

MR4420436 06F07 18F70 54H11

Picado, Jorge (P-CMBR-CM); **Pultr, Aleš** (CZ-KARL-AM)

Uniformities and a quantale structure on localic groups. (English summary)

Bull. Belg. Math. Soc. Simon Stevin **28** (2022), no. 4, 561–578.

The present paper continues the study of the authors on the theory of point-free topology (which “forget[s] about points, and consider[s] a structure mimicking the behavior of classical open sets”) as in, e.g., [J. Picado and A. Pultr, *Frames and locales*, Front. Math., Birkhäuser/Springer Basel AG, Basel, 2012; [MR2868166](#); *Separation in point-free topology*, Birkhäuser/Springer, Cham, 2021; [MR4241435](#)]. In particular, they are interested in a point-free analogue of the machinery producing uniform spaces from topological groups.

More precisely, the algebraic structure on a classical topological group very naturally induces uniformities on the underlying space, which appears to be also the case in the more general localic groups (observe that a localic group is a group object in the category **Loc** of locales, but the authors work with its dual form in the category **Frm** of frames [P. T. Johnstone, *Stone spaces*, Cambridge Stud. Adv. Math., 3, Cambridge Univ. Press, Cambridge, 1982; [MR0698074](#)]). The respective uniformity (as a system of entourages modeling neighborhoods of the diagonal) is based in a suitably defined associative binary operation $*$ on the underlying frame L of a localic group. It appears that one has a quantale $(L, *)$ naturally associated with a localic group (the G -quantale, which is a special case of involutive quantales [D. Kruml and J. Paseka, in *Handbook of algebra. Vol. 5*, 323–362, Handb. Algebr., 5, Elsevier/North-Holland, Amsterdam, 2008; [MR2523454](#)]). The present authors study this quantale as well as characterize those quantales that can be obtained in this way. Moreover, they consider the properties of (enriched) quantales which allow reconstruction of the group structure of the respective localic group. Additionally, the (localic) group homomorphisms are shown to correspond to quantale homomorphisms.

Lastly, the authors note that $(L, *)$ is just an ordered semigroup, which can be used as a set of values of a generalized metric. The localic group uniformities obtained are then shown to be metric uniformities of these generalized metrics (by analogy with the canonical uniformities of metric spaces).

The paper is well written, provides most of its required preliminaries (the rest can be easily found in the suggested references at the end of the paper), and will be of interest to all researchers studying point-free topology.

Sergey A. Solov'ov

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.