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Rings of real functions in pointfree topology. (English summary)

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Let L be a frame and $\mathcal{S}(L)$ be its sublocale lattice ordered by reverse inclusion, so that it is a frame. Denote by $\mathsf{F}(L)$ the ring of continuous real-valued functions on $\mathcal{S}(L)$. In the article [J. Pure Appl. Algebra **213** (2009), no. 6, 1064–1074; MR2498797 (2010a:06020)] by J. Gutiérrez García, T. Kubiak and J. Picado, it is well motivated why the ring $\mathsf{F}(L)$ should be viewed as the localic version of the ring of all real-valued, not necessarily continuous, functions on L. In the paper under review, new descriptions of the algebraic operations of $\mathsf{F}(L)$ are presented which greatly enhance our understanding of the posets of localic versions of lower and upper semicontinuous functions on a frame. Scales are put to good use in this regard. The authors proceed to study order-completeness properties of $\mathsf{F}(L)$, at each step appropriately inviting the reader to compare their results with those of B. Banaschewski and S. S. Hong [Comment. Math. Univ. Carolin. **44** (2003), no. 2, 245–259; MR2026162 (2004i:54016)]. To name a few of these results, it is shown that $\mathsf{F}(L)$ is order complete if and only if $\mathcal{S}(L)$ is extremally disconnected, and $\mathsf{F}(L)$ is σ-complete if and only if $\mathcal{S}(L)$ is basically disconnected. After thoroughly scrutinising the algebraic operations of the posets of lower and upper semicontinuous functions, the authors apply the material developed in the process to characterise idempotents in the ring $\mathsf{F}(L)$. Among other characterisations, idempotents of $\mathsf{F}(L)$ are precisely the characteristic functions of complemented sublocales of L, whilst idempotents of $\mathsf{C}(L)$—the subring consisting of continuous functions—are exactly the characteristic functions of complemented closed sublocales of L. Another application is on pointfree strict insertion of functions. Results from earlier work in this regard are amplified by the better understanding of the algebraic operations developed in this paper. The paper closes with a brief exploration of frames L for which every member of $\mathsf{F}(\mathcal{S}^\alpha(L))$, where $\mathcal{S}^\alpha(L)$ is the α-dissolution of L, is continuous. They are precisely the α-soluble frames [see T. Plewe, J. Pure Appl. Algebra **154** (2000), no. 1-3, 273–293; MR1787603 (2001f:18010)], that is, those whose α-dissolution is Boolean. A truly handsome observation to conclude a well-written paper.

Reviewed by *Themba Dube*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.