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★**Separation in point-free topology.**

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As the title promises, this book deals with a thorough study of different types of separation axioms in pointfree topology, i.e., the realm of the category **Frm** of frames and frame homomorphisms (which constitutes the algebraic side) and its opposite category, the category **Loc** of locales and localic maps (representing the topological side of things). The present book can be seen as a continuation of the monograph [*Frames and locales*, Front. Math., Birkhäuser/Springer Basel AG, Basel, 2012; MR2868166] by the same authors, although it is written in a self-contained way, providing the necessary background in the Appendix, which contains, among other things, a treatment of the construction of the binary coproduct in **Frm** as part of a more general construction, exhibiting analogy with the tensor product.

When working in **Loc**, as in their previous book, the authors use their covariant treatment of locales, i.e., 'with the arrows pointing the same way as in **Top**', with a localic map being a meet-preserving map with a left Galois adjoint that preserves finite meets (making localic maps exactly the right Galois adjoints of frame homomorphisms). Throughout the book, they choose to work either on the topological (= localic) or algebraic (= frame) side of things, depending on which is more convenient for the topic at hand, which is elegant and pays off. An important role throughout the book is played by the lattice of sublocales of a given locale, which happens to be a coframe (i.e., its dual is a frame, isomorphic to the lattice of frame congruences or the lattice of nuclei), which is one of the examples of the richness of the pointfree setting (even for spatial frames), and here this covariant take on locales proves to be very useful, e.g., when calculating in the sublocale lattice of a frame or when computing images and pre-images of sublocales.

The search for and study of separation axioms has, from the very start of locale theory, been a topic of great interest and activity to which many people, including the authors of this monograph, have made important contributions. It showcases a lot of subtleties of pointfree topology, exhibiting behavior that nicely extends that of classical topology (often with very different techniques and proofs being used) but also contains a lot of surprises or new things happening. The authors do an extremely good job collecting results scattered throughout the literature and presenting them here in a uniform way (for many of these results, it is the first time that they appear in a research monograph), drawing attention to the parallelisms and differences between the classical and pointfree worlds. The book starts with a very good sketch of the history of its subject matter provided in the Introduction, and also throughout the book, a lot of useful historical comments and pointers to the literature are given. In order to not overload this review, we will not mention the authors/papers to which the concepts mentioned are due, since this is very well documented in the monograph under review.

Chapter I (Separation in Spaces) deals with an overview of some well-known facts of the classical separation axioms in **Top**, but the particular use of the T_D-axiom for the pointfree setting is also highlighted, since, for two T_D-spaces X and Y, an isomorphism of their open set lattices $\Omega(X)$ and $\Omega(Y)$ implies that X and Y are homeomorphic. Therefore, T_D is the condition under which subspaces of a given T_0-space X are correctly represented as sublocales of $\Omega(X)$. Throughout the book (unless explicitly stated otherwise), *topological spaces are always assumed to be* T_0, which is a very natural assumption when considering relations to pointfree topology; we also use this convention here.

Pointfree relatives of the classical lower separation axioms T_1 and T_2 (= Hausdorff)

are discussed in detail in Chapter II (Subfitness and Basics of Fitness), Chapter III (Axioms of Hausdorff Type) and Chapter IV (Summarizing Low Separation).

As the title of Chapter II suggests, the main ingredient is *subfitness* (sometimes also called *conjunctivity*), where a frame L is called *subfit* if for every $a, b \in L$ with $a \not\leq b$, there exists a $c \in L$ with $a \vee c = 1$ and $b \vee c \neq 1$. Subfitness for spatial frames turns out to be a mildly weaker relative of T_1 since for a space X being T_1 is equivalent to being T_D together with the subfitness of $\Omega(X)$. An important characterization of subfitness, exploiting the rich structure of the sublocale lattice of a given locale, is treated in Section 4 of Chapter II, namely that a frame L is subfit if and only if every open sublocale of L is a join of closed sublocales. Fitness is already introduced in this chapter as well, but the authors convincingly argue that fitness rather has to be considered as one of the higher separation axioms, being akin to regularity (see Chapter VI). Fitness and subfitness are related by, *inter alia*, the fact that a frame is fit if and only if all of its sublocales are subfit. Moreover, fitness is hereditary, whereas subfitness is not. It is also shown here that fitness of a frame L is equivalent to the condition that every closed sublocale of L is a meet of open ones.

Chapter III treats various approaches that have been taken to express concepts related to classical Hausdorffness (which we will call T_2 for the rest of this paragraph) to the pointfree setting. After an introductory paragraph which also mentions the rich history of this topic, Section 2 treats three variants of weak Hausdorff properties, which turn out to be equivalent under subfitness, and which, taken together with subfitness, are a conservative extension of T_2 (where a property $\widehat{P}$ for frames is called a conservative extension of a property P for spaces, if a space X has property P if and only if the frame $\Omega(X)$ has property $\widehat{P}$). Imitating the well-known characterization for T_2-spaces that the diagonal is a closed subspace in the product of the space with itself yields the notion of strong Hausdorffness for locales. As the choice of terminology suggests, strong Hausdorffness of a frame implies Hausdorffness, and it has a lot of nice properties which are treated in Sections 3, 9, 10 and 11 of this chapter, such as, e.g., the facts that compact strongly Hausdorff locales are regular, that compact sublocales of strongly Hausdorff locales are closed, that dense frame homomorphisms are monomorphisms in the category of strongly Hausdorff frames and that strongly Hausdorff locales form an epireflective subcategory of **Loc**. However, since for a space X the frames $\Omega(X \times X)$ and $\Omega(X) \oplus \Omega(X)$ in general do not coincide, the notion of strong Hausdorffness fails to be conservative; in general, for a T_0-space X, only the implication '$\Omega(X)$ strongly Hausdorff $\Rightarrow X$ is T_2' holds. In Sections 3, 6 and 7, a slightly weaker variant of strong Hausdorffness with very good behavior is discussed, which was originally obtained by different authors looking at the topic from different angles, but resulted in equivalent concepts: one way builds on a spatial approach treating T_2 as a form of weak regularity, and the other one replaces the localic product and diagonal by a modification in the definition of strong Hausdorffness. The resulting notions are equivalent to the following definition, calling a frame L Hausdorff, if for any $1 \neq a \not\leq b$ in L, there exists $u, v \in L$ with $u \not\leq a$, $v \not\leq b$ and $u \wedge v = 0$. This notion of Hausdorffness for locales is a conservative extension of T_2, which is hereditary and productive in **Loc**; as a matter of fact, the Hausdorff locales form a reflective subcategory of **Loc**. Chapter III also contains a short paragraph discussing the notion of point-Hausdorffness, which is weaker than Hausdorffness (where a frame is called point-Hausdorff if every semi-prime element is maximal).

Chapter IV very nicely rounds up and synthesizes this first part of the book on lower separation in the pointfree setting, providing, *inter alia*, some tables of the valid implications for easy reference. The higher separation axioms are the subject of the next four chapters: Chapter V (Regularity and Fitness), Chapter VI (Complete

Regularity), Chapter VII (Normality) and Chapter VIII (More on Normality and Related Properties).

Regularity, resp. complete regularity, of a frame L is defined via the rather below relation, resp. the completely below relation, in L by demanding that each $a \in L$ is the join of all elements rather below a, resp. all elements completely below a. These provide conservative extensions of the classical regularity, resp. complete regularity, for spaces. Regularity implies strong Hausdorffness (and hence all the Hausdorff-like properties discussed in Chapter III and Chapter IV) and in Chapter V, the authors dedicate two paragraphs to explaining how several facts concerning density and compactness, which can be proved using weaker separation, become easier to deal with in the presence of regularity. The authors then revisit the notion of fitness, already treated in Chapter II, and show that it deserves to be considered as a relaxation of regularity. The chapter on regularity is concluded by proving that the notions of fitness and regularity define reflective, resp. coreflective, subcategories of **Loc**, resp. **Frm**. Chapter V discusses a (spatial) example showing that complete regularity is properly stronger than regularity, as well as a method for constructing a class of non-spatial completely regular frames, exhibiting that even in the presence of high separation, the domain of applicability of the pointfree setting still considerably expands the classical one. Subsequent sections cover the equivalence of complete regularity with uniformizability and the structure of the lattice of cozero elements of a frame L (the pointfree counterpart to cozero sets in topology), which form a sub-σ-frame of L, and which generate L if and only if L is completely regular. The chapter closes with three sections that really showcase some advantages that thinking pointfreely can bring about. The first one shows a more desirable behavior on the pointfree side: the completely regular Lindelöf frames (resp. locales) constitute a coreflective (resp. reflective) subcategory of the category of completely regular Lindelöf frames (resp. locales). The second one indicates how, putting in extra work, localic thinking is a good setting for doing choice-free (or even sometimes fully constructive) topology, which is illustrated in the last two sections of Chapter VI, dealing with choice-free versions of complete regularity and compactification.

Normality of a frame L is defined in the obviously conservative way as the condition that for all $a, b \in L$ with $a \vee b = 1$, there are $u, v \in L$ such that $a \vee u = 1 = b \vee v$ and $u \wedge v = 0$. In this case, subfitness does the trick of replacing T_1, since normal subfit frames are completely regular. In the remainder of Chapter VII, a pointfree version of the Wallman compactification and its relation to normality are treated, as well as the notion of complete normality, which is shown to be equivalent to hereditary normality.

Chapter VII starts with sections on perfect normality and collectionwise normality as relatives of normality; paracompactness is not studied in this monograph since a chapter in the authors' earlier book cited above is devoted to it. The frame of reals $\mathcal{L}(\mathbb{R})$ being discussed in detail in the Appendix, the authors now have at their disposal the concept of a continuous real-valued function on a frame L, being a frame map $\mathcal{L}(\mathbb{R}) \to L$, and the very useful way of defining such functions via the technical concept of a trail. With $\mathrm{S}(L)$ denoting the sublocale lattice of L (which is a coframe), the notion of (arbitrary) real-valued function on a frame L is defined as a continuous real-valued function on the frame $\mathrm{S}(L)^{\mathrm{op}}$, i.e., a frame homomorphism $\mathcal{L}(\mathbb{R}) \to \mathrm{S}(L)^{\mathrm{op}}$, leading also to very natural definitions of upper- and lower-semicontinuous real-valued functions on L (such that continuous becomes equivalent to being upper and lower semicontinuous). The next section of this chapter then develops a pointfree version of the Katětov-Tong insertion theorem, stating that normality of a frame L is equivalent to the statement that whenever one considers an upper-semicontinuous real-valued function f on L and a lower-semicontinuous real-valued function g on L with $f \leq g$, a continuous real-valued function h on L with $f \leq h \leq g$ can be found (here, an obvious

identification of real-valued continuous functions with certain real-valued functions is tacitly assumed, as explained in detail in the monograph). Subsequently, pointfree versions of the characterizations of normality by Urysohn's separation lemma and Tietze's (bounded) extension theorem are now readily obtained as corollaries. The last two sections of this chapter deal with extremal disconnectedness for frames (the definition of which is obtained by formally dualizing the one of normality) and obtaining both generalized and dualized versions of the Katětov-Tong insertion theorem in this setting.

The penultimate chapter is Chapter IX (Scatteredness: Joins of Closed Sublocales). In Chapter II, it is proved that subfitness of a locale is equivalent to the fact that every open sublocale of it is a join of closed ones. It is also proved that fitness can be expressed in a somewhat dual way, namely that a locale is fit if and only if every one of its closed sublocales can be written as a meet of open ones. Surprisingly, and indicating that fitness is quite a lot stronger than subfitness, it is proved that a locale is fit if and only if *every* one of its sublocales is a meet of open ones. This obviously begs the question of how the property that 'every sublocale of L can be written as a join of closed ones' relates to this picture. This property turns out to be a very strong one, equivalent to L being scattered and fit, and also to L being scattered and subfit (where a frame is called scattered if its sublocale lattice is a Boolean algebra). In the remainder of the chapter, further interesting material such as, e.g., the relation to classical scatteredness for spaces, Simmons' sublocale theorem and a study of the 'Boolean cover' of a subfit frame is treated.

In the last Chapter X (Subfit, Open and Complete), supplementary results are added, further completing the overall picture. As an example, the Joyal-Tierney theorem is treated, which states that a localic map $f\colon M \to L$ is open (i.e., the image of every open sublocale of M is open in L) if and only if its left Galois adjoint $f^*\colon L \to M$ is a complete Heyting homomorphism. Then building upon earlier material, one can infer that for L subfit, a localic map $f\colon M \to L$ is open if and only if its left Galois adjoint $f^*\colon L \to M$ is a complete lattice homomorphism.

The book is very well written and pays great attention to detail, making it a pleasure to read and, on top of covering a vast body of material, it conveys a lot of insights through the many remarks, comments, cross-references and references to the literature appearing throughout the book. Also, the ordering of the discussed topics is very well thought through and the book contains an extensive bibliography and a handy index plus a list of symbols.

In summary, the monograph under review, together with its companion [op. cit.] can only be highly recommended; they constitute a comprehensive source of information and insights regarding pointfree topology which will be of great interest and value to researchers already working in the field, to mathematicians who want to study pointfree topology and to general topologists who want to see how pointfree topology relates to classical topology at the same time. *Mark Sioen*