that obviously $h'(a, b) = h_1(a) \wedge h_2(b)$) we easily check that it respects the relation R and hence we have, by 4.4.3, a frame homomorphism h such that $h \nu_R = g$ and hence $h \nu_R \lambda \iota_i = h_i$. Finally, the elements

$$\downarrow(a, b) = \downarrow(a, 1) \cap \downarrow(1, b) = \lambda \iota_1(a) \cap \lambda \iota_2(b)$$

obviously generate $\mathfrak{D}(L_1 \times L_2)$ by joins, and ν_R is onto, so that h is uniquely determined by $h\bar{\iota}_1$ and $h\bar{\iota}_2$. $\qquad \square$

5 Loc as a concrete category. Localic maps and sublocales

5.1 Since frame homomorphisms $h \colon M \to L$ preserve all joins they have uniquely defined right adjoints $f = h_* \colon L \to M$. We will use them for a concrete representation of the category **Loc** of locales [49, 50]. Thus, from now on we will speak of the meet preserving maps $f \colon L \to M$ between frames with left adjoints f^* that are frame homomorphisms as of *localic maps*. The category **Loc** will be that with frames as objects (in this context we often — although not always — speak of them as of *locales*) and localic maps as morphisms.

5.2 Proposition *A meet preserving map $f \colon L \to M$ is a localic map iff*

(a) $f(x) = 1$ *only if $x = 1$, and*
(b) *for all $y \in M$ and $x \in L$, $f(f^*(y) \to x) = y \to f(x)$ (this identity is often referred to as the Frobenius identity).*

Proof Set $h = f^*$. The point is in determining when h preserves finite meets. First, we have to have $h(1) = 1$; this makes in the adjunction the condition $1 = h(1) \leq x$ iff $1 \leq f(x)$, hence (a).

We have

$$h(x) \wedge h(y) \leq z \quad \text{iff} \quad h(x) \leq h(y) \rightarrow z \quad \text{iff} \quad x \leq f(h(y) \rightarrow z), \text{ and}$$
$$h(x \wedge y) \leq z \quad \text{iff} \quad x \wedge y \leq f(z) \quad \text{iff} \quad x \leq y \rightarrow f(z).$$

If $h(x) \wedge h(y) = h(x \wedge y)$ the first inequalities coincide and we have for all x, $x \leq f(h(y) \rightarrow z)$ iff $x \leq y \rightarrow f(z)$, hence $f(h(y) \rightarrow z) = y \rightarrow f(z)$. On the other hand, if $f(h(y) \rightarrow z) = y \rightarrow f(z)$ we have for all z, $h(x) \wedge h(y) \leq z$ iff $h(x \wedge y) \leq z$, hence $h(x) \wedge h(y) = h(x \wedge y)$. $\qquad\square$

5.2.1 Examples (1) For each continuous map $f \colon X \rightarrow Y$, the localic map right adjoint to $\Omega(f)$ is given by

$$\Omega(f)_*(U) = Y \smallsetminus \overline{f[X \smallsetminus U]}.$$

(2) Recall 3.6. A point of a locale L is a localic map $p \colon \mathbf{2} \rightarrow L$. Then $p(1) = 1$ and $p(0) = a \neq 1$ is a prime in L: $x \wedge y \leq a = p(0)$ iff $p^*(x) \wedge p^*(y) \leq 0$ hence $p^*(x) = 0$ or $p^*(y) = 0$, that is, $x \leq p(0) = a$ or $y \leq p(0) = a$.

5.3 Aside: spectrum in thus represented category of locales Recall 3.6.2. Let us represent points as primes. We have

Observation *Let $f \colon L \rightarrow M$ be a localic map. Then for every prime p in L, $f(p)$ is prime in M.*

(Indeed, since $p \neq 1$, $f(p) \neq 1$, and $a \wedge b \leq f(p)$ iff $f^*(a) \wedge f^*(b) \leq p$ iff $f^*(a) \leq p$ or $f^*(b) \leq p$ iff $a \leq f(p)$ or $b \leq f(p)$.)

Set
$$\Sigma(L) = \Big(\{p \mid p \in L, p \text{ prime}\}, \{\Sigma_a \mid a \in L\} \Big)$$

where $\Sigma_a = \{p \mid a \nleq p\}$ and take (using Observation) for $\Sigma(f) \colon \Sigma(L) \rightarrow \Sigma(M)$ simply the restriction of f. By the adjunction we have

$$(\Sigma(f))^{-1}[\Sigma_a] = \{p \mid a \nleq f(p)\} = \{p \mid f^*(a) \nleq p\} = \Sigma_{f^*(a)}$$

and we easily see that $\Sigma_0 = \emptyset$, $\Sigma_1 = \Sigma(L)$, $\Sigma_{a \wedge b} = \Sigma_a \cap \Sigma_b$ and $\Sigma_{\bigvee a_i} = \bigcup \Sigma_{a_i}$. Thus, the $\Sigma(L)$ are topological spaces and the $\Sigma(f)$ are continuous maps, and we have a functor **Loc** $\rightarrow$ **Top** (this time without any formal reversals). If we now define $\sigma_L(a) = \Sigma_a$ and $\rho_X \colon X \rightarrow \Sigma\Omega(X)$ by setting $\rho_X(x) = X \smallsetminus \overline{\{x\}}$ we can easily check all the equalities corresponding to those of 3.2.

5.4 Sublocales and the coframe $\mathsf{S}(L)$ Natural candidates for subobjects in a category are extremal monomorphisms. In **Frm**, extremal epimorphisms are precisely the onto frame homomorphisms and hence the extremal monomorphisms in our representation of **Loc** will be the adjoints to these, namely precisely the one-to-one localic maps (recall the identities (adj3) in any adjunction; they show that onto maps correspond to one-to-one maps and vice versa).

Thus, a natural subobject of a locale L is a subposet $S \subseteq L$ that is a frame in the induced order, such that the embedding map $j_S : S \subseteq L$ is a localic one. First of all, it is closed under meets and the left adjoint of j_S is obviously given by the formula

$$\nu_S(x) = \bigwedge\{s \mid s \in S, \ x \leq s\}$$

(understood as a map $L \to S$ it has to be a frame homomorphism; usually, however one considers it as a map $L \to L$ and one speaks of the *nucleus*[4] of S). By 5.2 we have that for every $s \in S$ and every $x \in L$, $x \to s \in S$, because in this case $x \to s = x \to j_S(s) = j_S(\nu_S(x) \to s) \in S$. This leads to the following definition.

A *sublocale* of a locale (frame) L is a subset $S \subseteq L$ such that

(S1) for every $M \subseteq S$, $\bigwedge M \in S$, and

(S2) for every $x \in L$ and every $s \in S$, $x \to s \in S$.

(We have already seen that if $j_S : S \subseteq L$ is a localic map then (S1) and (S2) hold. On the other hand, if S satisfies (S1) and (S2) then it is closed under meets and the Heyting operation, and hence it is a locale (with the same meets and the same Heyting operation as in L[5]). By (S1), j_S has a left adjoint ν_S as above. By (S2), $x \to s = j_S(x \to s)$, and for any u, $u \leq x \to s$ iff $x \leq u \to s$ iff $\nu_S(x) \leq u \to s$ iff $u \leq \nu_S(x) \to s$ so that

$$\nu_S(x) \to s = x \to s \qquad\qquad \text{(nucleus)}$$

and hence $j_S(\nu_S(x) \to s) = x \to s = x \to j_S(s)$ and j_S is a localic map by 5.2 — the condition with 1 is trivial since j_S is one-to-one.)

5.4.1 Obviously any intersection of sublocales is a sublocale so that we have a complete lattice

$$\mathsf{S}(L)$$

of sublocales of L. We immediately see that the join in $\mathsf{S}(L)$ is given by the formula

$$\bigvee_{i \in I} S_i = \{\bigwedge M \mid M \subseteq \bigcup_{i \in I} S_i\}$$

(every sublocale containing all S_i has to contain this set, and on the other hand this set is a sublocale by property 2.6(H2) of the Heyting operator).

By (S1) every sublocale contains the top 1. Thus, the smallest sublocale, corresponding to the classical empty subspace, is $\mathsf{O} = \{1\}$.

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[4] Nuclei in L are in a one-to-one correspondence with onto frame homomorphisms with domain L hence constitute an alternative representation for sublocales in L [50].

[5] The joins $\bigsqcup$ in S are given by $\bigsqcup_i s_i = \nu_S(\bigvee_i s_i)$: if $t \in S$ and $t \geq s_i$ for all i then $t \geq \bigvee_i s_i$ and $t = \nu_S(t) \geq \nu_S(\bigvee_i s_i)$.

5.4.2 Proposition $\mathsf{S}(L)$ *is a coframe.*

Proof We need to show that $(\bigcap_{i \in I} S_i) \vee T = \bigcap_{i \in I}(S_i \vee T)$. The inclusion $\subseteq$ is obvious. Hence, consider an $x \in \bigcap_{i \in I}(S_i \vee T)$. Then for every i there are $s_i \in S_i$ and $t_i \in T$ such that $x = s_i \wedge t_i$. Set $t = \bigwedge_i t_i$. We have

$$x = \bigwedge_i (s_i \wedge t_i) = \bigwedge_i s_i \wedge \bigwedge_i t_i = (\bigwedge_i s_i) \wedge t \leq s_i \wedge t \leq s_i \wedge t_i = x$$

so that $x = s_i \wedge t$ for all i. Then, by rule 2.6.2(7), all the $t \to s_i$ coincide; denote by s the common value. Since $s = t \to s_i \in S_i$, $s \in \bigcap_{i \in I} S_i$ and we conclude by 2.6.2(5) that $x = t \wedge s_i = t \wedge (t \to s_i) = t \wedge s \in (\bigcap_{i \in I} S_i) \vee T$. $\qquad\square$

5.5 Open and closed sublocales Each element $a \in L$ is associated with a *closed* sublocale $\mathfrak{c}(a)$ and an *open* sublocale $\mathfrak{o}(a)$,

$$\mathfrak{c}(a) = {\uparrow}a \quad \text{and} \quad \mathfrak{o}(a) = \{x \in L \mid a \to x = x\} = \{a \to x \mid x \in L\}^6$$

(the equivalence of the two expressions for $\mathfrak{o}(a)$ follows immediately from 2.6.2(6)).

5.5.1 Proposition $\mathfrak{o}(a)$ *and* $\mathfrak{c}(a)$ *are complements of each other.*

Proof If $x \in \mathfrak{o}(a) \cap \mathfrak{c}(a)$ we have $a \leq x = a \to x$, hence $a = a \wedge a \leq x$ and $x = a \to x = 1$ by 2.6.2(3). On the other hand, each $x \in L$ is by 2.6.2(8) equal to $(a \to x) \wedge (a \vee x) \in \mathfrak{o}(a) \vee \mathfrak{c}(a)$. $\qquad\square$

5.5.2 Proposition *We have the following formulas*

$$\mathfrak{o}(0) = \mathsf{O}, \quad \mathfrak{o}(1) = L, \quad \mathfrak{o}(a \wedge b) = \mathfrak{o}(a) \cap \mathfrak{o}(b) \quad \text{and} \quad \mathfrak{o}(\textstyle\bigvee_i a_i) = \bigvee_i \mathfrak{o}(a_i),$$

$$\mathfrak{c}(0) = L, \quad \mathfrak{c}(1) = \mathsf{O}, \quad \mathfrak{c}(a \wedge b) = \mathfrak{c}(a) \vee \mathfrak{c}(b) \quad \text{and} \quad \mathfrak{c}(\textstyle\bigvee_i a_i) = \bigcap_i \mathfrak{c}(a_i).$$

Proof We will prove the formulas for $\mathfrak{c}$, those for $\mathfrak{o}$ will then follow by De Morgan formulas. They are simple observations:

${\uparrow}0 = L$, ${\uparrow}1 = \{1\} = \mathsf{O}$, $x \geq \bigvee a_i$ iff $x \geq a_i$ for all i, and finally, $x \geq a \wedge b$ iff $x = (x \vee a) \wedge (x \vee b)$, that is, iff $x \in {\uparrow}a \vee {\uparrow}b$. $\qquad\square$

5.5.3 Proposition *A general sublocale S can be represented by open and closed sublocales as follows:*

$$S = \bigcap\{\mathfrak{c}(\nu_S(x)) \vee \mathfrak{o}(x) \mid x \in L\} = \bigcap\{\mathfrak{c}(y) \vee \mathfrak{o}(x) \mid \nu_S(x) = \nu_S(y)\}.$$

Proof I. If $s \in S$ then for arbitrary x, $x \to s \in S$. Hence by 2.6.2(8) and (nucleus)

$$s = (s \vee \nu_S(x)) \wedge (\nu_S(x) \to s) = (s \vee \nu_S(x)) \wedge (x \to s) \in \mathfrak{c}(\nu_S(x)) \vee \mathfrak{o}(x).$$

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[6] The reader might have expected ${\downarrow}a$. This subset of L is not a sublocale, but the intuition is not wide from the target: ${\downarrow}a$ is isomorphic to $\mathfrak{o}(a)$ which is the image of the localic map adjoint to the map $(x \mapsto a \wedge x)\colon L \twoheadrightarrow {\downarrow}a$ (see 6.6.1 below).

On the other hand, if a is in $\bigcap\{\mathfrak{c}(\nu_S(x)) \vee \mathfrak{o}(x) \mid x \in L\}$ then, in particular, it is in $\mathfrak{c}(\nu_S(a)) \vee \mathfrak{o}(a)$ and therefore $a = x \wedge (a \to y)$ with $x \geq \nu_S(a)$. Since $a \leq a \to y$ we have $a \leq y$, hence $a \to y = 1$, so that $a = x \geq \nu_S(a)$ and $a = \nu_S(a)$, that is, $a \in S$.

II. Since $\nu_S(\nu_S(x)) = \nu_S(x)$ it suffices, in view of I, to show that if $\nu_S(x) = \nu_S(y)$ then $S \subseteq \mathfrak{c}(y) \vee \mathfrak{o}(x)$. Let $s \in S$. We have $s = (s \vee y) \wedge (y \to s) = (s \vee y) \wedge (\nu_S(y) \to s) = (s \vee y) \wedge (\nu_S(x) \to s) = (s \vee y) \wedge (x \to s) \in \mathfrak{c}(y) \vee \mathfrak{o}(x)$. $\square$

5.6 Closure, density and Isbell's theorem. Interior Like in spaces we have the *closure* of a sublocale, the smallest closed sublocale containing S (cf. [16]). It is determined by a particularly simple formula, namely

$$\overline{S} = \mathfrak{c}(\wedge S) = \uparrow \wedge S$$

(a closed set containing S has to contain $\wedge S$ and has to be an up-set). Consequently we have also an extremely simple criterion of density:

5.6.1 Observation *A sublocale $S \subseteq L$ is dense in L iff it contains the bottom 0.*

5.6.2 For a frame L set

$$\mathfrak{B}(L) = \{a \in L \mid a = a^{**}\} = \{a^* \mid a \in L\}.$$

Obviously it is a sublocale: we have $a^* = a \to 0$ and $\wedge_i(a_i \to 0) = (\vee_i a_i) \to 0$ by 2.6(H4), making for (S1); (S2) follows from 2.6.2(6).

$\mathfrak{B}(L)$ is a Boolean algebra, the largest Boolean algebra in among the sublocales. It is called the *Booleanization* of L, and it is a very old construction known from algebraic logic (Glivenko [24]).

5.6.3 Theorem (Isbell's Density Theorem) *A sublocale $S \subseteq L$ is dense iff it contains $\mathfrak{B}(L)$. Thus, each locale L contains a smallest dense sublocale, namely $\mathfrak{B}(L)$.*

Proof A dense sublocale contains 0 and hence, by (S2), all the $x \to 0$, that is, all the sublocale $\mathfrak{B}(L)$ which is itself dense, since $0 = 1 \to 0$. $\square$

5.6.4 Notes (a) This fact has no counterpart in classical topology. So e.g. in the frame of reals the sublocales of rationals and irrationals have still a very rich intersection (which, then, cannot be represented as a classical subspace). In the next pages we will pay some attention to the relation of sublocales and subspaces of classical spaces.
(b) The Booleanization $\mathfrak{B}(\Omega(X))$ is an example of advantages of the point-free approach. It is in fact a very natural space, namely the space of regular open sets, that typically has no classical representation.

5.6.5 Interior Similarly one defines the *interior* of a sublocale S as the largest open sublocale contained in S. We have $\mathfrak{o}(a) \subseteq S$ iff $\mathfrak{c}(a) \supseteq S^{\#}$ iff $a \leq \wedge S^{\#}$ iff $\mathfrak{o}(a) \subseteq \mathfrak{o}(\wedge S^{\#})$ so that

$$\text{int } S = \mathfrak{o}(\wedge S^{\#}).$$

Note that this can be read in terms of the coHeyting difference as

$$\text{int } S = L \smallsetminus \overline{(L \smallsetminus S)}$$

in analogy with the classical relation between interior and closure (the dual formula does not hold, though; see e.g. [22] for more information).

5.7 Subspaces and sublocales I. The axiom T_D This is a preparatory subsection. We will proceed in the next section after we will know more about images and preimages; now we will discuss just the correctness of point-free representation of subspaces.

5.7.1 Sublocales induced by subspaces Consider a space X, a subspace $Y \subseteq X$ and the embedding mapping $j_Y \colon Y \subseteq X$. Then we have the onto frame homomorphism

$$\Omega(j_Y) = (U \mapsto U \cap Y) \colon \Omega(X) \to \Omega(Y)$$

with the adjoint localic map $\kappa_Y \colon \Omega(Y) \to \Omega(X)$, an extremal monomorphism in **Loc**, given by

$$\kappa_Y(V) = \text{int}\,((X \smallsetminus Y) \cup V)$$

(since $U \cap Y \subseteq V$ iff $U \subseteq (X \smallsetminus Y) \cup V$ and U is open). This suggests the natural representation of Y as the sublocale

$$S_Y = \kappa_Y[\Omega(Y)].$$

Such sublocales S_Y of (locales representing) spaces are usually referred to as the *induced sublocales*, more precisely, *sublocales induced by subspaces*.

5.7.2 The axiom T_D We have already seen that even in the most natural spaces like the Euclidean space of reals (we will learn later that in fact the contrary is rather rare) there are sublocales that are not (induced by) subspaces. This is in fact a very useful feature of point-free topology. There is, however, another hitch that has to be taken into account: the space has to have a certain very weak property to have the subspaces represented correctly.

The following property was introduced in Aull and Thron [1], and in the same year, in Thron [60], it was already used to prove one of the first results about the reconstruction of X from $\Omega(X)$. Since then it turned out to be a very important property in comparing classical and point-free theory (see e.g. [55]). A T_D-*space*[7] is a space X in which

> for every $x \in X$ there is an open $U \ni x$ such that $U \smallsetminus \{x\}$ is open. (T_D)

5.7.3 Proposition *The representation of subspaces in $\Omega(X)$ as above is correct in the sense that distinct subspaces induce distinct sublocales iff the space X satisfies T_D.*

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[7] The importance of this condition is comparable with that of sobriety. Note that in a way these two conditions are dual to each other: while sobriety requires that we cannot *add* a point to X without changing $\Omega(X)$, T_D says that we cannot *subtract* a point.

Proof $\Rightarrow$: Let X be a space in which T_D does not hold and let x be such that none of the $U \smallsetminus \{x\}$ with open $U \ni x$ is open. Then for $Y = X \smallsetminus \{x\}$ we have $\kappa_Y(U \cap Y) = U$ for any $U \in \Omega(X)$; indeed, if $x \in U$ then

$$\kappa_Y(U \cap Y) = \kappa_Y(U \smallsetminus \{x\}) = \mathrm{int}\,(U) = U,$$

otherwise $\kappa_Y(U \cap Y) = \kappa_Y(U) = \mathrm{int}\,(\{x\} \cup U) = U$. Hence $S_Y = \Omega(X) = S_X$.
$\Leftarrow$: Note that the nucleus of the S_Y is given by

$$\nu_Y(U) = \mathrm{int}\,((X \smallsetminus Y) \cup (U \cap Y)) \quad (U \in \Omega(X)).$$

Let T_D hold and let Y, Z be distinct subspaces (with, say, $Y \ni x \notin Z$). Choose an open $U \ni x$ with $V = U \smallsetminus \{x\}$ open. Then $\nu_Y(U) \neq \nu_Y(V)$ while $\nu_Z(U) = \nu_Z(V)$. Hence $\nu_Y \neq \nu_Z$ and thus the corresponding sublocales S_Y and S_Z are distinct. $\square$

5.8 Aside: spatialization as a sublocale Recall 3.7. The full subcategory of all spatial locales in **Loc** will be denoted by $\mathbf{Loc}_{\mathsf{sp}}$. Let $\mathrm{Pr}(L)$ denote the set of all primes p in a locale L and set

$$\mathrm{Sp}(L) = \{\textstyle\bigwedge A \mid A \subseteq \mathrm{Pr}(L)\}.$$

Obviously $\mathrm{Sp}(L) = L$ for a spatial L.

5.8.1 Lemma $\mathrm{Sp}(L)$ *is a sublocale of L.*

Proof Obviously $\mathrm{Sp}(L)$ is closed under meets. Now if $A \subseteq \mathrm{Sp}(L)$ then $x \to \bigwedge A = \bigwedge_{p \in A}(x \to p) \in \mathrm{Sp}(L)$ since $x \to p \in \mathrm{Pr}(L) \cup \{1\}$ for every $x \in L$ and $p \in \mathrm{Pr}(L)$: if $a \wedge b \leq x \to p$ then $a \wedge b \wedge x \leq p$; if $x \leq p$ we have $x \to p = 1$, else $a \wedge b \leq p$ and then, say, $a \leq p \leq x \to p$. $\square$

Note further that $\mathrm{Sp}(L) \in \mathbf{Loc}_{\mathsf{sp}}$ since $\mathrm{Pr}(\mathrm{Sp}(L)) = \mathrm{Pr}(L)$.

5.8.2 Lemma *If $f \colon L \to M$ is a localic map then we have a localic map* $\mathrm{Sp}(f) \colon \mathrm{Sp}(L) \to \mathrm{Sp}(M)$ *defined by* $\mathrm{Sp}(f)(a) = f(a)$.

Proof Since f preserves meets, we have by Observation 5.3, $f[\mathrm{Sp}(L)] \subseteq \mathrm{Sp}(M)$, and hence we have a map $\mathrm{Sp}(L) \to \mathrm{Sp}(M)$ defined as in the statement; obviously it preserves meets. Since we have for the embeddings $j_L \colon \mathrm{Sp}(L) \subseteq L$, $f \cdot j_L = j_M \cdot \mathrm{Sp}(f)$, that is, $j_L^* \cdot f^* = \mathrm{Sp}(f)^* \cdot j_M^*$, we have, for any $a \in \mathrm{Sp}(L)$ and $b \in \mathrm{Sp}(M)$, by (nucleus),

$$\mathrm{Sp}(f)(\mathrm{Sp}(f)^*(b) \to a) = \mathrm{Sp}(f)(\mathrm{Sp}(f)^*(j_M^*(b)) \to a) = f(j_L^*(f^*(b)) \to a) =$$
$$= f(f^*(b) \to a) = b \to f(a) = b \to \mathrm{Sp}(f)(a),$$

and $\mathrm{Sp}(f)$ is a localic map. $\square$

Thus we have a functor
$$\mathrm{Sp} \colon \mathbf{Loc} \to \mathbf{Loc}_{\mathsf{sp}}$$

(clearly a coreflection of **Loc** on **Loc**$_{\mathrm{sp}}$). Recall the representation of the adjointness unit $\sigma_L = (a \mapsto \Sigma_a) \colon L \to \Omega\Sigma(L)$ of the spectrum from 5.3. Restricting it to $\mathsf{Sp}(L)$, we get a description of the spatialization of a locale L (3.9) as a sublocale of L:

5.8.3 Proposition $\sigma_L \colon \mathsf{Sp}(L) \to \Omega\Sigma(L)$ *is a frame isomorphism.*

Proof We have already mentioned in 5.3 that $\Sigma_1 = \mathsf{Pr}(L)$ and $\Sigma_{a \wedge b} = \Sigma_a \cap \Sigma_b$. It is also easy to check that

$$\Sigma_{\bigwedge \mathsf{Sp}(L)} = \emptyset \quad \text{and} \quad \Sigma_{\bigsqcup_i a_i} = \Sigma_{\bigwedge \{p \in \mathsf{Pr}(L) \mid \bigvee_i a_i \leq p\}} = \bigcup_i \Sigma_{a_i}$$

so that we have a frame homomorphism. σ_L is clearly one-to-one in $\mathsf{Sp}(L)$; it is onto since

$$\Sigma_a = \Sigma_{\bigwedge \{p \in \mathsf{Pr}(L) \mid a \leq p\}} \quad \text{for every } a \in L. \ \square$$

6 Images and preimages. Localic maps as continuous ones. Open maps

6.1 Proposition *Let $f \colon L \to M$ be a localic map. For every subblocale $S \subseteq L$ the image $f[S]$ is a sublocale of M.*

Proof Trivially, $f[S]$ is closed under meets. Now take an $s \in S$ and an arbitrary $x \in M$. We have $x \to f(s) = f(f^*(x) \to s) \in f[S]$ since $f^*(x) \to s \in S$. $\square$

6.1.1 An (epi, extremal mono) factorization In consequence we have in **Loc** the factorizations

$$L \xrightarrow{\ g = (x \mapsto f(x))\ } f[L] \xrightarrow{\ j = \subseteq\ } M.$$

Indeed, g obviously preserves meets, hence it has a left adjoint, and we have $f^* = g^* j^*$ with f^* and j^* frame homomorphisms, j^* onto, and hence g^* is a frame homomorphism.

6.2 By the formula for join in $\mathsf{S}(L)$ we have, for each subset $A \subseteq L$ closed under meets, the largest sublocale contained in A,

$$A_{\mathsf{sl}} = \bigvee \{S \in \mathsf{S}(L) \mid S \subseteq A\}.$$

The preimage $f^{-1}[S]$ of a sublocale is obviously closed under meets, but the condition (S2) typically fails. We set

$$f_{-1}[S] = f^{-1}[S]_{\mathsf{sl}}$$

and call this sublocale the *localic preimage* of S.

Conventions We will sometimes work with both $f^{-1}[S]$ and $f_{-1}[S]$. To avoid confusion we will speak of the former as of the *set preimage*. Further, $f^{-1}[M]$ is

closed under meets for any meet-preserving f and any M that is closed under meets and hence we have a sublocale $f_{-1}[M] = f^{-1}[M]_{\mathrm{sl}}$ for any such M. We will refer to such a situation stating that $f_{-1}[M]$ *makes sense*.

6.2.1 Proposition *Localic preimages of closed resp. open sublocales are closed resp. open. More precisely, we have $f_{-1}[\mathfrak{c}(a)] = f^{-1}[\mathfrak{c}(a)] = \mathfrak{c}(f^*(a))$ and $f_{-1}[\mathfrak{o}(a)] = \mathfrak{o}(f^*(a))$.*

Proof I. $x \in f^{-1}[\uparrow a]$ iff $f(x) \geq a$ iff $x \geq f^*(a)$.

II. For a general element $f^*(a) \to x$ of $\mathfrak{o}(f^*(a))$ we have $f(f^*(a) \to x) = a \to f(x) \in \mathfrak{o}(a)$, hence $\mathfrak{o}(f^*(a)) \subseteq f^{-1}[\mathfrak{o}(a)]$.

   Now let S be a sublocale contained in $f^{-1}[\mathfrak{o}(a)]$; we will show that $S \subseteq \mathfrak{o}(f^*(a))$. Set $b = f^*(a)$ and take an $s \in S$. We have $(b \to s) \to s \in S$ and hence $f((b \to s) \to s) \in \mathfrak{o}(a)$ so that, using 2.6.2(6), we compute

$$f((b \to s) \to s) = a \to f((b \to s) \to s) = f(f^*(a) \to ((b \to s) \to s)) =$$
$$= f((b \wedge (b \to s)) \to s) = f((b \wedge s) \to s) = f(1) = 1$$

and since for a localic map f, $f(x) = 1$ only if $x = 1$ we see that $(b \to s) \to s = 1$. But then $b \to s \leq s$, and since always $s \leq b \to s$ we conclude that $s \in \mathfrak{o}(b)$. $\square$

6.3 Proposition *For any localic map $f : L \to M$ we have the adjunction*

$$\mathsf{S}(L) \underset{f_{-1}[-]}{\overset{f[-]}{\underset{\bot}{\rightleftarrows}}} \mathsf{S}(M).$$

Hence, the image map $f[-]$ preserves all joins and the preimage map $f_{-1}[-]$ preserves all meets.

Proof We have $f[S] \subseteq T$ iff $S \subseteq f^{-1}[T]$ iff $S \subseteq f_{-1}[T]$, the first being the standard set-theoretical image-preimage adjunction, the second because S is a sublocale. $\square$

Note. It can be further proved that $f_{-1}[-]$ is a coframe homomorphism that preserves complements while $f[-]$ is a colocalic map [50].

6.4 Points, sublocales and subspaces Each sublocale contains, trivially, the top element, and the sublocale $\mathsf{O} = \{1\}$ plays the role of the void subspace. We have an easy

Proposition *The sublocales containing just one non-trivial element are the $P = \{p, 1\}$ with p prime.*

Proof If p is prime and if $x \to p \neq 1$ then $x \not\leq p$ and since $x \wedge (x \to p) \leq p$ then $x \to p \leq p$ and hence $x \to p = p$ by 2.6.2(1).

   If $x \to p \in \{p, 1\}$ and if $x \wedge y \leq p$ then either $x \to p = 1$ and $x \leq p$ or $x \to p = p$ and $y \leq x \to p = p$. $\square$

6.4.1 The sublocales $P = \{p, 1\}$ with p prime are called *one-point sublocales*, or simply *point sublocales*. Note that this is in agreement with the representation of points as primes. From formula (spatial) and the formula for join in $\mathsf{S}(L)$ we immediately obtain

Observation *A frame L is spatial iff $L = \bigvee \{P \mid P$ point sublocale of $L\}$.*

In particular in a space X we have the one-point sublocales

$$P_x^X = \{X \smallsetminus \overline{\{x\}}, X\}$$

of $\Omega(X)$ and we obtain

6.4.2 Observation $\Omega(X) = \bigvee \{P_x^X \mid x \in X\}$.

(Note that here we have simply used the fact that an open $U \subseteq X$ is the intersection $\bigcap \{X \smallsetminus \overline{\{x\}} \mid x \notin U\}$. Thus, if X is not sober we actually have not needed all the prime like in the previous statement.)

6.4.3 Induced sublocales of $\Omega(X)$ in terms of point sublocales Let X be a topological space and $Y \subseteq X$ a subspace. Recall from 5.7.1 the sublocale $S_Y \subseteq \Omega(X)$ induced by Y. We have

Theorem *In $\mathsf{S}(\Omega(X))$, $S_Y = \bigvee \{P_y^X \mid y \in Y\}$.*

Proof We have $S_Y = \kappa_Y[\Omega(Y)]$ where κ_Y is the localic map adjoint to the embedding homomorphism $\Omega(j) = (U \mapsto U \cap Y) \colon \Omega(X) \to \Omega(Y)$. From 5.7.1 we know that $\kappa_Y(V) = \mathrm{int}\,((X \smallsetminus Y) \cup V)$ (the largest open $U \subseteq X$ such that $U \cap Y = V$). By 6.4.2, $\Omega(Y) = \bigvee \{P_y^Y \mid y \in Y\}$ and hence by 6.3, $S_Y = \bigvee \{\kappa_Y(P_y^Y) \mid y \in Y\}$ and it suffices to prove that $\kappa_Y(P_y^Y) = P_y^X$, that is, that for $y \in Y$,

$$\mathrm{int}\,((X \smallsetminus Y) \cup (Y \smallsetminus \overline{\{y\}}^Y)) = X \smallsetminus \overline{\{y\}}$$

which, since the closure in Y is the intersection of the closure in X with Y, amounts to $\mathrm{int}\,((X \smallsetminus Y) \cup (Y \smallsetminus \overline{\{y\}})) = X \smallsetminus \overline{\{y\}}$. The inclusion $\supseteq$ is trivial. Now let U be open and $U \subseteq (X \smallsetminus Y) \cup (Y \smallsetminus \overline{\{y\}})$. We have to prove that $U \subseteq X \smallsetminus \overline{\{y\}}$. Suppose the contrary. Then there is a $z \in U \cup \overline{\{y\}}$, hence $y \in U$ which is a contradiction: y is neither in $X \smallsetminus Y$ nor in $Y \smallsetminus \overline{\{y\}}$. □

6.4.4 Note All sublocales of spaces are induced sublocales only exceptionally. For a T_D-space (recall 5.7.2) the following statements are equivalent.

(1) *All sublocales of $\Omega(X)$ are induced by subspaces,*
(2) *$\mathsf{S}(\Omega(X))$ is Boolean,*
(3) *X is scattered, that is, every infinite subset of X contains an isolated point.*

(See [57, 47, 52, 2].)

6.5 Geometry of localic maps Localic maps were introduced in a rather formal way: in the first step the category **Frm** was just turned upside down to formally obtain covariance; in the second step one gained concreteness by another formal measure, namely by taking Galois adjoints. It may come as a pleasant surprise that thus formally obtained maps are characterized among non-structured maps like classical continuous maps, namely by preserving closedness and openness (the latter in some strict sense) by preimages[8]. In classical spaces it suffices to assume one, obtaining the other for free, here we will have to assume both explicitly: the complements of closed sublocales have to be formed in $\mathsf{S}(L)$ and not set-theoretically as in classical topology, and $\mathsf{S}(L)$ is not quite so simple as the Boolean algebra of all subsets.

6.5.1 Lemma *Let L, M be frames and let $f : L \to M$ be a mapping such that for every closed sublocale $B \subseteq M$ the* (*set-theoretical*) *preimage $f^{-1}[B]$ is closed. Then f preserves meets* (*and hence has a left adjoint*).

Proof In particular, preimages of up-sets are up-sets and hence f is continuous in the Alexandroff (quasidiscrete) topology of the posets L, M, and consequently monotone.

Next, for every $b \in M$ we have an $a \in L$ such that $f^{-1}[{\uparrow}b] = {\uparrow}a$. The a is obviously uniquely determined; let us denote it by $h(b)$. The equality ${\uparrow}h(b) = f^{-1}[{\uparrow}b]$ can be rewritten as

$$h(b) \le x \quad \text{iff} \quad b \le f(x). \tag{$*$}$$

Realizing that h is monotone (if $b \le b'$ we have ${\uparrow}b \supseteq {\uparrow}b'$ and hence ${\uparrow}h(b) \supseteq {\uparrow}h(b')$ and $h(b) \le h(b')$) we conclude that $(*)$ makes f a right Galois adjoint, hence a mapping preserving all meets. $\qquad\square$

6.5.2 Theorem *Let L, M be frames. Then a mapping $f : L \to M$ is localic iff*

for every closed A, $f^{-1}[A]$ is closed, $f^{-1}[\mathsf{O}] = \mathsf{O}$, and

for every open U, $f_{-1}[U] = f^{-1}[U^{\mathsf{c}}]^{\mathsf{c}}$ (and hence it is open).

(Note that because of the first condition and 6.5.1 the use of the symbol $f_{-1}[U]$ makes sense — recall the convention in 6.2.)

Proof Every localic map satisfies the conditions by 6.2.1. Thus, let $f : L \to M$ be a plain map satisfying the conditions. Since $f^{-1}[\mathsf{O}] = \mathsf{O}$ we have $f(a) = 1$ only if $a = 1$ and by 6.5.1 we know there is a right adjoint h, hence it remains to prove that $f(h(a) \to x) = a \to f(x)$.

Consider $B = {\uparrow}a$ so that $B^{\mathsf{c}} = \mathfrak{o}(a)$. Thus, $A = f^{-1}[B] = {\uparrow}h(a)$ and by the second assumption we have $\mathfrak{o}(h(a)) \subseteq f^{-1}[\mathfrak{o}(a)]$. Consequently

$$f(h(a) \to x) = a \to y \tag{$*$}$$

---

[8] More precisely: a map is localic iff each closed sublocale has a closed preimage whose complement is contained in the preimage of the complement of the original sublocale, and the least subocales are preserved.

for some y and we have to prove that $a \to y = a \to f(x)$, that is, by 2.6.2(7), that
$a \wedge y = a \wedge f(x)$.

$\geq$: Trivially, $f(x) \leq f(h(a) \to x) = a \to y$, and hence $a \wedge f(x) \leq y$.

$\leq$: Using the adjunction inequality $\mathrm{id} \geq hf$ and $(*)$ we have

$$h(a) \to x \geq hf(h(a) \to x) = h(a \to y) =$$
$$= h(\bigvee\{u \mid u \wedge a \leq y\}) = \bigvee\{h(u) \mid u \wedge a \leq y\},$$

hence $\bigvee\{h(u) \mid u \wedge a \leq y\} \leq h(a) \to x$ and so (recall rule (5) of 2.6.2)

$$\bigvee\{h(a) \wedge h(u) \mid u \wedge a \leq y\} = h(a) \wedge \bigvee\{h(u) \mid u \wedge a \leq y\} \leq$$
$$\leq h(a) \wedge (h(a) \to x) \leq x.$$

Consequently, $h(a \wedge y) \leq h(a) \wedge h(y) \leq x$ and finally $a \wedge y \leq f(x)$. $\square$

6.6 Joyal-Tierney Theorem This is a very interesting useful characterisation of
open localic maps, that is, of localic maps $f \colon L \to M$ such that the image $f[\mathfrak{o}(a)]$
of every open sublocale is open.

6.6.1 It will be technically of advantage to replace embeddings of open sublocales
$\mathfrak{o}(a)$ by isomorphic representations by means of the frames $\downarrow a$ as indicated in the
following diagram where j_a is the embedding $\mathfrak{o}(a) \subseteq L$ and the dotted isomorphism
consists of $(x \mapsto a \wedge x) \colon \mathfrak{o}(a) \to \downarrow a$ and $(x \mapsto a \to x) \colon \downarrow a \to \mathfrak{o}(a)$.
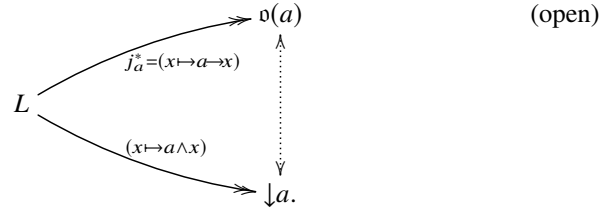
6.6.2 Theorem (Joyal and Tierney [40]) *A localic map $f \colon L \to M$ is open iff the
adjoint frame homomorphism $h = f^*$ is a complete Heyting homomorphism, that is,
if it preserves (also) all meets and the Heyting operation.*

Proof For each $a \in L$ we have a uniquely defined $\phi(a)$ such that $f[\mathfrak{o}(a)] = \mathfrak{o}(\phi(a))$
resulting in the decomposition $f \cdot j_a = j_{\phi(a)} \cdot g$ (where $j_a \colon \mathfrak{o}(a) \subseteq L$ and
$j_{\phi(a)} \colon \mathfrak{o}(\phi(a)) \subseteq M$ are the embeddings). Obviously this map $\phi \colon L \to M$ is mono-
tone. In terms of the adjoining frame homomorphism we thus have $j_a^* \cdot h = g^* \cdot j_{\phi(a)}^*$.
Replacing the j^*'s isomorphically as in (open) we obtain a commutative diagram

Thus the openness of f is characterized by the existence of a monotone $\phi \colon L \to M$ such that

$$x \wedge \phi(a) = y \wedge \phi(a) \quad \text{iff} \quad h(x) \wedge a = h(y) \wedge a$$

or, equivalently,

$$x \wedge \phi(a) \leq y \quad \text{iff} \quad h(x) \wedge a \leq h(y). \tag{$*$}$$

For $x = 1$, in particular, $\phi(a) \leq y$ iff $a \leq h(y)$ so that ϕ is a left adjoint of h and hence h preserves all meets. Further, we have by $(*)$, for arbitrary a, $a \leq h(x) \to h(y)$ iff $a \wedge h(x) \leq h(y)$ iff $x \wedge \phi(a) \leq y$ iff $\phi(a) \leq x \to y$ iff $a \leq h(x \to y)$ and hence $h(x) \to h(y) = h(x \to y)$.

On the other hand, if h preserves the Heyting operation, we have $x \wedge \phi(a) \leq y$ iff $\phi(a) \leq x \to y$ iff $a \leq h(x \to y) = h(x) \to h(y)$ iff $h(x) \wedge a \leq h(y)$, hence $(*)$. $\square$

For a thorough investigation of extensions of theorems 6.5.2 and 6.6.2 to the algebraic (not necessarily complete) setting of Heyting semilattices or algebras see Erné, Picado and Pultr [21].

7 Examples

In this final section we will, first, show a few examples of point-free reasoning; in particular we present the Banaschewski-Mulvey compactification [8, 9], illustrating on the one hand that the point-free techniques can be simpler than the classical ones, and on the other hand that one can obtain better facts than in the classical setting.

At the end we will mention, without proofs, a few more examples of facts that are more satisfactory than the classical ones.

7.1 Consider the relation between open sets of a space defined by $V \prec U$ iff $\overline{V} \subseteq U$. Thus, obviously, $x \in V \prec U$ is the same as saying that there are disjoint open sets $V \ni x$, $W \supseteq X \smallsetminus U$, and hence the property of regularity of a space X can be expressed by stating that each open set $U \subseteq X$ is the union $\bigcup \{V \mid V \prec U\}$.

This can be extended to the point-free context. Define, in a locale L, the relation $\mathfrak{o}(b) \prec \mathfrak{o}(a)$ between open sublocales iff $\overline{\mathfrak{o}(b)} \subseteq \mathfrak{o}(a)$, and declare L as *regular* if

$$\forall a \in L, \quad \mathfrak{o}(a) = \bigvee \{\mathfrak{o}(b) \mid \mathfrak{o}(b) \prec \mathfrak{o}(a)\}.$$

Recall 5.6. Since $\overline{\mathfrak{o}(b)} = \mathfrak{c}(b^*)$, then $\mathfrak{o}(b) \prec \mathfrak{o}(a)$ iff $\mathfrak{c}(b^*) \cap \mathfrak{c}(a) = 0$, that is, $b^* \vee a = 1$. Hence L is regular iff

$$\forall a \in L, \quad a = \bigvee \{b \in L \mid b \prec a\}$$

where $b \prec a$ (b is *rather below* a) in L iff $b^* \vee a = 1$ (or, equivalently, if there is a u such that $b \wedge u = 0$ and $u \vee a = 1$).

Regularity in frames is a very expedient property, easier to work with than e.g. variants of the Hausdorff property (see below in 7.6.2) and hence it has often appeared

in point-free reasoning (from its early stages) whenever one needed spaces with a "non-trivial separation". In the following we will present a few examples; technically we will typically work in frames, with localic interpretation added.

Compare the facts in 7.3.1, 7.2.3, and 7.2.4 below with the standard facts from classical topology concerning Hausdorff resp. compact Hausdorff spaces. Also in the point-free context they hold more generally, but the necessary techniques are much more involved. The simplicity of the proofs presented here indicates why the regularity is so popular in point-free topology.

7.1.1 Lemma *If $a_1, a_2 \prec b$ then $a_1 \vee a_2 \prec b$, if $a \prec b_1, b_2$ then $a \prec b_1 \wedge b_2$, and if $a \prec b$ then $b^* \prec a^*$.*

Proof If $a_i^* \vee b = 0$ then $(a_1 \vee a_2)^* \vee b = (a_1^* \wedge a_2^*) \vee b = 1$. If $a^* \vee b_i = 1$ then $a^* \vee (b_1 \wedge b_2) = 1$. If $a^* \vee b = 1$ then $a^* \vee b^{**} = 1$.   □

7.1.2 Lemma *If $h \colon L \to M$ is a frame homomorphism and $x \prec y$ then $h(x) \prec h(y)$. Consequently each sublocale of a regular frame is regular.*

Proof Apply h to the equalities $x \wedge u = 0, u \vee y = 1$. For the second statement consider the onto homomorphism h adjoint to the embedding $S \subseteq L$. For $b = h(a) \in S$ we have $b = h(\bigvee \{x \mid x \prec a\}) = \bigvee \{h(x) \mid x \prec a\}$.   □

7.2 A localic map $f \colon L \to M$ is *dense* if $\overline{f[L]} = M$, which is the same as saying that $f(0) = 0$. For the adjoint frame homomorphism we then have $h(x) = 0$ iff $x \leq f(0) = 0$ which leads to defining a *frame homomorphism* to be *dense* if $h(x) = 0$ implies that $x = 0$, and (so far for technical reasons only) as *codense* if $h(x) = 1$ implies that $x = 1$.

7.2.1 Lemma *Let a homomorphism $h \colon M \to L$ be codense and let M be regular. Then h is one-to-one.*

Proof Let $h(a) = h(b)$ and let $a \nleq b$. Then there is an $x \prec a$, $x \nleq b$. Since $x^* \vee a = 1$ we have $h(x^* \vee b) = h(x^* \vee a) = 1$, hence $x^* \vee b = 1$. Consequently, $x = x \wedge (x^* \vee b) = x \wedge b$ and $x \leq b$, a contradiction.   □

7.2.2 Compactness The concept of *compactness* is naturally extended to the point-free context: a *cover* of a frame L is a subset $A \subseteq L$ such that $\bigvee A = 1$, and a frame L is *compact* if every cover of L has a finite subcover.

7.2.3 Proposition *Let M be regular, L compact, and let a homomorphism $h \colon M \to L$ be dense. Then it is one-to-one. Thus, a dense localic map $f \colon L \to M$ with L compact and M regular is onto.*

Proof Let $h(a) = 1$. By regularity, $a = \bigvee \{b \mid b \prec a\}$, hence $1 = h(a) = \bigvee \{h(b) \mid b \prec a\}$ and by compactness there are b_i, $i = 1, \dots, n$, such that $1 = \bigvee_{i=1}^n h(b_i)$. Set $c = \bigvee_{i=1}^n b_i$. Then $h(c) = 1$ and by 7.1.1 $c \prec a$, and we have $c^* \vee a = 1$. Now $h(c^*) = h(c) \wedge h(c^*) = 0$ and by density $c^* = 0$ and we obtain that $a = 1$. Hence h is also codense, and it is one-to-one by 7.2.1.   □

7.2.4 Corollary *Each compact sublocale of a regular frame is closed.*

(Decompose the embedding mapping $S \subseteq L$ into the embedding mappings $S \subseteq \overline{S} \subseteq L$ and use 7.1.2.)

7.3 Theorem (Banaschewski Coequalizer Theorem) *Let $h_1, h_2 \colon M \to L$ be frame homomorphisms and let M be regular. Set $c = \bigvee \{h_1(x) \wedge h_2(y) \mid x \wedge y = 0\}$. Then*

$$\gamma = (x \mapsto x \vee c) \colon L \to {\uparrow}c$$

is the coequalizer of h_1 and h_2.

    In a localic formulation: the equalizer of any two localic maps $f_1, f_2 \colon L \to M$ with regular M is a closed sublocale of L, namely $\mathfrak{c}(\bigvee \{f_1^*(x) \wedge f_2^*(y) \mid x \wedge y = 0\})$.

Proof First, let us prove that $\gamma h_1 = \gamma h_2$. By symmetry, it suffices to show that $h_1(a) \leq h_2(a) \vee c$. Let $x \prec a$. Then $h_1(x) \wedge h_2(x^*) \leq c$ and hence $h_1(x) = h_1(x) \wedge (h_2(x^*) \vee h_2(a)) \leq c \vee h_2(a)$. Since by regularity $a = \bigvee \{x \mid x \prec a\}$ we obtain $h_1(a) = \bigvee \{h_1(x) \mid x \prec a\} \leq h_2(a) \vee c$.

    Now let $gh_1 = gh_2$ for some homomorphism $g \colon L \to K$. Then

$$g(c) = \bigvee \{gh_1(x) \wedge gh_2(y) \mid x \wedge y = 0\} = \bigvee \{gh_1(x \wedge y) \mid x \wedge y = 0\} = 0$$

so that we can define $\widetilde{g} \colon {\uparrow}c \to K$ by setting $\widetilde{g}(x) = g(x)$ to obtain $\widetilde{g} \cdot \gamma = g$. $\qquad \square$

7.3.1 Corollary *Let M be regular and let the localic maps $f_1, f_2 \colon L \to M$ coincide on a dense sublocale of L. Then $f_1 = f_2$.*

7.4 Complete regularity While in the case of regularity we have just presented some parallels with the classical result, here we will be able to present an example of a considerable improvement, namely an extension of the Stone-Čech compactification that is not only technically very simple, but also choice-free (!).

    In a frame L let us say that an element x is *completely below* y and write $x \lll y$ if there are x_d for diadically rationals d between 0 and 1 such that

$$x = x_0, \ y = x_1 \ \text{and} \ x_d \prec x_e \ \text{for} \ d < e. \tag{$*$}$$

A frame is *completely regular* if

$$\forall a \in L, \quad a = \bigvee \{b \mid b \lll a\}.$$

7.4.1 Notes (a) Again, similarly like with regularity, a space X is completely regular in the classical sense iff $\Omega(X)$ is completely regular as just defined. A continuous real function $f \colon X \to \mathbb{R}$ such that $f(x) = 0$ for all x in an open U and f constantly 1 on $X \setminus U$ can be obtained by inserting $x \in V \lll U$ and setting $f(y) = \inf \{d \mid y \in V_d\}$ (similarly like in the construction of the function separating two closed sets in Urysohn's Lemma).
(b) Instead of the set D of diadically rational numbers we can take any countable order-dense subset D' of the unit interval. The point is just in creating an interpolative sub-relation of $\prec$ and for this D is particularly transparent.

(c) One immediately sees that $\ll$ is the largest interpolative sub-relation of $\prec$ (meaning: an R such that for aRb there is always a c with $aRcRb$ — this is not necessarily true for $\prec$ itself). The construction of the largest interpolative subrelation given by the formula ($*$) is not quite choice-free (it needs the Axiom of Countably Dependent Choice). This can be avoided by defining $\ll$ simply as *the largest interpolative sub-relation of* $\prec$ (the union of all such subrelations). All we need (in particular the properties in 7.4.2 below) can be proved for thus defined $\ll$ to obtain a fully choice-free theory (see Banaschewski and Pultr [12]); it is of course, more involved.

7.4.2 From 7.1.1 and quite similarly like in 7.1.2 we obtain

Facts. (a) *$\ll$ is interpolative and if $a_1, a_2 \ll b$ then $a_1 \vee a_2 \ll b$, if $a \ll b_1, b_2$ then $a \ll b_1 \wedge b_2$, and if $a \ll b$ then $b^* \ll a^*$.*
(b) *Each sublocale of a completely regular frame is completely regular.*

7.5 A point-free Stone-Čech compactification (Banaschewski and Mulvey [8, 9])
For a frame L set

$$\mathfrak{J}(L) = \{J \mid J \text{ (non-empty) ideal in } L\}$$

ordered by inclusion.

7.5.1 Lemma $\mathfrak{J}(L)$ *is a compact frame.*

Proof Since intersections of ideals are ideals, it is a complete lattice. It is easy to check that the join in $\mathfrak{J}(L)$ is given by the formula

$$\bigvee_{i \in I} J_i = \{\bigvee M \mid M \text{ finite, } M \subseteq \bigcup_{i \in I} J_i\}.$$

Trivially, $(\bigvee J_i) \cap K \supseteq \bigvee(J_i \cap K)$ and if $x_1 \vee \cdots \vee x_n \in (\bigvee J_i) \cap K$ with $x_j \in J_{i_j}$ then $x_j \in K$ (an ideal is a down-set), hence $x_j \in J_{i_j} \cap K$, and $x_1 \vee \cdots \vee x_n \in \bigvee(J_i \cap K)$.

Finally, if $\bigvee J_i = L$ we have in particular $1 \in \bigvee J_i$, hence $1 = x_1 \vee \cdots \vee x_n$ with $x_j \in J_{i_j}$; but then $1 \in \bigvee_{j=1}^n J_{i_j}$ and hence $L = \bigvee_{j=1}^n J_{i_j}$. Thus $\mathfrak{J}(L)$ is compact. $\square$

7.5.2 An ideal J is *regular* if for every $x \in J$ there is a $y \in J$ such that $x \ll y$. In a completely regular frame L we have the regular ideals (recall the interpolation)

$$\sigma(a) = \{x \in L \mid x \ll a\}$$

and we have, for every ideal J,

$$J = \bigvee\{\sigma(a) \mid a \in J\}.$$

7.5.3 Lemma *Let L be completely regular. Then the set*

$$\mathfrak{R}(L) = \{J \in \mathfrak{J}(L) \mid J \text{ is regular}\}$$

is a compact completely regular frame.

Proof Obviously any intersection of regular ideals is a regular ideal, and from 7.4.2 we easily infer that a join of regular ideals is regular as well. Thus, $\mathfrak{R}(L)$ is a subframe of $\mathfrak{J}(L)$, and as (obviously) a subframe of a compact frame is compact, it is compact.

It remains to be proved that it is completely regular. We have, for each $J \in \mathfrak{R}(L)$,

$$J = \bigvee\{\sigma(a) \mid a \in J\} = \bigvee\{\sigma(b) \mid \exists a \in J, \ b \ll a\},$$

and since obviously $x \leq x' \ll y' \leq y$ implies $x \ll y$, it remains to be proved that

$$b \ll a \text{ in } L \quad \Rightarrow \quad \sigma(b) \ll \sigma(a) \text{ in } \mathfrak{R}(L).$$

Interpolate $b \ll x \prec y \ll a$. Then we have $y \in \sigma(a)$ and (recall 7.4.2) $x^* \in \sigma(b^*)$ and hence $1 = x^* \vee y \in \sigma(b^*) \vee \sigma(a)$ and $\sigma(b^*) \vee \sigma(a) = L$; on the other hand, trivially $\sigma(b^*) \cap \sigma(b) = \{0\}$ since $x \in \sigma(b^*) \cap \sigma(b)$ makes $x \leq b \wedge b^*$. $\qquad\square$

7.5.4 $\mathfrak{R}(L)$ as a compactification of L Define $v = v_L \colon \mathfrak{R}(L) \to L$ by setting $v(J) = \bigvee J$ and consider $\sigma_L = (a \mapsto \sigma(a)) \colon L \to \mathfrak{R}(L)$. We have $v(\sigma(a)) = a$ and $L \subseteq \sigma(v(L))$, hence v is a left Galois adjoint of σ, and hence it preserves all joins. Further, since J, K are down-sets,

$$v(J) \cap v(K) = \bigvee\{x \wedge y \mid x \in J, y \in K\} \subseteq$$
$$\subseteq \bigvee\{z \mid z \in J \cap K\} = v(J \cap K) \subseteq v(J) \cap v(K)$$

and hence v is a frame homomorphism (and σ is a localic map). Since $\bigvee J = 0$ only if $J = \{0\}$ and v is obviously onto, we have (recall 7.2) that

   each σ_L is a dense embedding of L into $\mathfrak{R}(L)$.

The construction $\mathfrak{R}$ can be extended to a functor by setting $\mathfrak{R}(h)(J) = {\downarrow}h[J]$ and it is easy to check that v is a natural transformation. Thus, to show we have here

   a compactification akin to the Stone-Čech compactification of spaces

it suffices to show that for L compact the homomorphism v_L is an isomorphism, and since we already know that generally $v(\sigma(a)) = a$ and $L \subseteq \sigma(v(L))$, it suffices to prove that for a compact L, $\sigma(v(J)) \subseteq J$. Thus, let $x \in \sigma(v(J))$. In particular $x \prec \bigvee J$, hence $x^* \vee \bigvee J = 1$ and hence, by compactness, there are $y_1, \ldots, y_n$ in J such that $x^* \vee y_1 \vee \cdots \vee y_n = 1$. J is an ideal, hence $y = y_1 \vee \cdots \vee y_n \in J$, and $x^* \vee y = 1$, hence $x \leq y$ and we conclude that $x \in J$.

7.5.5 Comments Note that the construction is much simpler than the construction of the compactification in classical spaces. Further note that we have not used the Axiom of Choice and not even the rule of Excluded Middle. Hence the construction is fully constructive. Furthermore, it is easy to see that if a reflection is constructive then also the fact that the limits, in particular the products, are in the smaller category, is constructive. Thus, in contrast with the situation in classical spaces where the compactness of products of (in this case Hausdorff) compact spaces is compact is

equivalent with a choice principle[9], products of compact completely regular locales in **Loc** are always compact.

Now the reader may start to doubt whether this compactification is at all closely related to the Stone-Čech one. We have been so far careful in stating that it is *akin* to that. But this was just a cagey formulation in a situation when we could not comment about it properly. In fact, by the Hofmann-Lawson duality [28, 4], in particular the products of completely regular spaces correspond exactly to the products of the corresponding locales, hence the Banaschewski-Mulvey compactification does extend the Stone-Čech one. How is this possible? (Not very) roughly speaking, Tychonoff theorem in locales ([31, 42]; see also [33, 17]) does not require any choice principle for proving the compactness: products in **Loc** of compact spaces (more precisely, of their corresponding spatial locales) are always compact, without the Axiom of Choice; however, they can be non-spatial.

7.6 A glimpse of other separation axioms (see [54] for more information) Normality can be immediately copied from classical topology: a frame is *normal* if

$$\forall a, b \text{ s.t. } a \vee b = 1 \ \ \exists u, v \text{ s.t. } a \vee u = 1, v \vee b = 1 \text{ and } u \wedge v = 0. \qquad \text{(norm)}$$

To present just a simple fact:

7.6.1 Proposition *In a normal frame the relation $\prec$ interpolates.*

Proof Let $a \prec b$. Then there is an x with $a \wedge x = 0$ and $x \vee b = 1$. By normality, there are u, v with $x \vee u = 1 = v \vee b$ and $u \wedge v = 0$ which makes $a \prec u \prec b$.   □

7.6.2 Lower separation, in particular subfitness About mimicking the Hausdorff axiom let us just mention that it is a complex area, with more candidates [54]. An interesting fact is that the candidates that are conservative, that is, applied to classical spaces agree with the classical Hausdorff property, do not behave as good as the so called strong Hausdorff property, which is not conservative but parallels very well properties of Hausdorff spaces.

Instead of T_1 we have a very expedient, weaker, subfitness. A frame is *subfit* if

$$a \not\leq b \quad \Rightarrow \quad \exists c, \ a \vee c = 1 \neq b \vee c. \qquad \text{(sfit)}$$

Obviously, T_1-spaces are subfit (if $x \in U \smallsetminus V$ set $W = X \smallsetminus \{x\}$ to obtain $U \cup W = X \neq V \cup W$). This property goes back to Wallman, 1938 (in a pioneering article [61] of point-free thinking, published long before point-free topology started to develop; it was later rediscovered [29], and only recently really appreciated [44, 35, 20, 54]). To show a simple application, let us prove the following

7.6.3 Proposition *A normal subfit frame is regular and hence, by 7.6.1, completely regular.*

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[9] With the Boolean Ultrafilter Theorem; the compactness of products for general spaces is equivalent with the full Axiom of Choice. Even the theorem for general frames is choice-free, but this is technically much more involved [31, 42, 5].

Proof Suppose a normal subfit L is not regular. Then there is an $a \neq b = \bigvee\{x \mid x \prec a\}$. Since $a \not\leq b$ there is a c with $a \vee c \neq b \vee c$. By normality there are u, v with $u \wedge v = 0$, $u \vee a = 1$ and $v \vee c = 1$. But then $v \prec a$, hence $v \leq b$ and we obtain a contradiction $b \vee c \geq v \vee c = 1$. $\square$

(In classical topology one usually speaks of normal T_1-spaces being (completely) regular, but of course subfitness suffices in the classical context as well.)

Subfitness has a lot of useful consequences. Here let us just mention a slightly surprising formula for pseudocomplement (where, in fact we will use even slightly less).

7.6.4 Proposition *In a subfit frame we have $a^* = \bigwedge\{x \mid x \vee a = 1\}$.*

Proof Set $b = \bigwedge\{x \mid x \vee a = 1\}$. If $x \vee a = 1$ then $a^* = a^* \wedge (x \vee a) = a^* \wedge x$, hence $a^* \leq x$ and we see that $a^* \leq b$. Thus, if $b \neq a^*$ we have $a \wedge b \neq 0$ and hence there is a $c \neq 1$ such that $c \vee (a \wedge b) = 1$. Then $c \vee a = 1$ and $c \vee b = 1$, by the former $b \leq c$, and by the latter $c = c \vee b = 1$, a contradiction. $\square$

Note. Thus, in a subfit frame we can compute the pseudocomplement by a formula for supplement. It is not generally a supplement, though: for that we would need the coframe distributivity. But of course we have the consequence that

   a subfit frame that is also a coframe is a Boolean algebra.

7.7 A few more examples We will finish with a few examples of point-free facts that are more satisfactory than the classical counterparts. We will present them in an easily understandable form and provide references, but will not go into details.

7.7.1 In 7.5.5 we have shown, using a choice-free reflection, that compactness of the product of completely regular locales is choice-free. In fact

   this holds for any frames whatsoever

but the proof is more involved [31, 42, 5].

7.7.2 The structure of frames can be naturally enriched, like that of classical spaces. Thus we have, e.g., a theory of uniform frames [53, 50], with the concept of completeness and completion quite parallel to the classical counterparts. But (after 7.7.1 not quite surprisingly)

   completion in the point-free context is fully constructive (see [11, 7]).

7.7.3 The reader may remember the concept of *paracompactness* that comes in many equivalent forms (the usual one requiring a locally finite refinement for every cover, another stating that the system of all covers constitutes a uniformity), has many useful applications, but behaves very badly (even a product of a paracompact space with a metric one may not be paracompact). Contrasting with this,

   the subcategory of paracompact locales is reflective in **Loc** [29, 10].

There is also an elegant characterisation (not holding classically) stating that

> *a frame is paracompact iff it admits a complete uniformity* [29]

(see also [56, 10, 50]).

7.7.4 Further one has that

> *the subcategory of Lindelöf locales is reflective in* **Loc** [45]

(the very important subcategory of *Lindelöf spaces* is not reflective in **Top**).

7.7.5 Extending the concept of a topological group one has the *localic groups* (standardly considering theory of groups over the category **Loc** instead of over **Top**) with properties similar to the classical ones (natural uniformities, etc.). But there is a fact that is fundamentally different (and somehow more satisfactory considering the classical zero group obtained as $\mathbb{R}/\mathbb{Q}$, dividing a group by an infinitely smaller one), the Closed Subgroup Theorem

> *every localic subgroup of a localic group L is closed in L* [30, 36, 37].

7.7.6 In classical measure theory, one has to restrict measure in the Euclidean space $\mathbb{R}^n$ to special measurable subsets in order to avoid Vitali and Banach-Tarski paradoxes. Instead, by enlarging the powerset $\mathcal{P}(\mathbb{R}^n)$ of subsets of $\mathbb{R}^n$ to the lattice of sublocales $\mathsf{S}(\Omega(\mathbb{R}^n))$ (recall 5.7.3 and 6.4.4), the point-free approach produces an isometry-invariant measure on all sublocales of $\Omega(\mathbb{R}^n)$, consistent with Lebesgue measure (Simpson [58]). In particular,

> *every subset in $\mathbb{R}^n$ is assigned a measure*

via the inclusion of $\mathcal{P}(\mathbb{R}^n)$ in $\mathsf{S}(\Omega(\mathbb{R}^n))$. The contradictions are avoided because disjoint subsets need not be disjoint as sublocales: although the intersection of two such sublocales has no points, they nevertheless overlap in $\mathsf{S}(\Omega(\mathbb{R}^n))$.

Acknowledgements

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