WEIL ENTOURAGES

IN POINTFREE TOPOLOGY

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ABSTRACT

Uniform structures for frames and their generalizations (quasi-uniformities and nearnesses) are the subject of this thesis. Weil's notion of *entourage* is extended to this framework and it is proved that this is a basic concept on which that structures may be axiomatized. On the other hand, it is shown that uniform frames may also be described by gauge structures, that is, certain families of metric diameters.

CONTENTS

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INTRODUCTION

Mathematics place too much importance on the theorems people prove, and not enough on the definitions they devise. (The relative importance we ascribe is clear: theorems often have people's names attached; definitions rarely do). Yet it is definitions that give us the concepts that make thinking effective and make the theorems possible.

— S. Maurer

Sometimes I don't understand how people came across the concept of "fun"; it was probably only abstracted as an opposite to sadness.

— F. Kafka

In the thirties, Stone presented in the famous papers [71] and [72] two revolutionary ideas. Firstly, he concluded that ideals are very important in lattice theory, by showing that Boolean algebras are actually special instances of rings: the concept of Boolean algebra is equivalent to a certain type of ring — nowadays called *Boolean ring*.

Afterwards, following his maxim "*one must always topologize*" [73], he linked topology to lattice theory by establishing a representation theorem for Boolean algebras:

> *Every Boolean algebra is isomorphic to the Boolean algebra of open-closed sets of a totally disconnected compact Hausdorff space* (or, in other words, *compact zero-dimensional Hausdorff spaces*).

This theorem has had a great influence in many areas of modern mathematics (the reader may see a detailed description of it in the book [43] by Johnstone), namely in the study of topological concepts from a lattice theoretical point of view, initiated with Wallman in 1938 and followed with McKinsey and Tarski (1944), Nöbeling (1954), Lesieur (1954), Ehresmann (1957), Dowker and Papert (1966), Banaschewski (1969), Isbell (1972), Simmons (1978), Johnstone (1981), Pultr (1984), among others.

Ehresmann and Bénabou were the first (in 1957) to look at complete lattices with an appropriate distributive law — finite meets distribute over arbitrary joins — as "generalized" topological spaces. They called these lattices *local lattices*, meanwhile, named "*frames*" after Dowker and Papert.

Frame theory is lattice theory applied to topology. This approach to topology takes the lattices of open sets as the basic notion — it is a "*pointfree topology*". There, one investigates typical properties of lattices of open sets that can be expressed without reference to points.

Usually one thinks of frames as generalized spaces:

> "*The generalized spaces will be called locales. "Generalized" is imprecise, since arbitrary spaces are not determined by their lattices of open sets; but the "insertion" from spaces to locales is full and faithful on Hausdorff spaces*" (Isbell [39]).

Nevertheless, the frame homomorphisms — which should preserve finite meets and arbitrary joins — may only be interpreted as "generalized continuous maps" when considered in the dual category. Isbell, in his celebrated paper of 1972 "Atomless parts of spaces", was the first to stress this and to point out the need for a separate terminology for the dual category of frames, whose objects he named "*locales*", as cited above.

Johnstone in [43], [44] and [45] gives us a detailed account of these historical developments and of the advantages of this new way of doing topology as opposed to the classical one. As Isbell states in his review of [45] in "*Zentralblatt für Mathematik*",

> "*this paper is an argument that topology is better modeled in the category of locales than in topological spaces or another of their variants, with indication of how the millieu should be regarded and supporting illustrations*".

Frame theory has the advantage that many results which require the Axiom of Choice (or some of its variants) in the topological setting may be proved constructively in the frame setting. Examples are the Tychonoff Theorem [42], the construction of the Stone-Čech compactification [8] or the construction of the Samuel compactification [10]. By that reason, locales are the "right" spaces for topos theory [53].

Sometimes the frame situation differs from the classical one. In general, when this happens, the frame situation is more convenient. For example, coproducts of paracompact frames are paracompact [39] while products of paracompact spaces are not necessarily paracompact. Another example: coproducts of regular frames preserve the Lindelöf property [17], products of regular spaces do not.

One may also look to frames as the type of algebras behind the "logic of verifiable sentences". This is the approach of Vickers in "*Topology via Logic*" [74]:

> "*The traditional — spatial — motivation for general topology and its axioms relies on abstracting first from Euclidean space to metric spaces, and then abstracting out, for no obvious reason, certain properties of their open sets. I believe that the localic view helps to clarify these axioms, by interpreting them not as set theory (finite intersections and arbitrary unions), but as logic (finite conjunctions and arbitrary disjunctions: hence the title)*".
>
> ..."*I have tried to argue directly from these logical intuitions to the topological axioms, and to frames as the algebraic embodiment of them*".

The study of structured frames started with Isbell [39], who considered the notion of frame uniformity in the form of a system of covers, later developed by Pultr in [63] and [64], who also defined metric diameters for frames, using them in analogy with pseudometrics in the spatial setting. Subsequently, Frith [29] studied uniform-like structures from a more categorical point of view, introducing in frame theory other topological structures such as quasi-uniformities and proximities. Most of these concepts are formulated in terms of covers, and indeed Frith even stated that

> "*families of covers constitute the only tool that works for frames*" [29].

Nevertheless, Fletcher and Hunsaker [23] recently presented an equivalent notion of frame uniformity in terms of certain families of maps from the frame into itself.

This dissertation has its origin in the following suggestion of Professor Bernhard Banaschewski:

> "*We usually consider uniformities given by covers, as done by Tukey for spaces, but there should also be a theory (deliberately put aside by Isbell in "Atomless parts of spaces") of uniformities by entourages, in the style of Bourbaki*".

So, the study of structured frames via entourages in the style of Weil is the subject of this thesis. We begin with uniform structures (Chapter I) and in Chapters III and IV we investigate the corresponding natural generalizations of quasi-uniform and nearness structures. In parallel, with the aim of completing for frames a picture analogous to the one for spaces, we characterize uniform structures in terms of "gauge structures", that is, families of metric diameters satisfying certain axioms.

The language of this thesis is almost entirely algebraic and we never employ the geometric approach made possible by the language of *locales*. This point of view is corroborated by Madden [54]:

> "*There are differing opinions about this, and I appreciate that there are some very good reasons for wanting to keep the geometry in view. On the other hand, the algebraic language seems to me, after much experimentation, to afford the simplest and most streamlined presentation of results. Also, I think readers will not have much difficulty finding the geometric interpretations themselves, if they want them. After all, this ultimately comes down to just "reversing all the arrows"*".

Throughout this dissertation we also adopt the categorical point of view. The language of category theory has proved to be an adequate tool in the approach to the type of conceptual problems we are interested in, namely in the selection of the best axiomatizations for some frame structures and in the study of the relationship between them. Furthermore, this insight allows us to understand and to put in perspective the real meaning of these approaches. As Herrlich and Porst state in the preface to "*Category Theory at Work*":

> "*Some mathematical concepts appear to be "unavoidable", e.g. that of natural numbers. For other concepts such a claim seems debatable, e.g., for the concepts of real numbers or of groups. Other concepts — within certain limits — seem to be quite arbitrary, their use being based more on historical accidents than on structural necessities. A good example is the concept of topological spaces: compare such "competing" concepts as metric spaces, convergence spaces, pseudotopological spaces, uniform spaces, nearness spaces, frames respectively locales, etc. What are the structural "necessities" or at least "desirabilities"? Category theory provides a language to formulate such questions with the kind of precision needed to analyse advantages and disadvantages of various alternatives. In particular, category theory enables us to decide whether certain mathematical "disharmonies" are due to inherent structural features or rather to chance ocurrences, and in the latter case helps to "set things right"* ".

This is the point of view adopted in this work as well as in the articles [59], [60], and [61] on which it relies. Another thread of our work is the search, in each setting, for an adjunction between structured versions of the "open" and "spectrum" functors. These functors acted as a categorical guide of the accuracy of the choosen axiomatizations.

We describe now this thesis in more detail:

Chapter 0 introduces well-known basic definitions and results needed in the body of the dissertation.

Chapter I, which is the core of this work, presents a theory of frame uniformities in the style of Weil (Section 4) which is equivalent to the ones of Isbell [39] and Fletcher and Hunsaker [23], as is proved in Theorem 5.14. The chapter ends with an application of this theory to the study, in the setting of frames, of an important theorem of the theory of uniform spaces, due to Efremovič. Our approach via entourages reveals to be the right language to yield the result analogous to the one of Efremovič in the context of frames.

After Chapter I it would be natural to investigate the non-symmetric structures (as well as the nearness structures) that arise from our theory of uniformities. However the existence of another characterization of uniform spaces due to Bourbaki [14] and the notion of metric diameter introduced by Pultr [64] lead us to investigate a way of describing frame uniformities in terms of those diameters. This is what it is done in the beginning of Chapter II. As an aplication of this characterization, inspired by the paper [2] of Adámek and Reiterman, it is shown in Corollary 4.20 that the category of uniform frames is fully embeddable in a (final and universal) completion of the category of metric frames. Therefore, metric frames provide a categorical motivation for uniform frames.

In Chapters III and IV we come back to the main stream of our work. The third section of Chapter III contains the axiomatization of a theory of quasi-uniformities via Weil entourages. Theorem 4.15 shows that this is an equivalent theory to the existing one in the literature.

In the final chapter we proceed to another level of generality by studying nearness structures in frames using Weil entourages. In this case the corresponding spatial structures, which seem unnoticed so far, appear as a topic worthy of study. Although distinct from the classical nearness spaces of Herrlich [33], this class of spaces forms a nice topological category (Proposition 5.1) which unifies several topological and uniform concepts (Propositions 5.4, 5.5, 5.6, 5.8 and Corollary 5.15). The notion of Weil entourage is therefore a basic topological concept by means of which various topological ideas may be expressed. In the last section we study proximal frames. Theorem 6.10 gives a new characterization of this type of frames in terms of Weil entourages. The infinitesimal relations of Efremovič [19] are also studied in the context of frames and the chapter ends with a remark which once more exhibits the adequacy of Weil entourages to bring out the meaning, in the context of frames, of the spatial results formulated in terms of entourages.

At the end of each chapter there is a section with additional references and comments.

An appendix (on page 141) contains two diagrams which summarize the relations between the various categories of spaces and frames presented along the text. We also list categories (page 151), symbols (page 153) and definitions (page 157) used throughout.

Our concern in relating the concepts introduced here with the ones already existing in the literature as well as in motivating the ideas developed in frames with the spatial situations, justifies the extensive bibliography included.

Proofs of already known results will usually be ommited.

We think that all results included in Chapters I, II, III and IV for which no reference is given are original.

In general, choice principles as the Axiom of Choice or the Countable Dependent Axiom of Choice are used without mention.

CHAPTER 0

PRELIMINARIES

This chapter is a summary of the relevant background from the literature which will be required in the remaining chapters. All results are well known. Our main reference for information on frames is [43], and on category theory [52] and [1]. Any introductory text such as, for instance, [77] is a good reference for topology.

1. Frames and topological spaces

In the same way as the notion of Boolean algebra appears as an abstraction of the power set $\mathcal{P}(X)$ of a set X, the notion of frame arises as an abstraction from the topology $\mathcal{T}$ of a topological space $(X, \mathcal{T})$: a *frame* is a complete lattice L satisfying the distributive law

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$$

for all $x \in L$ and $S \subseteq L$. A *frame homomorphism* is a map between frames which preserves finitary meets (including the unit, or top, 1) and arbitrary joins (including the zero, or bottom, 0). Observe that the lattice structure of a frame does not differ from that of a complete Heyting algebra since, by the Freyd's Adjoint Functor Theorem, a complete lattice satisfies the above distributive law if and only if it is an Heyting algebra.

The category of frames and frame homomorphisms will be denoted by Frm.

As we pointed above, the motivating examples of frames are topologies: for every topological space $(X, \mathcal{T})$ the lattice $\mathcal{T}$ of open sets of $(X, \mathcal{T})$, where the join operator is the union and the meet operator is the interior of the intersection, is a frame. Any frame of this type is called *spatial*. There are non-spatial frames: any non-atomic complete Boolean algebra is not a topology of some set. Nevertheless there is an important relationship between frames and topological spaces which we describe below. The category of topological spaces and continuous maps will be denoted by Top.

Definitions 1.1.

(1) The contravariant functor $\Omega : \mathsf{Top} \longrightarrow \mathsf{Frm}$ which assigns to each topological space $(X, \mathcal{T})$ its frame $\mathcal{T}$ of open sets and to each continuous function $f : (X, \mathcal{T}) \longrightarrow (X', \mathcal{T}')$ the frame map $\Omega(f) : \mathcal{T}' \longrightarrow \mathcal{T}$ given by $\Omega(f)(U) = f^{-1}(U)$, where $U \in \mathcal{T}'$, is called the *open functor* from Top to Frm.

(2) Let L be a frame. The *spectrum* of L is the set ptL of all frame homomorphisms $p : L \longrightarrow \mathbf{2}$ ($\mathbf{2}$ denotes the two-point frame $\{0, 1\}$), the so-called *points* of L, with the *spectral topology* $\mathcal{T}_{ptL} = \{\Sigma_x : x \in L\}$ where $\Sigma_x = \{p \in ptL \mid p(x) = 1\}$. The contravariant functor $\Sigma : \mathsf{Frm} \longrightarrow \mathsf{Top}$ which assigns to each frame its spectrum $\Sigma(L) = (ptL, \mathcal{T}_{ptL})$ and to each frame map $f : L \longrightarrow L'$ the continuous map $\Sigma(f) : \Sigma(L') \longrightarrow \Sigma(L)$ given by $\Sigma(f)(p) = p \cdot f$, where p is a point of L', is called the *spectrum functor* from Frm to Top.

Theorem 1.2. (Papert and Papert [58], Isbell [39]) *The open and spectrum functors and the natural transformations*

$$\eta : 1_{\mathsf{Top}} \longrightarrow \Sigma\Omega \quad \textit{and} \quad \xi : 1_{\mathsf{Frm}} \longrightarrow \Omega\Sigma$$

given by

$$\eta_{(X,\mathcal{T})}(x)(U) = \begin{cases} 1 & \textit{if } x \in U \\ 0 & \textit{if } x \notin U \end{cases}$$

and

$$\xi_L(x) = \Sigma_x,$$

define a dual adjunction between the category of topological spaces and the category of frames. ■

This dual adjunction restricts to a dual equivalence between the full subcategories of sober spaces [43] and spatial frames, and this is the "largest" duality contained in the given dual adjunction since these are the fixed objects of the adjunction. One may think of frames as generalized topological spaces, taking into account that a topological space is essentially determined by the frame of its open sets when it is sober but, beyond that point, spaces and frames diverge.

Since Ω is contravariant, when thinking topologically on frames one usually works in the dual category of Frm, called the category Loc of *locales*. Now $\Omega : \mathsf{Top} \longrightarrow \mathsf{Loc}$ and $\Sigma : \mathsf{Loc} \longrightarrow \mathsf{Top}$ are covariant. Throughout this thesis, however, we shall always stay in Frm.

We list some notions and properties of frames that will be relevant in the sequel.

A subset M of a frame L is a *subframe* of L if $0, 1 \in M$ and M is closed under finite meets and arbitrary joins.

A frame L is called:

- *compact* if $1 = \bigvee S$ implies $1 = \bigvee S'$ for some finite $S' \subseteq S$;
- *regular* if, for every $x \in L$, $x = \bigvee\{y \in L \mid y \prec x\}$, where $y \prec x$ means that $y \wedge z = 0$ and $x \vee z = 1$ for some $z \in L$. In terms of the *pseudocomplement*

$$y^* := \bigvee\{z \in L \mid z \wedge y = 0\},$$

  this is equivalent to saying that $y^* \vee x = 1$.
- *normal* if $x \vee y = 1$ implies the existence of u and v in L satisfying $u \wedge v = 0$ and $x \vee u = 1 = y \vee v$.

Note that, for any topological space $(X, \mathcal{T})$, the frame $\Omega(X, \mathcal{T})$ is compact, regular or normal if and only if $(X, \mathcal{T})$ is, respectively, compact, regular or normal in the usual topological sense.

Further, any compact regular frame is normal.

In any frame the DeMorgan formula

$$\left(\bigvee_{i\in I} x_i\right)^* = \bigwedge_{i\in I} x_i^*$$

holds.

For infima we have only the trivial inequality

$$\bigvee_{i\in I} x_i^* \leq \left(\bigwedge_{i\in I} x_i\right)^*.$$

Note also that in case $f : L \longrightarrow L'$ is a frame homomorphism we have only the trivial inequality $f(x^*) \leq f(x)^*$. Clearly, if L is a *Boolean frame*, that is, a Boolean algebra, the equality $f(x^*) = f(x)^*$ holds for all $x \in L$. Note that a frame L is Boolean if and only if $x \vee x^* = 1$ for every $x \in L$ or, equivalently, $x^{**} = x$ for every $x \in L$.

2. Biframes and bitopological spaces

Similarly, as topological spaces motivate frames, bitopological spaces (first studied by Kelly [47]) motivate the notion of biframe. This idea is due to Banaschewski, Brümmer and Hardie [7]. A *biframe* is a triple (L_0, L_1, L_2) in which L_0 is a frame and L_1 and L_2 are subframes of L_0 such that each element of L_0 is the join of finite meets from $L_1 \cup L_2$. A *biframe map* $f : (L_0, L_1, L_2) \longrightarrow (L_0', L_1', L_2')$ is a frame map from L_0 to L_0' which maps L_i into L_i' ($i \in \{1, 2\}$).

We denote the categories of bitopological spaces and bicontinuous maps and of biframes and biframe maps by BiTop and BiFrm, respectively. The dual adjunction between topological spaces and frames may be extended to one between bitopological spaces and biframes [7]:

- The contravariant *open functor* $\Omega : \mathsf{BiTop} \longrightarrow \mathsf{BiFrm}$ assigns to each bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$ the biframe $(\mathcal{T}_1 \vee \mathcal{T}_2, \mathcal{T}_1, \mathcal{T}_2)$ where $\mathcal{T}_1 \vee \mathcal{T}_2$ is the coarsest topology finer than $\mathcal{T}_1$ and $\mathcal{T}_2$. For a bicontinuous map $f : (X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow (X', \mathcal{T}_1', \mathcal{T}_2')$, $\Omega(f) : \Omega(X', \mathcal{T}_1', \mathcal{T}_2') \longrightarrow \Omega(X, \mathcal{T}_1, \mathcal{T}_2)$ is given by $\Omega(f)(U) = f^{-1}(U)$ for $U \in \mathcal{T}_1' \vee \mathcal{T}_2'$.

- The contravariant *spectrum functor* $\Sigma : \mathsf{BiFrm} \longrightarrow \mathsf{BiTop}$ is defined as follows: for a biframe $L = (L_0, L_1, L_2)$, $\Sigma(L) = (ptL_0, \{\Sigma_x : x \in L_1\}, \{\Sigma_y : y \in L_2\})$ where each Σ_x, for $x \in L_1 \cup L_2$, is the set $\{p \in ptL_0 : p(x) = 1\}$. For a biframe map $f : L \longrightarrow L'$, the bicontinuous map $\Sigma(f) : \Sigma(L') \longrightarrow \Sigma(L)$ is given by $\Sigma(f)(\xi) = \xi \cdot f$.

- These functors define a dual adjunction between BiTop and BiFrm, with the adjunction units

$$\eta_{(X,\mathcal{T}_1,\mathcal{T}_2)} : (X, \mathcal{T}_1, \mathcal{T}_2) \longrightarrow \Sigma\Omega(X, \mathcal{T}_1, \mathcal{T}_2)$$

and

$$\xi_{(L_0,L_1,L_2)} : (L_0, L_1, L_2) \longrightarrow \Omega\Sigma(L_0, L_1, L_2)$$

given by

$$\eta_{(X,\mathcal{T}_1,\mathcal{T}_2)}(x)(U) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \notin U \end{cases}$$

and

$$\xi_{(L_0,L_1,L_2)}(x) = \Sigma_x.$$

For more details on biframes see [7] and [69].

3. Quotients of frames

Since a frame is an algebraic structure, there is a convenient way of constructing frame quotients as quotients by congruences ([42], [48]):

Let R be a binary relation on a frame L. An element $x \in L$ is *R-saturated* if, for every y, z, w in L,

$$y \wedge w \leq x \Longleftrightarrow z \wedge w \leq x$$

whenever $(y, z) \in R$. We are using the terminology of [49]; these are the *R-coherent* elements for Kříž [48] and the *R-compatible* elements for Banaschewski [3]. Meets of R-saturated elements are R-saturated and hence one can define a map $\kappa : L \longrightarrow L$ being $\kappa(x)$ the least R-saturated element above x. Put $L/R := \kappa(L)$.

Theorem 3.1. (Kříž [48])

(a) *L/R endowed with the meets from L and joins defined by $\bigsqcup_{i\in I} x_i = \kappa(\bigvee_{i\in I} x_i)$ is also a frame and $\kappa : L \longrightarrow L/R$ is a frame homomorphism such that $\kappa(x) = \kappa(y)$ whenever $(x, y) \in R$.*

(b) *If $f : L \longrightarrow L'$ is a frame homomorphism such that $(x, y) \in R$ implies $f(x) = f(y)$, then there is exactly one frame homomorphism $g : L/R \longrightarrow L'$ such that $g \cdot \kappa = f$.* ■

4. Down-sets and filters

For a subset A of a preordered set $(L, \leq)$, let

$$\downarrow A := \{x \in L \mid x \leq a \text{ for some } a \in A\}$$

and

$$\uparrow A := \{x \in L \mid a \leq x \text{ for some } a \in A\}.$$

The set A is said to be a *down-set* (respectively, an *upper-set*) of L if $\downarrow A = A$ (respectively, $\uparrow A = A$). Since the intersection of two down-sets is a down-set, the set $\mathcal{D}(L)$ of all down-sets of L is a complete lattice (it is even a frame).

We shall denote by $[x, y]$ the intersection $\uparrow\{x\} \cap \downarrow\{y\}$.

Assuming that $(L, \leq)$ has finite meets (in particular, containing a unit 1), a subset F of L is a *filter* of $(L, \leq)$ if it is an upper-set closed under finite meets (in particular, containing the unit 1). A subset F' of a filter F is a *basis* of F if $\uparrow F' = F$. A set $F' \subseteq L$ is a filter for some filter on L if and only if for every $x, y \in F'$ there exists $z \in F'$ such that $z \leq x \wedge y$. In this case the *filter generated* by F', that is, the filter for which F' is a basis, is the set $\uparrow F'$.

5. Binary coproducts of frames

Let L_1 and L_2 be frames. Recall (cf., e.g., [17] or [43]) that the coproduct of frames L_1 and L_2

$$L_1 \xrightarrow{u_{L_1}} L_1 \oplus L_2 \xleftarrow{u_{L_2}} L_2$$

can be constructed as follows: take the cartesian product $L_1 \times L_2$ with the usual order. One obtains $L_1 \oplus L_2$ as the frame $\mathcal{D}(L_1 \times L_2)/R$ where R consists of all pairs of the type

$$\Big(\downarrow\{(x,0)\},\emptyset\Big),$$
$$\Big(\downarrow\{(0,y)\},\emptyset\Big),$$
$$\Big(\downarrow\Big\{(\bigvee S,y)\Big\},\bigcup_{s\in S}\downarrow\Big\{(s,y)\Big\}\Big)$$

and

$$\Big(\downarrow\Big\{(x,\bigvee S)\Big\},\bigcup_{s\in S}\downarrow\Big\{(x,s)\Big\}\Big).$$

Equivalently, defining a C-*ideal* of $L_1 \times L_2$ as a down-set $A \subseteq L_1 \times L_2$ satisfying

$$\{x\}\times S \subseteq A \;\Rightarrow\; \Big(x,\bigvee S\Big)\in A$$

and

$$S\times\{y\} \subseteq A \;\Rightarrow\; \Big(\bigvee S,y\Big)\in A,$$

since the intersection of C-ideals is again a C-ideal, the set of all C-ideals of $L_1 \times L_2$ is a frame in which

$$\bigwedge_{i\in I} A_i = \bigcap_{i\in I} A_i$$

and

$$\bigvee_{i\in I} A_i = \bigcap\{B \mid B \text{ is a } C\text{-ideal and } \bigcup_{i\in I} A_i \subseteq B\};$$

this is the frame $L_1 \oplus L_2$. Observe that the case $S=\emptyset$ implies that every C-ideal contains the C-ideal $\downarrow(1,0)\cup\downarrow(0,1)$, which we shall denote by $\mathbb{O}_{L_1\oplus L_2}$ (or just by $\mathbb{O}$ whenever there is no ambiguity). This is the zero of $L_1 \oplus L_2$. Obviously, each $\downarrow(x,y)\cup\mathbb{O}$ is a C-ideal. It is denoted by $x\oplus y$. Finally put $u_{L_1}(x)=x\oplus 1$ and $u_{L_2}(y)=1\oplus y$. The following clear facts are useful:

- For every $E \in L_1 \oplus L_2$, $E = \bigvee\{x \oplus y \mid (x, y) \in E\}$, and so the C-ideals of the type $x \oplus y$ generate by joins the frame $L_1 \oplus L_2$;

- $x \oplus y = \mathbb{O}$ if and only if $x = 0$ or $y = 0$;

- $\bigvee\{x \oplus s \mid s \in S\} = x \oplus (\bigvee S)$ and $\bigvee\{s \oplus y \mid s \in S\} = (\bigvee S) \oplus y$;

- $\bigcap\{s \oplus t \mid s \in S, t \in T\} = (\bigwedge S) \oplus (\bigwedge T)$;

- $x \leq y$ and $z \leq w$ imply $x \oplus z \subseteq y \oplus w$;

- $\mathbb{O} \neq x \oplus y \subseteq z \oplus w$ implies $x \leq z$ and $y \leq w$.

Note that $L_1 \oplus \mathbf{2}$ is isomorphic to L_1: consider the coproduct diagram

$$L_1 \xrightarrow{1} L_1 \xleftarrow{\sigma} \mathbf{2}$$

where σ is the unique morphism $\mathbf{2} \longrightarrow L_1$. Under this isomorphism the element $x \oplus 1$ is identified with x and $x \oplus 0$ with 0.

For any frame homomorphisms $f_i : L_i \longrightarrow L_i'$, $(i \in \{1, 2\})$, we write $f_1 \oplus f_2$ for the unique morphism from $L_1 \oplus L_2$ to $L_1' \oplus L_2'$ that makes the following diagram commutative

$$\begin{tikzcd}
L_1 \arrow[r, "u_{L_1}"] \arrow[d, "f_1"'] & L_1 \oplus L_2 \arrow[d, dashed, "f_1 \oplus f_2"] & L_2 \arrow[l, "u_{L_2}"'] \arrow[d, "f_2"] \\
L_1' \arrow[r, "u_{L_1'}"] & L_1' \oplus L_2' & L_2' \arrow[l, "u_{L_2'}"']
\end{tikzcd}$$

Obviously,

$$(f_1 \oplus f_2)\Big(\bigvee_{\gamma \in \Gamma} (x_\gamma \oplus y_\gamma)\Big) = \bigvee_{\gamma \in \Gamma} \Big(f_1(x_\gamma) \oplus f_2(y_\gamma)\Big).$$

6. Galois connections

We end this chapter with a notion that the reader already met, at least in the setting of partially ordered sets. A *Galois connection* between partially ordered sets A and B is defined as a pair (f, g) of order-preserving maps $f : B \longrightarrow A$ and $g : A \longrightarrow B$ with the property that, for all $a \in A$ and $b \in B$,

$$f(b) \leq a \text{ if and only if } b \leq g(a).$$

The latter condition is equivalent to

$$f \cdot g \leq 1_A \text{ and } 1_B \leq g \cdot f.$$

If we view A, B in the standard way as categories (see MacLane [52]) and f, g as functors, a Galois connection expresses an adjunction between A and B: (f, g) is a Galois connection if and only if g is left adjoint to f (and, in this case, one writes $g \dashv f$).

Unfortunately most of the interesting results about Galois connections are no longer valid for adjoint situations. Nevertheless for the following level of generality the interesting properties of Galois connections remain valid (see [36]):

Let $\mathcal{A}$ and $\mathcal{B}$ be concrete categories and let $G : \mathcal{A} \longrightarrow \mathcal{B}$ and $F : \mathcal{B} \longrightarrow \mathcal{A}$ be concrete functors; the pair (F, G) is called a *Galois correspondence* provided that

$$F \cdot G \leq 1_{\mathcal{A}} \text{ and } 1_{\mathcal{B}} \leq G \cdot F.$$

Notes on Chapter 0:

(1) We cite the first chapter of *Stone Spaces* [43] as a reference for the basic notions of Lattice Theory we use throughout. The *Compendium of Continuous Lattices* [31] is also a useful reference for the type of lattice theoretical results we work with.

(2) The adjunction of Theorem 1.2 was first constructed in the Seminar of Erhesmann (1958) by Papert and Papert [58] and later developed by Isbell in [39].

(3) The description of factorization of frames and Theorem 3.1 are due to Kříž [48]. The main idea of his approach belongs to Johnstone [42] who formulated the theorem in a slightly less general way.

CHAPTER I

WEIL UNIFORM FRAMES

It is well known that a uniformity on a set X may be described in any of three equivalent ways: as a family of reflexive binary relations ("entourages") on X, as a family of covers ("uniform covers") of X, or as a family of pseudometrics ("gauges") on X.

In pointfree topology, the notion of uniformity — in the form of a system of covers — was introduced by Isbell in [39], and later developed by Pultr in [63] and [64]. Since there is a current interest in uniform frames, it would seem desirable to complete the picture for frames, that is, to have characterizations of frame uniformities that are analogous to those spatial ones given in terms of Weil entourages and pseudometrics. This is the main motivation for this thesis.

Recently, another equivalent notion of frame uniformity was given by Fletcher and Hunsaker in [23], which they called "entourage uniformity". The purpose of this first chapter is to present another characterization of frame uniformities in the style of Weil. We formulate and investigate a definition of entourage uniformity — alternative to that one of Fletcher and Hunsaker — which, in our opinion, is more likely to the Weil pointed entourage uniformity, since it is expressed in terms of coproducts of frames (i.e., products of locales). We identify this new notion with the existing ones by proving the (concrete) isomorphism between the respective categories.

1. Uniform spaces

Uniformities on a set X were introduced in the thirties by André Weil, in terms of subsets of $X \times X$ containing the diagonal

$$\Delta_X := \{(x,x) \,:\, x \in X\},$$

called “entourages” or “surroundings”. The classical account of this subject is in Chapter II of Bourbaki [15].

Definitions 1.1. (Weil [76]) Let X be a set.

(a) A subset E of $X \times X$ is called an *entourage* of X if it contains the diagonal Δ_X.

(b) A *uniformity* on X is a set $\mathcal{E}$ of entourages of X such that:

(UW1) $\mathcal{E}$ is a filter of the complete lattice $(WEnt(X), \subseteq)$ of all entourages of X;

(UW2) for each $E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ such that the entourage

$$F \circ F := \{(x,y) \in X \times X \mid \text{ there is } z \in X \text{ such that } (x,z),(z,y) \in F\}$$

is contained in E;

(UW3) for every $E \in \mathcal{E}$ the set

$$E^{-1} := \{(x,y) \in X \times X \mid (y,x) \in E\}$$

is also in $\mathcal{E}$.

The pair $(X, \mathcal{E})$ is then called a *uniform space*.

(c) A map $f : (X,\mathcal{E}) \longrightarrow (X', \mathcal{E}')$, where $(X,\mathcal{E})$ and $(X',\mathcal{E}')$ are uniform spaces, is *uniformly continuous* if, for every $E \in \mathcal{E}'$, $(f \times f)^{-1}(E) \in \mathcal{E}$.

A *basis of a uniformity* $\mathcal{E}$ is a subfamily $\mathcal{E}'$ of $\mathcal{E}$ such that $\uparrow \mathcal{E}' = \mathcal{E}$. A collection $\mathcal{E}'$ of entourages of X is therefore a basis of some uniformity if and only if it is a filter basis of $(WEnt(X), \subseteq)$ satisfying (UW2) and the condition

for every $E \in \mathcal{E}'$ there exists $F \in \mathcal{E}'$ such that $F^{-1} \subseteq E$.

If $(X, \mathcal{E})$ is a uniform space a topology $\mathcal{T}_\mathcal{E}$ on X (the *uniform topology*) is defined as follows:

> $A \in \mathcal{T}_\mathcal{E}$ if for every $x \in A$ there exists $E \in \mathcal{E}$ such that $E[x] := \{y \in X \mid (x, y) \in E\} \subseteq A$.

For $A, B \subseteq X$, we write $A \overset{\mathcal{E}}{\lhd} B$ if $E \circ (A \times A) \subseteq B \times B$ for some $E \in \mathcal{E}$. Since $A \times A \subseteq E \circ (A \times A)$, the order relation $\overset{\mathcal{E}}{\lhd}$ is stronger than the inclusion $\subseteq$. The following result is well known:

Proposition 1.2. (Weil [76]) *Let $(X, \mathcal{E})$ be a uniform space. Then, for every $A \in \mathcal{T}_\mathcal{E}$, $A = \bigcup\{B \in \mathcal{T}_\mathcal{E} \mid B \overset{\mathcal{E}}{\lhd} A\}$.* ∎

There is another equivalent axiomatization of the notion of uniformity, due to Tukey [75], in which the basic term is the one of "uniform cover" of X:

Definitions 1.3. (Tukey [75]) Let X be a set.

(a) A subset $\mathcal{U}$ of $\mathcal{P}(X)$ is a *cover* of X if $\bigcup_{U \in \mathcal{U}} U = X$. For $A \subseteq X$,

$$st(A, \mathcal{U}) := \bigcup\{U \in \mathcal{U} \mid U \cap A \neq \emptyset\}$$

is called the *star of A in $\mathcal{U}$*. A cover $\mathcal{U}$ *refines* a cover $\mathcal{V}$, and in this case one writes $\mathcal{U} \leq \mathcal{V}$, if for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subseteq V$.

(b) A *covering uniformity* on X is a set μ of covers of X such that:

(U1) μ is a filter of the preordered set $(Cov(X), \leq)$ of all covers of X;

(U2) for each $\mathcal{U} \in \mu$ there is $\mathcal{V} \in \mu$ such that the cover

$$\mathcal{V}^* := \{st(V, \mathcal{V}) \mid V \in \mathcal{V}\}$$

refines $\mathcal{U}$.

A *basis of a covering uniformity* μ is a subfamily μ' of μ such that $\uparrow \mu' = \mu$. Evidently, μ' is a basis for some covering uniformity if and only if it is a filter basis of $(Cov(X), \leq)$ such that, for every $\mathcal{U} \in \mu'$, there exists $\mathcal{V} \in \mu'$ satisfying $\mathcal{V}^* \leq \mathcal{U}$.

Theorem 1.4. (Tukey [75]) *The family μ of all uniform covers of a uniform space $(X,\mathcal{E})$, that is, the family of all covers refined by a cover of the form $\{E[x]$ for some $E\in\mathcal{E}$ and $x\in X$, is a filter basis for a covering uniformity on X.*

Conversely, given any covering uniformity μ on a set X, the family of all sets

$$\bigcup_{U\in\mathcal{U}}(U\times U),$$

for $\mathcal{U}\in\mu$, is a basis for a uniformity on X, whose uniform covers are precisely the elements of μ. ■

From a categorical point of view this means that there is a concrete isomorphism between the category of uniform spaces of Weil and the category of uniform spaces of Tukey (which are concrete categories over the category of sets). Informally, this means that the description of these "structured sets", although distinct, are essentially the same and we may substitute one structure for the other with no problem.

Thus the uniform covers also describe a uniformity and one may define a uniform space as a pair (X,μ) formed by a set X and a family μ of covers of X satisfying axioms (U1) and (U2) of Definition 1.3.

With this language, the uniform topology induced by $(X,\mathcal{U})$ is the set of all $A\subseteq X$ such that for every $x\in A$ there exists $\mathcal{U}\in\mu$ satisfying $st(x,\mathcal{U})\subseteq A$. Proposition 1.2 has now the following formulation:

Proposition 1.5. (Tukey [75]) *Let (X,μ) be a uniform space given in terms of covers and let $\mathcal{T}_\mu$ be the associated topology on X. Then, for every $A\in\mathcal{T}_\mu$, $A=\bigcup\{B\in\mathcal{T}_\mu \mid B\overset{\mu}{\lhd}A\}$, where $B\overset{\mu}{\lhd}A$ means that there is $\mathcal{U}\in\mu$ such that $st(B,\mathcal{U})\subseteq A$.* ■

A map $f:(X,\mu)\longrightarrow(X',\mu')$, where (X,μ) and (X',μ') are uniform spaces, is uniformly continuous if and only if, for every $\mathcal{U}\in\mu'$,

$$f^{-1}[\mathcal{U}]:=\{f^{-1}(U)\mid U\in\mathcal{U}\}$$

belongs to μ.

2. Covering uniform frames

It was Tukey's approach to uniform spaces via covers that was first studied in the pointfree context of frames. In [39] Isbell introduced uniformities on frames, as the precise translation into frame terms of Tukey's notion of a uniformity of a space expressed in terms of open covers, later developed in detail by Pultr ([63], [64]). Subsequently, Frith [29] studied uniform-type structures from a more categorical point of view, also making use of frame covers.

From here on throughout the remainder of this thesis, L will always denote a frame.

A set $U \subseteq L$ is a *cover* of L if $\bigvee_{x \in U} x = 1$. For $x \in L$, the element

$$st(x, U) := \bigvee \{y \in U \mid y \wedge x \neq 0\}$$

is called the *star of x in U*. The set of covers of L can be preordered: a cover U *refines* a cover V, written $U \leq V$, if for each $x \in U$ there is $y \in V$ with $x \leq y$. This is a preordered set with meets and joins: take for $U \wedge V$ the cover $\{x \wedge y \mid x \in U, y \in V\}$ and for $U \vee V$ just the union $U \cup V$.

For each family $\mathcal{U}$ of covers of L let us consider the order relation in L defined by

$$x \overset{\mathcal{U}}{\triangleleft} y \text{ (read "}x\text{ is }\mathcal{U}\text{-strongly below }x\text{") if there is } U \in \mathcal{U} \text{ such that } st(x, U) \leq y.$$

Note that these orders are indeed stronger than $\leq$ because $x \leq st(x, U)$ for every cover U.

Definition 1.3 and Proposition 1.5 motivate the notion of uniform frame:

Definitions 2.1. (Isbell [39])

(a) Let L be a frame. A family $\mathcal{U}$ of covers of L is a *uniformity basis* provided that:

(U1) $\mathcal{U}$ is a filter basis of the preordered set $(Cov(L), \leq)$ of all covers of L;

(U2) every $U \in \mathcal{U}$ has a star-refinement, i.e., for every $U \in \mathcal{U}$ there is $V \in \mathcal{U}$ with

$$V^* := \{st(x, V) \mid x \in V\} \leq U;$$

(U3) for every $x \in L$, $x = \bigvee \{y \in L \mid y \overset{\mathcal{U}}{\triangleleft} x\}$.

A *uniformity* on L is a filter $\mathcal{U}$ of covers of L generated by some uniformity basis. The pair $(L,\mathcal{U})$ is then called a *uniform frame*.

(b) Let $(L,\mathcal{U})$ and $(L',\mathcal{U}')$ be uniform frames. A frame homomorphism $f : L \longrightarrow L'$ is a *uniform homomorphism* if, for every $U \in \mathcal{U}$, $f(U) = \{f(x) \mid x \in U\} \in \mathcal{U}'$.

We denote by UFrm the category of uniform frames and uniform homomorphisms.

The category UFrm can be related to the category Unif of uniform spaces and uniformly continuous maps via the open and spectrum functors [29]:

- The open functor $\Omega : \mathsf{Unif} \longrightarrow \mathsf{UFrm}$ assigns to each uniform space (X, μ) the uniform frame $(\mathcal{T}_\mu, \mathcal{U}_{\mathcal{T}_\mu})$, where $\mathcal{T}_\mu$ is the topology induced by μ and $\mathcal{U}_{\mathcal{T}_\mu}$ is the collection of the $\mathcal{T}_\mu$-open covers of μ. If $f : (X, \mu) \longrightarrow (X', \mu')$ is uniformly continuous, then $\Omega(f) : \Omega(X', \mu') \longrightarrow \Omega(X, \mu)$ defined by $\Omega(f)(U) = f^{-1}(U)$ is a uniform homomorphism.

- The spectrum functor $\Sigma : \mathsf{UFrm} \longrightarrow \mathsf{Unif}$ assigns to each uniform frame $(L,\mathcal{U})$ the uniform space (ptL, μ_{ptL}), being μ_{ptL} the filter of covers of ptL generated by $\Big\{\{\Sigma_x : x \in U\} \mid U \in \mathcal{U}\Big\}$, where $\Sigma_x = \{p \in ptL \mid p(x) = 1\}$ as defined previously. If $f : (L,\mathcal{U}) \longrightarrow (L',\mathcal{U}')$ is a uniform homomorphism then $\Sigma(f) : \Sigma(L',\mathcal{U}') \longrightarrow \Sigma(L,\mathcal{U})$, given by $\Sigma(f)(p) = p \cdot f$, is uniformly continuous.

Theorem 2.2. (Frith [29]) *The two above contravariant functors Ω and Σ define a dual adjunction, with the adjunction units*

$$\eta_{(X,\mu)} : (X, \mu) \longrightarrow \Sigma\Omega(X, \mu) \text{ and } \xi_{(L,\mathcal{U})} : (L,\mathcal{U}) \longrightarrow \Omega\Sigma(L,\mathcal{U})$$

given by $\eta_{(X,\mu)}(x)(U) = 1$ if and only if $x \in U$ and $\xi_{(L,\mathcal{U})}(x) = \Sigma_x$. ■

3. Entourage uniform frames

In [39] the author also suggested a theory of frame uniformities by entourages but, intentionally, put it aside:

"*Entourages ought to work, but not in the present state of knowledge of product locales*".

In a recent article [23] Fletcher and Hunsaker were the first to approach this problem and presented a new theory of uniformities, which they proved to be equivalent to the covering one, and where the basic term is the one of "entourage" which names certain maps from the frame into itself. Let us briefly recall it:

Let $\mathcal{O}(L)$ be the collection of all order-preserving maps from L to L. The pair $(\mathcal{O}(L), \leq)$ where $\leq$ is defined pointwisely, that is,

$$e \leq f \equiv \ \forall\, x \in L \; e(x) \leq f(x)$$

is a frame; the join $\vee$ and the meet $\wedge$ are also defined pointwisely:

$$(e \vee f)(x) = e(x) \vee f(x) \text{ and } (e \wedge f)(x) = e(x) \wedge f(x) \text{ for every } x \in L.$$

If $e \in \mathcal{O}(L)$ and $x \in L$, then x is *e-small* if $x \leq e(y)$ whenever $x \wedge y \neq 0$. A map e of $\mathcal{O}(L)$ is an *entourage* of L if the set of all e-small elements of L is a cover. Note that, in this case, $x \leq e(x)$ for every $x \in L$ (and, consequently, $e^n \leq e^{n+1}$, for every natural n) . Indeed, we have

$$x = x \wedge \bigvee\{y \in L \mid y \text{ is } e\text{-small}\} = \bigvee\{x \wedge y \mid y \text{ is } e\text{-small and } x \wedge y \neq 0\} \leq e(x).$$

For any set $\mathcal{M}$ of entourages of L and $x, y \in L$, the relation $x \overset{\mathcal{M}}{\triangleleft} y$ means that there exists $e \in \mathcal{M}$ with $e(x) \leq y$. Obviously, $x \overset{\mathcal{M}}{\triangleleft} y$ implies $x \leq y$.

Definitions 3.1. (Fletcher and Hunsaker [23])

(a) Let L be a frame. A family $\mathcal{M}$ of entourages of L is an *entourage uniformity basis* provided that:

(UE1) for any $e, f \in \mathcal{M}$ there exists a join-homomorphism $g \in \mathcal{M}$ such that $g \leq e \wedge f$;

(UE2) for each $e \in \mathcal{M}$ there is $f \in \mathcal{M}$ such that $f \cdot f \leq e$;

(UE3) for every $e \in \mathcal{M}$ and $x, y \in L$, $x \wedge e(y) = 0$ if and only if $e(x) \wedge y = 0$;

(UE4) for every x in L, $x = \bigvee\{y \in L \mid y \overset{\mathcal{M}}{\triangleleft} x\}$.

A subset $\mathcal{M}$ of $\mathcal{O}(L)$ is called an *entourage uniformity* on L if it is generated by an entourage uniformity basis $\mathcal{M}'$, i.e.,

$$\mathcal{M} = \{e \in \mathcal{O}(L) \mid \text{ there is } f \in \mathcal{M}' \text{ such that } f \leq e\}.$$

The pair $(L, \mathcal{M})$ is then called an *entourage uniform frame.*

(b) Let $(L, \mathcal{M})$ and $(L', \mathcal{M}')$ be entourage uniform frames. An *entourage uniform homomorphism* $f : (L, \mathcal{M}) \longrightarrow (L', \mathcal{M}')$ is a frame map $f : L \longrightarrow L'$ such that, for every $e \in \mathcal{M}$, there exists $g \in \mathcal{M}'$ with $g \cdot f \leq f \cdot e$.

The category of entourage uniform frames and entourage uniform homomorphisms will be denoted by EUFrm.

4. Weil uniform frames

Now, following the hint of Isbell [39], let us present an alternative approach to entourage uniformities, expressed in terms of the coproduct $L \oplus L$, showing this way that entourages in the style of Weil do work in the pointless context.

Consider the cartesian product $L \times L$ with the usual order. If A and B are down-sets of $L \times L$, we denote by $A \cdot B$ the set

$$\{(x, y) \in L \times L \mid \text{ there is } z \in L \setminus \{0\} \text{ such that } (x, z) \in A \text{ and } (z, y) \in B\}$$

and by $A \circ B$ the C-ideal generated by $A \cdot B$.

The operation $\circ$ (which in general is not commutative) is associative and so bracketing is unnecessary for repeated compositions such as

$$A^n = A \circ A \circ \cdots \circ A \qquad (n \text{ factors}).$$

Further we have:

For any $A \in \mathcal{D}(L \times L)$, $A^{-1} := \{(x, y) \in L \times L \mid (y, x) \in A\}$ and A is *symmetric* if $A = A^{-1}$. The element

$$\bigvee\{y \in L \mid (y, y) \in A, y \wedge x \neq 0\}$$

will be denoted by $st(x, A)$. In the following proposition we list some obvious properties of these operators.

Proposition 4.1. *Let $x, y \in L$ and $A, B \in \mathcal{D}(L \times L)$. Then:*

(a) $(x \oplus y)^{-1} = y \oplus x$;

(b) $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$;

(c) $A \circ \mathbb{O} = \mathbb{O} \circ A = \mathbb{O}$;

(d) $st(x, A) \oplus y \subseteq A \circ (x \oplus y)$ *and* $x \oplus st(y, A) \subseteq (x \oplus y) \circ A$. ■

The map

$$\begin{array}{cccc} k_0 & : & \mathcal{D}(L \times L) & \longrightarrow & \mathcal{D}(L \times L) \\ & & A & \longmapsto & \Big\{(x, \bigvee S) \mid \{x\} \times S \subseteq A\Big\} \bigcup \Big\{(\bigvee S, y) \mid S \times \{y\} \subseteq A\Big\} \end{array}$$

is a prenucleus, that is, for all $A, B \in \mathcal{D}(L \times L)$, $A \subseteq k_0(A)$, $k_0(A) \cap B \subseteq k_0(A \cap B)$ and $k_0(A) \subseteq k_0(B)$ whenever $A \subseteq B$. Consequently (cf. [4])

$$Fix(k_0) := \{A \in \mathcal{D}(L \times L) \mid k_0(A) = A\} = L \oplus L$$

is a closure system, and the associated closure operator is then given by

$$k(A) = \bigcap \{B \in L \oplus L \mid A \subseteq B\},$$

which is the C-ideal generated by A. The following technical lemma will play a crucial role in the sequel.

Lemma 4.2. *Let $A, B \in \mathcal{D}(L \times L)$. Then:*

(a) $k(A) \circ k(B) = A \circ B$;

(b) $k(A^{-1}) = k(A)^{-1}$.

Proof.

(a) It suffices to show that $k(A) \cdot k(B) \subseteq k(A \cdot B)$. For this, consider the non-empty set

$$\mathbf{E} = \{E \in \mathcal{D}(L \times L) \mid A \subseteq E \subseteq k(A), E \cdot B \subseteq k(A \cdot B)\}$$

and let us prove that $k_0(E) \in \mathbf{E}$ whenever $E \in \mathbf{E}$.

So, consider $(x,y) \in k_0(E) \cdot B$ and $z \neq 0$ such that $(x,z) \in k_0(E)$ and $(z,y) \in B$. If $(x,z) = (x, \bigvee S)$ for some S with $\{x\} \times S \subseteq E$, there is a non-zero $s \in S$ such that $(x,s) \in E$ and $(s,y) \in B$ and, therefore, $(x,y) \in E \cdot B \subseteq k(A \cdot B)$. On the other hand, if $(x,z) = (\bigvee S, z)$ for some S with $S \times \{z\} \subseteq E$, $(s,y) \in E \cdot B$ for every $s \in S$ and, therefore, $(x,y) \in k_0(E \cdot B) \subseteq k(A \cdot B)$.

Moreover, for any non-void $\mathbf{F} \subseteq \mathbf{E}$, $\bigcup_{F \in \mathbf{F}} F \in \mathbf{E}$, since $(\bigcup_{F \in \mathbf{F}} F) \cdot B \subseteq \bigcup_{F \in \mathbf{F}} (F \cdot B)$. Therefore $S = \bigcup_{E \in \mathbf{E}} E$ belongs to $\mathbf{E}$, i.e., $\mathbf{E}$ has a largest element S. But $k_0(S) \in \mathbf{E}$ so $S = k_0(S)$, i.e., S is a C-ideal. Hence $k(A) = S \in \mathbf{E}$ and, consequently, $k(A) \cdot B \subseteq k(A \cdot B)$. By symmetry, $A \cdot k(B) \subseteq k(A \cdot B)$.

In conclusion, we have $k(A) \cdot k(B) \subseteq k(A \cdot k(B)) \subseteq k^2(A \cdot B) = k(A \cdot B)$, as desired.

(b) Consider $A \in \mathcal{D}(L \times L)$ and let

$$\mathbf{E} = \{E \in \mathcal{D}(L \times L) \mid A^{-1} \subseteq E \subseteq k(A^{-1}), E \subseteq k(A)^{-1}\}.$$

The set $\mathbf{E}$ is non-empty and $k_0(E) \in \mathbf{E}$ whenever $E \in \mathbf{E}$. Moreover, for any non-void $\mathbf{F} \subseteq \mathbf{E}$, $\bigcup_{F \in \mathbf{F}} F \in \mathbf{E}$. Therefore $\mathbf{E}$ has a largest element S which must be $k(A^{-1})$ since $S = k_0(S)$ and $A^{-1} \subseteq S \subseteq k(A^{-1})$. This says that $k(A^{-1}) \subseteq k(A)^{-1}$. Since A is arbitrary, $k(A)^{-1} \subseteq k(A^{-1})$, i.e., $k(A^{-1}) = k(A)^{-1}$.

■

The map k can also be constructed from k_0, by transfinite induction over the class Ord of ordinals:

If one defines, for each $A \in \mathcal{D}(L \times L)$ and each ordinal β,

- $k_0^0(A) = A$,

- $k_0^\beta(A) = k_0(k_0^\alpha(A))$ if $\beta = \alpha + 1$,
- $k_0^\beta(A) = \bigvee\{k_0^\alpha(A) \mid \alpha < \beta\}$ if β is a limit ordinal,

then

$$k = \bigvee_{\beta \in Ord} k_0^\beta.$$

Thus, the preceding lemma can be proved by transfinite induction. The approach we followed (inspired by [4] and [5]) avoids the use of ordinals.

Definition 4.3. We define a *Weil entourage* of L as an element E of $L \oplus L$ for which there exists a cover U of L satisfying

$$\bigvee_{x \in U} (x \oplus x) \subseteq E.$$

This is equivalent to saying that

$$\{x \in L \mid (x, x) \in E\}$$

is a cover of L.

The collection $WEnt(L)$ of all Weil entourages of L may be partially ordered by inclusion. This is a complete lattice.

Proposition 4.4. *Let E be a Weil entourage. Then:*

(a) *for any $x \in L$, $x \leq st(x, E)$;*

(b) *$E^n \subseteq E^{n+1}$ for every natural n;*

(c) *for any $A \in \mathcal{D}(L \times L)$, $A \subseteq (E \circ A) \cap (A \circ E)$.*

Proof.

(a) Consider $x \in L$. We have

$$x = x \wedge \bigvee\{y \in L \mid (y, y) \in E\} = \bigvee\{x \wedge y \mid (y, y) \in E, x \wedge y \neq 0\} \leq st(x, E).$$

(b) It suffices to prove that $E \subseteq E^2$.

Consider $(x, y) \in E$. By (a), $y \leq st(y, E)$. But, by Proposition 4.1 (d), $x \oplus st(y, E) \subseteq (x \oplus y) \circ E \subseteq E^2$. Consequently, $(x, y) \in E^2$.

(c) Let $(x, y) \in A$. The cases $x = 0$ or $y = 0$ are trivial. If $x, y \neq 0$, since $x = \bigvee\{x \wedge z \mid (z, z) \in E, x \wedge z \neq 0\}$ and, for any $(z, z) \in E$ with $x \wedge z \neq 0$, $(z, y) \in E \circ A$, we have, by definition of C-ideal, that $(x, y) \in E \circ A$.

Similarly, $A \subseteq A \circ E$. ■

Let $\mathcal{E}$ be a set of Weil entourages. By definition, we write $x \overset{\mathcal{E}}{\lhd} y$ whenever $E \circ (x \oplus x) \subseteq y \oplus y$ for some $E \in \mathcal{E}$.

Definition 1.1 and Proposition 1.2 suggests us the introduction of the following definitions:

Definitions 4.5.

(a) Let L be a frame and let $\mathcal{E} \subseteq WEnt(L)$. We say that the pair $(L, \mathcal{E})$ is a *Weil uniform frame* if:

(UW1) $\mathcal{E}$ is a filter of $(WEnt(L), \subseteq)$;

(UW2) for each $E \in \mathcal{E}$ there is $F \in \mathcal{E}$ such that $F \circ F \subseteq E$;

(UW3) for any $E \in \mathcal{E}$, E^{-1} is also in $\mathcal{E}$;

(UW4) for any $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}$.

We say in these circumstances that the family $\mathcal{E}$ is a *Weil uniformity* on L. A *Weil uniformity basis* is just a filter basis of some Weil uniformity.

(b) Let $(L, \mathcal{E})$ and $(L', \mathcal{E}')$ be Weil uniform frames. A *Weil uniform homomorphism* $f : (L, \mathcal{E}) \longrightarrow (L', \mathcal{E}')$ is a frame map $f : L \longrightarrow L'$ such that $(f \oplus f)(E) \in \mathcal{E}'$ whenever $E \in \mathcal{E}$.

These are the objects and morphisms of the category WUFrm.

It is useful to note that the symmetric Weil entourages E of $\mathcal{E}$ form a basis for $\mathcal{E}$. In fact, if $E \in \mathcal{E}$ then $E^{-1} \in \mathcal{E}$ so $E \cap E^{-1}$ is a symmetric Weil entourage of $\mathcal{E}$ contained in E.

In case $\mathcal{E}$ is a Weil uniformity the order relation $\overset{\mathcal{E}}{\lhd}$ may be expressed in several equivalent ways:

Proposition 4.6. *Let $\mathcal{E}$ be a Weil uniformity on L and let $x, y \in L$. The following assertions are equivalent:*

(i) $x \overset{\mathcal{E}}{\triangleleft} y$;

(ii) $(x \oplus x) \circ E \subseteq y \oplus y$, *for some* $E \in \mathcal{E}$;

(iii) $E \circ (x \oplus 1) \subseteq y \oplus 1$, *for some* $E \in \mathcal{E}$;

(iv) $(1 \oplus x) \circ E \subseteq 1 \oplus y$, *for some* $E \in \mathcal{E}$;

(v) $st(x, E) \leq y$, *for some* $E \in \mathcal{E}$.

Proof. We only prove that statements (i) and (v) are equivalent because the proofs that each one of (ii), (iii) and (iv) is equivalent to (v) are similar.

(i) $\Rightarrow$ (v): For any z with $(z, z) \in E$ and $z \wedge x \neq 0$, the pair (z, x) belongs to $E \circ (x \oplus x) \subseteq y \oplus y$. Therefore $z \leq y$.

(v) $\Rightarrow$ (i): In order to show that $x \overset{\mathcal{E}}{\triangleleft} y$ it suffices to prove that $F \circ (x \oplus x) \subseteq y \oplus y$ for any symmetric $F \in \mathcal{E}$ such that $F^2 \subseteq E$. So, consider $a, b, c \in L$ such that $(a, b) \in F$, $(b, c) \leq (x, x)$ and $a, b, c \neq 0$. Then $(a, b) \in F^2$ and, by the symmetry of F, $(b, a) \in F^2$, which forces $(a \vee b, a \vee b) \in F^2 \subseteq E$, as (a, a) and (b, b) also belong to F^2. Thus $(a \vee b, a \vee b) \in E$ and, therefore, $a \leq st(x, E)$, since $(a \vee b) \wedge x \geq b \neq 0$. Hence $a \leq y$ and $c \leq x \leq st(x, E) \leq y$ which implies that $(a, c) \in y \oplus y$. ■

In particular, it follows from this proposition that $\overset{\mathcal{E}}{\triangleleft}$ is stronger than $\leq$. We call $\overset{\mathcal{E}}{\triangleleft}$ the *strong inclusion* for $\mathcal{E}$, and when $x \overset{\mathcal{E}}{\triangleleft} y$ we say that x is *$\mathcal{E}$-strongly below* y.

Remark 4.7. Trivially, $x \overset{\mathcal{E}}{\triangleleft} y$ also implies that $E \circ (1 \oplus x) \subseteq 1 \oplus y$ for some $E \in \mathcal{E}$ (although the reverse implication is not true). Therefore, condition (UW4) of Definition 4.5 could be formulated in the following equivalent way:

> For each $J \in L \oplus L$, $J = \bigvee\{I \in L \oplus L \mid I \overset{\mathcal{E}}{\sqsubseteq} J\}$, where $I \overset{\mathcal{E}}{\sqsubseteq} J$ means that $E \circ I \subseteq J$ for some $E \in \mathcal{E}$.

Indeed, for every $J = \bigvee_{\gamma \in \Gamma}(a_\gamma \oplus b_\gamma) \in L \oplus L$, we have

$$a_\gamma \oplus b_\gamma = \left(\bigvee\{x \in L \mid x \overset{\mathcal{E}}{\triangleleft} a_\gamma\}\right) \oplus \left(\bigvee\{y \in L \mid y \overset{\mathcal{E}}{\triangleleft} b_\gamma\}\right) = \bigvee\{x \oplus y \mid x \overset{\mathcal{E}}{\triangleleft} a_\gamma, y \overset{\mathcal{E}}{\triangleleft} b_\gamma\},$$

since, for every $\gamma \in \Gamma$, $a_\gamma = \bigvee\{x \in L \mid x \overset{\mathcal{E}}{\lhd} a_\gamma\}$ and $b_\gamma = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} b_\gamma\}$. But $x \overset{\mathcal{E}}{\lhd} a_\gamma$ and $y \overset{\mathcal{E}}{\lhd} b_\gamma$ imply, respectively, that $E_1 \circ (x \oplus 1) \subseteq a_\gamma \oplus 1$ and $E_2 \circ (1 \oplus y) \subseteq 1 \oplus b_\gamma$, for some $E_1, E_2 \in \mathcal{E}$; thus

$$E \circ (x \oplus y) \subseteq \Big(E \circ (x \oplus 1)\Big) \cap \Big(E \circ (1 \oplus y)\Big) \subseteq (a_\gamma \oplus b_\gamma),$$

for $E = E_1 \cap E_2 \in \mathcal{E}$. Consequently,

$$\bigvee\{x \oplus y \mid x \overset{\mathcal{E}}{\lhd} a_\gamma, y \overset{\mathcal{E}}{\lhd} b_\gamma\} \subseteq \bigvee\{I \in L \oplus L \mid I \overset{\mathcal{E}}{\sqsubseteq} (a_\gamma \oplus b_\gamma)\}.$$

Conversely, for every $x \in L$,

$$x \oplus 1 = \{I \in L \oplus L \mid I \overset{\mathcal{E}}{\sqsubseteq} (x \oplus 1)\} \leq \Big(\bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}\Big) \oplus 1,$$

because $I \overset{\mathcal{E}}{\sqsubseteq} (x \oplus 1)$ implies that, for every $(a,b) \in I$, $a \overset{\mathcal{E}}{\lhd} x$, i.e., $I \subseteq (\bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}) \oplus 1$. Hence $x \leq \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}$.

The order $\overset{\mathcal{E}}{\lhd}$ has the same nice properties as its corresponding order $\overset{\mathcal{U}}{\lhd}$ for covering uniformities:

Proposition 4.8. *Assume that $\mathcal{E}$ is a basis for a filter of $(WEnt(L), \subseteq)$. Then the relation $\overset{\mathcal{E}}{\lhd}$ is a sublattice of $L \times L$ satisfying the following properties:*

(a) *for any x, y, z, w in L, $x \leq y \overset{\mathcal{E}}{\lhd} z \leq w$ implies $x \overset{\mathcal{E}}{\lhd} w$;*

(b) *$x \overset{\mathcal{E}}{\lhd} y$ implies $x \prec y$.*

Furthermore, we have:

(c) *if $\mathcal{E}$ is a basis for a Weil uniformity, the relation $\overset{\mathcal{E}}{\lhd}$ interpolates, that is, there exists $z \in L$ such that $x \overset{\mathcal{E}}{\lhd} z \overset{\mathcal{E}}{\lhd} y$ whenever $x \overset{\mathcal{E}}{\lhd} y$;*

(d) *for any morphism $f : (L, \mathcal{E}) \longrightarrow (L', \mathcal{E}')$ of* WUFrm*, if $x \overset{\mathcal{E}}{\lhd} y$ then $f(x) \overset{\mathcal{E}'}{\lhd} f(y)$.*

Proof. If $E_1 \circ (x_1 \oplus x_1) \subseteq y_1 \oplus y_1$ and $E_2 \circ (x_2 \oplus x_2) \subseteq y_2 \oplus y_2$ with $E_1, E_2 \in \mathcal{E}$ then, immediately,

$$\begin{aligned}(E_1 \cap E_2) \circ (x_1 \wedge x_2 \oplus x_1 \wedge x_2) &\subseteq \Big(E_1 \circ (x_1 \oplus x_1)\Big) \cap \Big(E_2 \circ (x_2 \oplus x_2)\Big) \\ &\subseteq (y_1 \oplus y_1) \cap (y_2 \oplus y_2) \\ &\subseteq (y_1 \wedge y_2) \oplus (y_1 \wedge y_2).\end{aligned}$$

Since $\mathcal{E}$ is a filter basis, there exists $F \in \mathcal{E}$ such that $F \subseteq E_1 \cap E_2$, and so $x_1 \wedge x_2 \overset{\mathcal{E}}{\lhd} y_1 \wedge y_2$ whenever $x_1 \overset{\mathcal{E}}{\lhd} y_1$ and $x_2 \overset{\mathcal{E}}{\lhd} y_2$. On the other hand, if $st(x_1, E_1) \leq y_1$ and $st(x_2, E_2) \leq y_2$ for $E_1, E_2 \in \mathcal{E}$, then $st(x_1 \vee x_2, E_1 \cap E_2) \leq y_1 \vee y_2$ because

$$st(x_1 \vee x_2, E_1 \cap E_2) \leq st(x_1, E_1) \vee st(x_2, E_2).$$

Hence, $x_1 \overset{\mathcal{E}}{\lhd} y_1$ and $x_2 \overset{\mathcal{E}}{\lhd} y_2$ imply that $x_1 \vee x_2 \overset{\mathcal{E}}{\lhd} y_1 \vee y_2$ and the relation $\overset{\mathcal{E}}{\lhd}$ is a sublattice of $L \times L$. Obviously $0 \overset{\mathcal{E}}{\lhd} 0$ and $1 \overset{\mathcal{E}}{\lhd} 1$ so this sublattice is bounded.

(a) It is obvious.

(b) Let $E \in \mathcal{E}$ be such that $E \circ (x \oplus x) \subseteq y \oplus y$. Then $st(x, E) \leq y$. Also

$$1 = \bigvee \{w \in L \mid (w, w) \in E\} = st(x, E) \vee \bigvee \{w \in L \mid (w, w) \in E, w \wedge x = 0\}.$$

Denote $\bigvee \{w \in L \mid (w, w) \in E, w \wedge x = 0\}$ by z. Since $z \wedge x = 0$ and $z \vee y \geq z \vee st(x, E) = 1$, z separates x and y.

(c) Suppose that $x \overset{\mathcal{E}}{\lhd} y$. Then there is some $E \in \mathcal{E}$ with $st(x, E) \leq y$. Consider $F \in \mathcal{E}$ satisfying $F^2 \subseteq E$. Of course, $x \overset{\mathcal{E}}{\lhd} st(x, F)$. We claim that $st(x, F) \overset{\mathcal{E}}{\lhd} y$. Since $st(x, F^2) \leq st(x, E) \leq y$, it suffices to show that $st(st(x, F), F) \leq st(x, F^2)$, which is always true for any C-ideal F:

In fact, $st(st(x, F), F) = \bigvee \{y \in L \mid (y, y) \in F, y \wedge st(x, F) \neq 0\}$. Consider $y \in L$ with $(y, y) \in F$ and $y \wedge st(x, F) \neq 0$. Then there is $z \in L$ such that $(z, z) \in F$, $z \wedge x \neq 0$ and $z \wedge y \neq 0$. Therefore $(y, y \wedge z) \in F$ and $(y \wedge z, z) \in F$ thus $(y, z) \in F^2$. Similarly, $(z, y) \in F^2$. Also $(y, y), (z, z) \in F^2$. But F^2 is a C-ideal so $(y \vee z, y \vee z) \in F^2$. In conclusion, $(y \vee z, y \vee z) \in F^2$ and $(y \vee z) \wedge x \geq z \wedge x \neq 0$, hence $y \leq st(x, F^2)$ and $st(st(x, F), F) \leq st(x, F^2)$ as we claimed.

(d) Assume that $E \circ (x \oplus x) \subseteq y \oplus y$ for some $E \in \mathcal{E}$. An application of Lemma 4.2 (a) yields

$$(f \oplus f)(E) \circ (f(x) \oplus f(x)) = \left(\bigcup_{(a,b) \in E} (f(a) \oplus f(b)) \right) \circ (f(x) \oplus f(x)).$$

If $(x', y') \leq (f(a), f(b))$ for some $(a, b) \in E$, $(y', z') \leq (f(x), f(x))$ and $x', y', z' \neq 0$, then $f(b \wedge x) \neq 0$. Thus $b \wedge x \neq 0$ and $(a, x) \in E \circ (x \oplus x) \subseteq y \oplus y$, i.e., $a, x \leq y$.

So, $(f \oplus f)(E) \circ (f(x) \oplus f(x)) \subseteq f(y) \oplus f(y)$. Since $(f \oplus f)(E) \in \mathcal{E}'$ we conclude that $f(x) \overset{\mathcal{E}'}{\lhd} f(y)$. ■

In particular, it follows from (b) that L is necessarily a regular frame. From (c) it follows that, under the Countable Dependent Axiom of Choice, L is completely regular.

We proceed to consider some examples of Weil uniform frames.

Examples 4.9.

(a) Let (L, d) be a metric frame (see Section II.2 of this thesis or, for more information, [64]). For any real $\epsilon > 0$ let $E_\epsilon = \bigvee\{x \oplus x \mid d(x) < \epsilon\}$. The family $(E_\epsilon)_{\epsilon>0}$ is a basis for a Weil uniformity on L.

(b) For every (Weil) uniform space $(X, \mathcal{E})$, let $\mathcal{T}_\mathcal{E}$ be the topology induced by $\mathcal{E}$ on X and consider the collection of all Weil entourages $\bigvee_{x\in X}(E[x] \oplus E[x])$, with $E \in \mathcal{E}$. Immediately, for any $E, F \in \mathcal{E}$,

$$\bigvee_{x\in X}\Big((E\cap F)[x] \oplus (E \cap F)[x]\Big) \subseteq \Big(\bigvee_{x\in X}(E[x] \oplus E[x])\Big) \cap \Big(\bigvee_{x\in X}(F[x] \oplus F[x])\Big).$$

Furthermore, for any $E, F \in \mathcal{E}$ with F symmetric and $F^2 \subseteq E$,

$$\Big(\bigvee_{x\in X}(F[x] \oplus F[x])\Big) \circ \Big(\bigvee_{x\in X}(F[x] \oplus F[x])\Big) \subseteq \bigvee_{x\in X}(E[x] \oplus E[x]),$$

and

$$\Big(\bigvee_{x\in X}(F[x] \oplus F[x])\Big) \circ (V \oplus V) \subseteq U \oplus U$$

whenever $U, V \subseteq X$ and $E \circ (V \times V) \subseteq U \times U$. Therefore the above collection of Weil entourages is a basis for a Weil uniformity $\mathcal{E}_{\mathcal{T}_\mathcal{E}}$ on $\mathcal{T}_\mathcal{E}$.

(c) Let B be a complete Boolean algebra. For each $b \in B$, the C-ideal

$$E_b = (b \oplus b) \vee (b^* \oplus b^*)$$

is a Weil entourage of B. By Lemma 4.2 (a),

$$E_b \circ E_b = ((b \oplus b) \cup (b^* \oplus b^*)) \circ ((b \oplus b) \cup (b^* \oplus b^*)).$$

Then, easily, $E_b \circ E_b = E_b$. Similarly, $E_b \circ (b \oplus b) = b \oplus b$, that is, b is $\{E_b\}$-strongly below b. Hence, we have a subbasis for a Weil uniformity on B, i.e., taking all finite meets of Weil entourages of this type, we form a basis for a Weil uniformity.

(d) In the same way that the categorical notion of group [52] in the category of topological spaces yields the topological groups, it defines the "localic groups" [40] in the category of locales. So, a *localic group* is a cogroup in the category of frames, that is, a frame L endowed with a multiplication $\mu : L \longrightarrow L \oplus L$, an inverse $\imath : L \longrightarrow L$ and a unit point $\varepsilon : L \longrightarrow \mathbf{2}$ satisfying the identities

$$(\mu \oplus 1_L) \cdot \mu = (1_L \oplus \mu) \cdot \mu,$$

$$(\varepsilon \oplus 1_L) \cdot \mu = 1_L = (1_L \oplus \varepsilon) \cdot \mu$$

and

$$\nabla \cdot (\imath \oplus 1_L) \cdot \mu = \nabla \cdot (1_L \oplus \imath) \cdot \mu = \sigma \cdot \varepsilon,$$

where $\nabla : L \oplus L \longrightarrow L$ is the codiagonal and σ is the morphism $\mathbf{2} \longrightarrow L$. Note that the usual properties for groups

$$\varepsilon \cdot \imath = \varepsilon,$$

$$\imath \cdot \imath = \imath$$

and

$$\mu \cdot \imath = \tau \cdot (\imath \oplus \imath) \cdot \mu,$$

where τ is the unique map from $L \oplus L$ to $L \oplus L$ satisfying $\tau \cdot u_L = u'_L$ and $\tau \cdot u'_L = u_L$ (u_L and u'_L are the injections of the coproduct), are also valid here. These groups have Weil uniformities that arise in a similar way as in the spatial setting of topological groups. For any $x \in L$ such that $\varepsilon(x) = 1$ put

$$E^l_x := (1_L \oplus \imath)(\mu(x)) \quad \text{and} \quad E^r_x := (\imath \oplus 1_L)(\mu(x)).$$

Proposition 4.10. *$\mathcal{E}^l := \{E^l_x \mid x \in L, \varepsilon(x) = 1\}$ and $\mathcal{E}^r := \{E^r_x \mid x \in L, \varepsilon(x) = 1\}$ are bases for Weil uniformities.*

Proof. We only show that $\mathcal{E}^r$ is a Weil uniformity basis. The proof for $\mathcal{E}^l$ is similar.

Each E^r_x is a Weil entourage because $\nabla \cdot (\imath \oplus 1_L) \cdot \mu(x) = 1$. In fact, any C-ideal A satisfying $\nabla(A) = 1$ is a Weil entourage: if $A = \bigvee_{\gamma\in\Gamma}(a_\gamma \oplus b_\gamma)$ and $\nabla(A) = \bigvee_{\gamma\in\Gamma}(a_\gamma \wedge b_\gamma) = 1$ then $\{a_\gamma \wedge b_\gamma \mid \gamma \in \Gamma\}$ is a cover of L and $\bigvee_{\gamma\in\Gamma}(a_\gamma \wedge b_\gamma \oplus a_\gamma \wedge b_\gamma) \subseteq A$.

Obviously, $E^r_x \cap E^r_y = E^r_{x\wedge y}$. Since $\varepsilon(x \wedge y) = 1$ whenever $\varepsilon(x) = \varepsilon(y) = 1$, $\mathcal{E}^r$ is a filter basis of $(WEnt(L), \subseteq)$.

The symmetry is a consequence of the fact that, for every x, $(E^r_x)^{-1} = E^r_{\imath(x)}$ which we prove next. Put $\mu(x) = \bigvee_{\gamma\in\Gamma}(x_\gamma \oplus y_\gamma)$. Then $E^r_x = \bigvee_{\gamma\in\Gamma}(\imath(x_\gamma) \oplus y_\gamma)$. On the other hand, since $\mu \cdot \imath = \tau \cdot (\imath \oplus \imath) \cdot \mu$, we have $\mu(\imath(x)) = \bigvee_{\gamma\in\Gamma}(\imath(y_\gamma) \oplus \imath(x_\gamma))$ and, therefore, $E^r_{\imath(x)} = \bigvee_{\gamma\in\Gamma}(y_\gamma \oplus \imath(x_\gamma))$.

Now, consider E^r_x with $\varepsilon(x) = 1$. We have

$$\varepsilon = 1_{\mathbf{2}} \oplus \varepsilon = (1_{\mathbf{2}} \oplus \varepsilon) \cdot (\varepsilon \oplus 1_L) \cdot \mu = (\varepsilon \oplus \varepsilon) \cdot \mu.$$

Thus $(\varepsilon \oplus \varepsilon) \cdot \mu(x) = 1$, that is, $\bigvee\{\varepsilon(a) \oplus \varepsilon(b) \mid (a, b) \in \mu(x)\} = 1$. Therefore there is some $(a, b) \in \mu(x)$ with $\varepsilon(a) = \varepsilon(b) = 1$. Also $(a \wedge b, a \wedge b) \in \mu(x)$ and $\varepsilon(a \wedge b) = 1$. Denote $a \wedge b$ by y. We claim that $E^r_y \circ E^r_y \subseteq E^r_x$. In fact $E^r_y \circ E^r_y$ is the Weil entourage

$$\Big(\bigvee_{(a,b)\in\mu(y)}(\imath(a)\oplus b)\Big)\circ\Big(\bigvee_{(a,b)\in\mu(y)}(\imath(a)\oplus b)\Big) = \Big(\bigcup_{(a,b)\in\mu(y)}(\imath(a)\oplus b)\Big)\circ\Big(\bigcup_{(a,b)\in\mu(y)}(\imath(a)\oplus b)\Big).$$

Take $(\imath(a), b)$ with $(a, b) \in \mu(y)$ and $(\imath(c), d)$ with $(c, d) \in \mu(y)$ such that $b \wedge \imath(c) \neq 0$. From the inclusion $y \oplus y \subseteq \mu(x)$ it follows that

$$\mu(y) \oplus \mu(y) \subseteq (\mu \oplus \mu)(\mu(x)) = (1_L \oplus \mu \oplus 1_L) \cdot (\mu \oplus 1_L) \cdot (\mu(x)).$$

Therefore

$$a \oplus b \oplus c \oplus d \subseteq (1_L \oplus \mu \oplus 1_L) \cdot (\mu \oplus 1_L) \cdot (\mu(x)).$$

Applying $1_L \oplus \nabla \cdot (1_L \oplus \imath) \oplus 1_L$ to both sides we get

$$a \oplus (b \wedge \imath(c)) \oplus d \subseteq (1_L \oplus \sigma \oplus 1_L) \cdot (\mu(x)) = \bigvee_{(a,b)\in\mu(x)}(a \oplus 1_L \oplus b).$$

Since $b \wedge \imath(c) \neq 0$, then $(a, d) \in \mu(x)$ and $(\imath(a), d) \in E^r_x$. Hence $E^r_y \circ E^r_y \subseteq E^r_x$.

Finally, let us check the admissibility condition (UW4). From the identity $(1_L \oplus \varepsilon) \cdot \mu = 1_L$, it follows that, for every $x \in L$,

$$x = \bigvee\{a \oplus \varepsilon(b) \mid a \oplus b \subseteq \mu(x)\} = \bigvee\{a \in L \mid \exists b \in L : \varepsilon(b) = 1 \text{ and } a \oplus b \subseteq \mu(x)\},$$

so it remains to show that a is $\mathcal{E}^r$-strongly below x whenever there is some $b \in L$ satisfying $\varepsilon(b) = 1$ and $a \oplus b \subseteq \mu(x)$. By Proposition 4.6 it suffices to prove that $st(a, E^r_b) \leq x$. So, assume $(c, c) \in E^r_b$ and $c \wedge a \neq 0$. Then $\imath(c) \oplus c \subseteq \mu(b)$. On the other hand, $a \oplus b \subseteq \mu(x)$. Consequently,

$$\begin{aligned} a \oplus b \oplus a &\subseteq (\mu \oplus 1_L)(x \oplus a) \\ \Rightarrow a \oplus \mu(b) \oplus a &\subseteq (1_L \oplus \mu \oplus 1_L) \cdot (\mu \oplus 1_L)(x \oplus a) \\ \Rightarrow a \oplus \imath(c) \oplus a &\subseteq (1_L \oplus \mu \oplus 1_L) \cdot (\mu \oplus 1_L)(x \oplus a). \end{aligned}$$

By the associativity of μ,

$$\begin{aligned} (1_L \oplus \mu \oplus 1_L) \cdot (\mu \oplus 1_L) &= \Big((1_L \oplus \mu) \cdot \mu\Big) \oplus 1_L \\ &= \Big((\mu \oplus 1_L) \cdot \mu\Big) \oplus 1_L \\ &= (\mu \oplus 1_L \oplus 1_L) \cdot (\mu \oplus 1_L). \end{aligned}$$

Thus,

$$a \oplus \imath(c) \oplus c \oplus a \subseteq (\mu \oplus 1_L \oplus 1_L) \cdot (\mu \oplus 1_L)(x \oplus a).$$

Applying $(\nabla \cdot (1_L \oplus \imath)) \oplus 1_L \oplus 1_L$ to both sides we get

$$\begin{aligned} a \wedge c \oplus c \oplus a &\subseteq (\sigma \cdot \varepsilon \oplus 1_L \oplus 1_L) \cdot (\mu \oplus 1_L)(x \oplus a) \\ &= ((\sigma \cdot \varepsilon \oplus 1_L) \cdot \mu \oplus 1_L)(x \oplus a) \\ &= (((\sigma \oplus 1_L) \cdot (\varepsilon \oplus 1_L) \cdot \mu) \oplus 1_L)(x \oplus a) \\ &= (\sigma \oplus 1_L \oplus 1_L)(x \oplus a). \end{aligned}$$

But $\sigma \oplus 1_L(x) = 1 \oplus x$ so $(a \wedge c) \oplus c \oplus a \subseteq 1 \oplus x \oplus a$. Since $a \wedge c \neq 0$, we finally obtain $c \oplus a \subseteq x \oplus a$. In conclusion, for every $(c, c) \in E^r_b$ with $c \wedge a \neq 0$, $c \oplus a \subseteq x \oplus a$. Hence $st(a, E^r_b) \oplus a \subseteq x \oplus a$ and $st(a, E^r_b) \leq x$. ■

We point out that, as for topological groups, the map $\imath$ is a Weil uniform frame isomorphism between the two structures: $\imath$ is a frame isomorphism, and, for any $x \in L$,

$$(\imath \oplus \imath)(E^r_x) = (\imath \oplus \imath)(\imath \oplus 1_L)(\mu(x)) = (1_L \oplus \imath)(\mu(x)) = E^l_x.$$

The open and spectrum functors may be adapted to this setting and act as a categorical guide of the correctness of the notion of Weil uniform frame. It suffices to define the open functor $\Omega : \mathsf{Unif} \longrightarrow \mathsf{WUFrm}$ by assigning to each uniform space $(X, \mathcal{E})$ the Weil uniform frame $\Omega(X, \mathcal{E}) = (\mathcal{T}_{\mathcal{E}}, \mathcal{E}_{\mathcal{T}_{\mathcal{E}}})$ of Example 4.9 (b) and to each uniformly continuous map $f : (X, \mathcal{E}) \longrightarrow (X', \mathcal{E}')$ the Weil uniform homomorphism $\Omega(f) : \Omega(X', \mathcal{E}') \longrightarrow \Omega(X, \mathcal{E})$ defined by $\Omega(f)(U) = f^{-1}(U)$.

On the other way round, let $(L, \mathcal{E})$ be a Weil uniform frame. For each $E \in \mathcal{E}$, let

$$\Sigma_E := \bigcup_{(x,y)\in E} (\Sigma_x \times \Sigma_y),$$

and denote by $\mathcal{E}_{ptL}$ the family of entourages of ptL each of which contains some member of $\{\Sigma_E \mid E \in \mathcal{E}\}$.

Proposition 4.11. *For $(L, \mathcal{E}) \in \mathsf{WUFrm}$, $\Sigma(L, \mathcal{E}) = (ptL, \mathcal{E}_{ptL})$ is a uniform space.*

Proof. Each Σ_E is in fact an entourage since $\bigvee\{x \in L \mid (x, x) \in E\} = 1$ implies that, for every $p \in ptL$, $\bigvee\{p(x) \mid (x, x) \in E\} = 1$, i.e., that there is some $(x, x) \in E$ with $p(x) = 1$.

The symmetry of $\mathcal{E}_{ptL}$ is a consequence of the fact that $(\Sigma_E)^{-1} = \Sigma_{E^{-1}}$.

Let $E, F \in \mathcal{E}$; trivially, $\Sigma_E \cap \Sigma_F = \Sigma_{E\cap F}$ and $E \subseteq F$ implies $\Sigma_E \subseteq \Sigma_F$. Thus $\{\Sigma_E \mid E \in \mathcal{E}\}$ is a filter basis of $(WEnt(L), \subseteq)$.

Finally, suppose $F^2 \subseteq E \in \mathcal{E}$. Then $\Sigma_F \circ \Sigma_F \subseteq \Sigma_E$. To see this, suppose $(p, q), (q, r) \in \Sigma_F$ and $(x, y), (x', y') \in F$ with $p(x) = 1, r(y') = 1$ and $q(y) = q(x') = 1$. Then $x' \wedge y \neq 0$ so $(x, y') \in F^2 \subseteq E$ and $(p, r) \in \Sigma_E$. ■

Lemma 4.12. *Suppose that $f : L \longrightarrow L'$ is a frame map and let $E \in L \oplus L$. Then*

$$\bigcup_{(x,y)\in f\oplus f(E)} (\Sigma_x \times \Sigma_y) \subseteq \bigcup_{(x,y)\in E} (\Sigma_{f(x)} \times \Sigma_{f(y)}).$$

Proof. This lemma can be proved in a very similar way to Lemma 4.2; just consider the set

$$\mathbf{E} = \left\{ V \in \mathcal{D}(L' \times L') \mid U \subseteq V \subseteq k(U), \bigcup_{(x,y)\in V} (\Sigma_x \times \Sigma_y) \subseteq \bigcup_{(x,y)\in E} (\Sigma_{f(x)} \times \Sigma_{f(y)}) \right\}$$

where

$$U = \bigcup_{(x,y)\in E} (f(x) \oplus f(y)) \in \mathcal{D}(L' \times L'),$$

and check that $U \in \mathbf{E}$, $k_0(V) \in \mathbf{E}$ whenever $V \in \mathbf{E}$ and $\bigcup_{i\in I} V_i \in \mathbf{E}$ whenever all V_i belong to $\mathbf{E}$. The essencial fact that makes things work is the observation that $p \in \Sigma_{x_i}$ for some $i \in I$ whenever $p \in \Sigma_x$ and $x = \bigvee_{i\in I} x_i$. ■

For any Weil uniform homomorphism $f : (L, \mathcal{E}) \longrightarrow (L', \mathcal{E}')$ define

$$\Sigma(f) : \Sigma(L', \mathcal{E}') \longrightarrow \Sigma(L, \mathcal{E})$$

by $\Sigma(f)(p) = p \cdot f$.

Proposition 4.13. *$\Sigma(f)$ is uniformly continuous.*

Proof. Let $E \in \mathcal{E}$. Then

$$\begin{aligned} (\Sigma(f) \times \Sigma(f))^{-1}(\Sigma_E) &= \{(p,q) \in ptL \times ptL \mid (p \cdot f, q \cdot f) \in \Sigma_E\} \\ &= \{(p,q) \in ptL \times ptL \mid \exists (x,y) \in E : p \in \Sigma_{f(x)}, q \in \Sigma_{f(y)}\}. \end{aligned}$$

An application of Lemma 4.12 yields

$$\Big(\Sigma(f) \times \Sigma(f)\Big)^{-1}(\Sigma_E) \supseteq \Sigma_{(f\oplus f)(E)} \in \mathcal{E}_{ptL'}.$$

Hence $\Big(\Sigma(f) \times \Sigma(f)\Big)^{-1}(\Sigma_E) \in \mathcal{E}_{ptL'}$. ■

In conclusion, Σ is a contravariant functor from WUFrm to Unif.

Theorem 4.14. *The contravariant functors Ω and Σ constructed above define a dual adjunction between the categories Unif and WUFrm.*

Proof. For any uniform space $(X,\mathcal{E})$ let $\eta_{(X,\mathcal{E})} : (X,\mathcal{E}) \longrightarrow \Sigma\Omega(X,\mathcal{E})$ be defined by $\eta_{(X,\mathcal{E})}(x)(U) = 1$ if and only if $x \in U$. We check that $\eta_{(X,\mathcal{E})}$ is uniformly continuous. Consider $E \in \mathcal{E}$ and let $\overline{E}$ be the Weil entourage $\bigvee_{x\in X}(E[x] \oplus E[x])$ of $pt\mathcal{T}_{\mathcal{E}}$. It suffices to verify that $(\eta_{(X,\mathcal{E})} \times \eta_{(X,\mathcal{E})})^{-1}(\Sigma_{\overline{E}}) \in \mathcal{E}$. An easy computation is sufficient:

$$\begin{aligned}(\eta_{(X,\mathcal{E})} \times \eta_{(X,\mathcal{E})})^{-1}(\Sigma_{\overline{E}}) &= \Big\{(x,y) \mid \Big(\eta_{(X,\mathcal{E})}(x), \eta_{(X,\mathcal{E})}(y)\Big) \in \bigcup_{(U,V)\in\overline{E}} (\Sigma_U \times \Sigma_V)\Big\} \\ &= \{(x,y) \mid \exists (U,V) \in \overline{E} : x \in U, y \in V\} \\ &\supseteq E.\end{aligned}$$

Since, for every $f : (X,\mathcal{E}) \longrightarrow (X',\mathcal{E}')$, $\eta_{(X',\mathcal{E}')}(f(x)) = \eta_{(X,\mathcal{E})}(x)\cdot\Omega(f)$, the maps $\eta_{(X,\mathcal{E})}$ define a natural transformation $\eta : 1_{\mathsf{Unif}} \longrightarrow \Sigma\Omega$. Besides, each morphism $\eta_{(X,\mathcal{E})}$ is universal from $(X,\mathcal{E})$ to Σ. Let $f : (X,\mathcal{E}) \longrightarrow \Sigma(L,\mathcal{F})$ be a uniformly continuous map. Then there is a unique morphism $\overline{f} : (L,\mathcal{F}) \longrightarrow \Omega(X,\mathcal{E})$ in WUFrm such that the diagram

$$\begin{tikzcd}
(X,\mathcal{E}) \arrow[r, "\eta_{(X,\mathcal{E})}"] \arrow[dr, "f"'] & \Sigma\Omega(X,\mathcal{E}) \arrow[d, dashed, "\Sigma(\overline{f})"] \\
& \Sigma(L,\mathcal{F})
\end{tikzcd}$$

commutes. Indeed, if one requires that, for every $x \in X$, $\Sigma(\overline{f}) \cdot \eta_{(X,\mathcal{E})}(x) = f(x)$, then, necessarily, for every $x \in X$ and $a \in L$, $x \in \overline{f}(a)$ if and only if $f(x)(a) = 1$, that is, for every $a \in L$, $\overline{f}(a)$ must be equal to $f^{-1}(\Sigma_a) \in \mathcal{T}_{\mathcal{E}}$. Furthermore, this $\overline{f}$ is a Weil uniform homomorphism. Obviously, it is a frame map. For any Weil entourage $F \in \mathcal{F}$, if we consider a symmetric $G \in \mathcal{F}$ such that $G^2 \subseteq F$ then $E := (f \times f)^{-1}(\Sigma_G) \in \mathcal{E}$. Thus, in order to prove that $(\overline{f} \oplus \overline{f})(F)$ is a Weil entourage of $\Omega(X,\mathcal{E})$, we only have to show that $\bigvee_{x\in X}(E[x] \oplus E[x]) \subseteq (\overline{f} \oplus \overline{f})(F)$. We have for each $x \in X$

$$E[x] = \Big\{y \in X \mid (f(x), f(y)) \in \bigcup_{(a,b)\in G} (\Sigma_a \times \Sigma_b)\Big\}$$

$$= \Big\{ y \in X \mid (x,y) \in \bigcup_{(a,b)\in G} (\overline{f}(a) \times \overline{f}(b)) \Big\}$$
$$= \bigcup \Big\{ \overline{f}(b) \mid (a,b) \in G, x \in \overline{f}(a) \Big\}.$$

On the other hand, for any $(a,b),(c,d) \in G$ with $x \in \overline{f}(a) \cap \overline{f}(b)$,

$$\Big(\overline{f}(b), \overline{f}(d)\Big) \in (\overline{f} \oplus \overline{)} f(G) \circ (\overline{f} \oplus \overline{f})(G) \subseteq (\overline{f} \oplus \overline{f})(G^2)$$

so

$$(E[x], E[x]) \in (\overline{f} \oplus \overline{f})(G^2) \subseteq (\overline{f} \oplus \overline{f})(F).$$

In conclusion, we have a dual adjunction, being $\eta_{(X,\mathcal{E})}$ one of the adjunction units. The other is given by

$$\begin{array}{rccc} \xi_{(L,\mathcal{E})} : & (L,\mathcal{E}) & \longrightarrow & \Omega\Sigma(L,\mathcal{E}) \\ & x & \longmapsto & \Sigma_x. \end{array}$$ ■

5. The isomorphism between the categories UFrm, WUFrm and EUFrm

The functor Ψ : UFrm $\longrightarrow$ WUFrm

Let U be a cover of L. We say that $x \in L$ is U-*small* if $x \leq st(z,U)$ whenever $x \wedge z \neq 0$, and that a pair $(x,y) \in L \times L$ is U-*small* if $x \vee y \leq st(z,U)$ whenever $x \wedge z \neq 0$ and $y \wedge z \neq 0$. Note that this does not imply that x and y are U-small. However, (x,x) is U-small if and only if x is U-small.

Now, let $\mathcal{U}$ be a family of covers of L. For each $U \in \mathcal{U}$ consider the Weil entourage

$$E_U := \bigvee_{x\in U} (x \oplus x),$$

and denote the set $\{E_U \mid U \in \mathcal{U}\}$ by $\mathcal{E}_{\mathcal{U}}$.

In the following lemma we conclude that one can control in terms of U-smallness the elements that go inside the Weil entourage E_U. This result will play a crucial role in our conclusion that Weil entourages also characterize frame uniformities.

Lemma 5.1. *For $U \in Cov(L)$, if $(x, y) \in E_U$ then (x, y) is U-small.*

Proof. Since E_U is symmetric it suffices to show that $x \leq st(z, U)$ whenever $(x, y) \in E_U$ and $y \wedge z \neq 0$. But this is consequence of the following result:

$$E_U \circ (x_1 \oplus x_2) \subseteq st(x_1, U) \oplus st(x_2, U) \text{ for every } x_1, x_2 \in L. \tag{5.1.1}$$

In fact, by 5.1.1, $E_U \circ (z \oplus z) \subseteq st(z, U) \oplus st(z, U)$ thus $(x, z) \in st(z, U) \oplus st(z, U)$, because $(x, y \wedge z) \in E_U$ and $(y \wedge z, z) \in z \oplus z$.

So, let us show 5.1.1:

According to Lemma 4.2 (a),

$$E_U \circ (x_1 \oplus x_2) = \Big(\bigcup_{z \in U} (z \oplus z)\Big) \circ (x_1 \oplus x_2).$$

Consider $(a, b) \in \bigcup_{z \in U}(z \oplus z)$ and $(b, c) \in x_1 \oplus x_2$ with $a, b, c \neq 0$. We have $(a, b) \leq (z, z)$ for some $z \in U$ and $z \wedge x_1 \geq b \neq 0$. Then $a \leq z \leq st(x_1, U) \leq y_1$ and, on the other hand, $c \leq x_2 \leq st(x_2, U) \leq y_2$, thus $(a, c) \in st(x_1, U) \oplus st(x_2, U)$. ■

Remark 5.2. In the sequel, we only need the following particular case of Lemma 5.1:

Let U be a cover of L. If $(x, x) \in E_U$ then x is U-small.

We are now able to prove that $\mathcal{E}_{\mathcal{U}}$ is a Weil uniformity basis whenever $\mathcal{U}$ is a uniformity basis.

Proposition 5.3. *Let $\mathcal{U}$ be a uniformity basis on a frame L. Then $\mathcal{E}_{\mathcal{U}}$ is a Weil uniformity basis on L.*

Proof. Let $E_U, E_V \in \mathcal{E}_{\mathcal{U}}$. Take $W \in \mathcal{U}$ such that $W \leq U \wedge V$. Clearly $E_W \subseteq E_U \cap E_V$, thus $\mathcal{E}_{\mathcal{U}}$ is a filter basis of Weil entourages of L.

Consider $E_U \in \mathcal{E}_{\mathcal{U}}$, and take $V \in \mathcal{U}$ such that $V^* \leq U$. Then $E_V \circ E_V \subseteq E_U$:

By Lemma 4.2 (a) it follows that

$$E_V \circ E_V = \Big(\bigcup_{x \in V} (x \oplus x)\Big) \circ \Big(\bigcup_{x \in V} (x \oplus x)\Big).$$

Let $(a,b) \leq (x,x)$ and $(b,c) \leq (y,y)$ where $x, y \in V$ and $b \neq 0$. Then $x \wedge y \neq 0$, $a \leq x \leq st(x,V)$ and $c \leq y \leq st(x,V)$. As $st(x,V) \in V^* \leq U$, this says that there is $u \in U$ such that $a \leq u$ and $c \leq u$ and, consequently, that $(a,c) \in E_U$.

The symmetry condition (UW3) is obviously satisfied since each E_U is symmetric.

Finally, let us check the admissibility condition (UW4). Assume $x \in L$. By hypothesis, $x = \bigvee\{y \in L \mid y \overset{\mathcal{U}}{\triangleleft} x\}$. We check (UW4) by showing that, for any $y \in L$ satisfying $y \overset{\mathcal{U}}{\triangleleft} x$, $y \overset{\mathcal{E}_\mathcal{U}}{\triangleleft} x$. So, consider $y \in L$ with $y \overset{\mathcal{U}}{\triangleleft} x$ and take $U \in \mathcal{U}$ satisfying $st(y,U) \leq x$. We claim that $st(y, E_U) \leq x$. Consider $(z,z) \in E_U$ such that $z \wedge y \neq 0$. By Remark 5.2, z is U-small thus $z \leq st(y,U) \leq x$. Hence $st(y, E_U) \leq x$. ■

In the sequel, for every uniformity $\mathcal{U}$, $\psi(\mathcal{U})$ denotes the Weil uniformity generated by $\mathcal{E}_\mathcal{U}$. The correspondence $(L,\mathcal{U}) \longrightarrow (L, \psi(\mathcal{U}))$ is functorial. Indeed, it is the function on objects of a functor Ψ : UFrm $\longrightarrow$ WUFrm whose function on morphisms is described in the following proposition:

Proposition 5.4. *Let* $(L,\mathcal{U})$ *and* $(L',\mathcal{U}')$ *be uniform frames and let* $f : (L,\mathcal{U}) \longrightarrow (L',\mathcal{U}')$ *be a uniform homomorphism. Then* $f : (L, \psi(\mathcal{U})) \longrightarrow (L', \psi(\mathcal{U}'))$ *is a Weil uniform homomorphism.*

Proof. It is obvious since, for every $U \in \mathcal{U}$,

$$(f \oplus f)(E_U) = \bigvee_{x \in U} (f(x) \oplus f(x)) = E_{f[U]}.$$ ■

Example 5.5. The Weil uniformities which we considered in localic groups (Example 4.9 (d)) are the Weil uniformities induced, via functor Ψ, by the left and right uniformities of [40] (Proposition 3.2) whose bases are given by, respectively,

$$\mathcal{U} = \{U(x) \mid \varepsilon(x) = 1\}, \quad \text{where } U(x) = \{y \in L \mid y \oplus \imath(y) \subseteq \mu(x)\},$$

and

$$\mathcal{V} = \{V(x) \mid \varepsilon(x) = 1\}, \quad \text{where } V(x) = \{y \in L \mid \imath(y) \oplus y \subseteq \mu(x)\}.$$

Let us see, for example, that $\mathcal{E}_\mathcal{V} = \{E_{V(x)} \mid \varepsilon(x) = 1\}$ and $\mathcal{E}^r$ generate the same Weil uniformity. Of course, $E_{V(x)} = \bigvee\{y \oplus y \mid \imath(y) \oplus y \subseteq \mu(x)\}$ is contained in $E^r_x = (\imath \oplus 1_L) \cdot \mu(x)$. On the other hand, for every $x \in L$, let $y \in L$ be such that $(E^r_y)^2 \subseteq E^r_x$, and consider the symmetric Weil entourage $E^r_{y\wedge\imath(y)} = E^r_y \cap E^r_{\imath(y)} = E^r_y \cap (E^r_y)^{-1}$. Then $E^r_{y\wedge\imath(y)}$ is contained in $E_{V(x)}$: for $(a,b) \in E^r_{y\wedge\imath(y)}$ with $a,b \neq 0$, $(b,a) \in E^r_{y\wedge\imath(y)}$ and $(a,a),(b,b) \in (E^r_{y\wedge\imath(y)})^2$, so $(a\vee b, a\vee b) \in (E^r_{y\wedge\imath(y)})^2 \subseteq E^r_x$. Thus $(a\vee b)\oplus(a\vee b) \subseteq E_{V(x)}$ and, therefore, $(a,b) \in E_{V(x)}$.

The functor $\Phi : \mathsf{WUFrm} \longrightarrow \mathsf{EUFrm}$

Let $\mathcal{E} \subseteq L \oplus L$. For each $E \in \mathcal{E}$ define $e_E : L \longrightarrow L$ by $e_E(x) = st(x,E)$ and denote the set $\{e_E \mid E \in \mathcal{E}\}$ by $\mathcal{M}_{\mathcal{E}}$. It is obvious that each e_E preserves arbitrary joins and that it is an entourage of L whenever E is a Weil entourage because x is e_E-small provided that $(x,x) \in E$.

Proposition 5.6. *Let $\mathcal{E}$ be a Weil uniformity basis on a frame L. Then $\mathcal{M}_{\mathcal{E}}$ is an entourage uniformity basis on L.*

Proof. Let us check conditions (UE1)-(UE4) of Definition 3.1.

(UE1) Let $e_E, e_F \in \mathcal{M}_{\mathcal{E}}$. In order to prove that $\mathcal{M}_{\mathcal{E}}$ is a filter basis just take e_G, for some Weil entourage G such that $G \subseteq E \cap F$.

(UE2) For $e_E \in \mathcal{M}_{\mathcal{E}}$ consider $F \in \mathcal{E}$ such that $F^2 \subseteq E$. In the proof of Proposition 4.8 (c) we observed that $st(st(x,F),F) \leq st(x,F^2)$. Hence $e_F^2 \leq e_E$.

(UE3) Let $E \in \mathcal{E}$ and $x, y \in L$. Then we have that

$$x \wedge e_E(y) = 0 \Leftrightarrow \bigvee\{x \wedge u \mid (u,u) \in E \text{ and } u \wedge y \neq 0\} = 0, \tag{5.6.1}$$

and, analogously,

$$e_E(x) \wedge y = 0 \Leftrightarrow \bigvee\{y \wedge u \mid (u,u) \in E \text{ and } u \wedge x \neq 0\} = 0. \tag{5.6.2}$$

Obviously 5.6.1 and 5.6.2 are equivalent.

(UE4) It is trivial, since $y \overset{\mathcal{E}}{\triangleleft} x$ if and only if $y \overset{\mathcal{M}_{\mathcal{E}}}{\triangleleft} x$. ∎

In what follows, if $\mathcal{E}$ is a Weil uniformity on L, then $\phi(\mathcal{E})$ denotes the entourage uniformity generated by $\mathcal{M}_{\mathcal{E}}$. The correspondence $(L,\mathcal{E}) \longmapsto (L,\phi(\mathcal{E}))$ is functorial:

Proposition 5.7. *Let $(L,\mathcal{E})$ and $(L',\mathcal{E}')$ be Weil uniform frames and let $f:(L,\mathcal{E}) \longrightarrow (L',\mathcal{E}')$ be a Weil uniform homomorphism. Then $f:(L,\phi(\mathcal{E})) \longrightarrow (L',\phi(\mathcal{E}'))$ is an entourage uniform homomorphism.*

Proof. Let $e_E \in \mathcal{M}_{\mathcal{E}}$, where $E \in \mathcal{E}$. Take a symmetric $F \in \mathcal{E}$ such that $F^2 \subseteq E$. Since f is a Weil uniform homomorphism, $(f \oplus f)(F) \in \mathcal{E}'$. In order to show that $f:(L,\phi(\mathcal{E})) \longrightarrow (L',\phi(\mathcal{E}'))$ is uniform it suffices to show that $e_{(f\oplus f)(F)} \cdot f \leq f \cdot e_E$.

So, fix $x \in L$ and take $y \in L'$ such that $(y,y) \in (f \oplus f)(F)$ and $y \wedge f(x) \neq 0$. Then $(y, y \wedge f(x)) \in (f \oplus f)(F)$ and $(y \wedge f(x), f(x)) \in f(x) \oplus f(x)$, and, consequently, $(y, f(x)) \in (f \oplus f)(F) \circ (f(x) \oplus f(x))$. Further, since F is of the form $\bigvee_{\gamma\in\Gamma}(a_\gamma \oplus b_\gamma)$, for some subset $\{(a_\gamma, b_\gamma) \mid \gamma \in \Gamma\}$ of $L \times L$, we have that

$$\begin{aligned} (f \oplus f)(F) \circ (f(x) \oplus f(x)) &= \Big((f \oplus f)\Big(\bigvee_{\gamma\in\Gamma}(a_\gamma \oplus b_\gamma)\Big)\Big) \circ (f(x) \oplus f(x)) \\ &= k\Big(\bigcup_{\gamma\in\Gamma}(f(a_\gamma) \oplus f(b_\gamma))\Big) \circ k\Big(\downarrow(f(x), f(x))\Big), \end{aligned}$$

so, by Lemma 4.2 (a),

$$(f \oplus f)(F) \circ (f(x) \oplus f(x)) = \Big(\bigcup_{\gamma\in\Gamma}(f(a_\gamma) \oplus f(b_\gamma))\Big) \circ (\downarrow(f(x), f(x))).$$

But

$$\Big(\bigcup_{\gamma\in\Gamma}(f(a_\gamma) \oplus f(b_\gamma))\Big) \circ (\downarrow(f(x), f(x)))$$

is contained in $(f \cdot e_E)(x) \oplus f(x)$:

For any

$$(a,b) \in \Big(\bigcup_{\gamma\in\Gamma}(f(a_\gamma) \oplus f(b_\gamma)) \cdot \downarrow(f(x), f(x))\Big) \setminus \mathbf{O},$$

there exist $c \in L \setminus \{0\}$ and $\gamma \in \Gamma$ such that $(a,c) \leq (f(a_\gamma), f(b_\gamma))$ and $(c,b) \leq (f(x), f(x))$. It follows that $a \leq f(a_\gamma \vee b_\gamma)$ and, therefore, that $a \leq (f \cdot e_E)(x)$. Indeed, $(a_\gamma \vee b_\gamma) \wedge x \neq 0$ because $f(b_\gamma \wedge x) \geq c \neq 0$ and, by the symmetry of F, $(a_\gamma \vee b_\gamma, a_\gamma \vee b_\gamma) \in F^2 \subseteq E$.

In conclusion, we have that

$$(y, f(x)) \in (f \oplus f)(F) \circ f(x) \oplus f(x) \subseteq (f \cdot e_E)(x) \oplus f(x).$$

Hence $y \leq (f \cdot e_E)(x)$ which implies that $e_{(f \oplus f)(F)}(f(x)) \leq f(e_E(x))$, as required. ■

We shall denote the functor defined above by Φ.

The functor $\Theta : \mathsf{EUFrm} \longrightarrow \mathsf{UFrm}$

For each entourage e of L let U_e be the cover of all e-small elements of L.

Proposition 5.8. (Fletcher and Hunsaker [23]) *Let $\mathcal{M}$ be an entourage uniformity basis on a frame L. Then $\mathcal{U}_{\mathcal{M}} := \{U_e \mid e \in \mathcal{M}\}$ is a uniformity basis on L.*

Proof. (U1) Consider $U_e, U_f \in \mathcal{U}_{\mathcal{M}}$ and let $g \in \mathcal{M}$ such that $g \leq e \wedge f$. Then it is obvious that $U_g \leq U_e \wedge U_f$.

(U2) Let $U_e \in \mathcal{U}_{\mathcal{M}}$ and take $f \in \mathcal{M}$ such that $f^3 \leq e$. We claim that $U_f^* \leq U_e$. Consider $st(x, U_f) \in U_f^*$. It suffices to show that $st(x, U_f)$ is e-small. So, consider $z \in L$ such that $z \wedge st(x, U_f) \neq 0$. Then there is $y \in U_f$ with $y \wedge x \neq 0$ and $y \wedge z \neq 0$. The f-smallness of x and y implies then that $x \leq f^2(z)$. Therefore, for every $y' \in U_f$ such that $y' \wedge x \neq 0$ we have $y' \leq f(x) \leq f^3(z) \leq e(z)$.

(U3) $\bigvee\{y \in L \mid y \overset{\mathcal{U}_{\mathcal{M}}}{\lhd} x\}$ is always below x because $y \overset{\mathcal{U}_{\mathcal{M}}}{\lhd} x$ implies $y \leq x$. In order to conclude that $x \leq \bigvee\{y \in L \mid y \overset{\mathcal{U}_{\mathcal{M}}}{\lhd} x\}$ it suffices to prove that $y \overset{\mathcal{M}}{\lhd} x$ implies $y \overset{\mathcal{U}_{\mathcal{M}}}{\lhd} x$. Let $e \in \mathcal{M}$ with $e(y) \leq x$. Then, immediately, $z \leq e(y) \leq x$ for every $z \in U_e$ such that $z \wedge y \neq 0$, that is, $st(y, U_e) \leq x$. ■

In the sequel, if $\mathcal{M}$ is an entourage uniformity, $\theta(\mathcal{M})$ denotes the uniformity generated by $\mathcal{U}_{\mathcal{M}}$.

On the other hand, with respect to morphisms we have:

Proposition 5.9. (Fletcher and Hunsaker [23]) *Let $(L, \mathcal{M})$ and $(L', \mathcal{M}')$ be entourage uniform frames and let $f : (L, \mathcal{M}) \longrightarrow (L', \mathcal{M}')$ be an entourage uniform homomorphism. Then $f : (L, \theta(\mathcal{M})) \longrightarrow (L', \theta(\mathcal{M}'))$ is a uniform homomorphism.*

Proof. Let $U \in \theta(\mathcal{M})$ and let $e \in \mathcal{M}$ such that $U_{e^3} \leq U$. By hypothesis, there exists $g \in \mathcal{M}'$ with $g \cdot f \leq f \cdot e$. We show that $U_g \leq f[U_{e^3}]$. Let x be a non-zero

g-small element of L'. Since $f[U_e]$ is a cover of L', there exists $y \in U_e$ satisfying $x \wedge f(y) \neq 0$. Consequently, $x \leq g \cdot f(y) \leq f \cdot e(y)$. But, as can be easily proved, the fact that y is e-small implies that $e(y)$ is e^3-small. In conclusion, $U_g \leq f[U_{e^3}] \leq f[U]$ and $f[U] \in \theta(\mathcal{M}')$. ■

We have this way a functor EUFrm $\longrightarrow$ UFrm which we denote by Θ.

The isomorphism

Finally, let us show that the functors Ψ, Φ and Θ define an isomorphism between the categories UFrm, WUFrm and EUFrm.

Lemma 5.10. *For any cover U of L we have that:*

(a) $U \leq U_{e_{E_U}}$;

(b) $U_{e_{E_U}} \leq U^*$.

Proof.

(a) Let $x \in U$. For any $y \in L$ satisfying $x \wedge y \neq 0$, since $(x, x) \in E_U$, $x \leq st(y, E_U) = e_{E_U}(y)$, that is, x is e_{E_U}-small and, therefore, $x \in U_{e_{E_U}}$.

(b) For any non-zero e_{E_U}-small member x of L, there exists $y \in U$ such that $x \wedge y \neq 0$. Then $x \leq st(y, E_U)$. But, for every $(z, z) \in E_U$, z is U-small, so $z \leq st(y, U)$ in case $z \wedge y \neq 0$. This means that $st(y, E_U) \leq st(y, U)$. Hence $x \leq st(y, U) \in U^*$.

■

The corresponding property for Weil entourages is the following:

Lemma 5.11. *For any symmetric Weil entourage E of L we have that:*

(a) $E \subseteq E_{U_{e_{E^2}}}$;

(b) $E_{U_{e_E}} \subseteq E^2$.

Proof.

(a) Consider $(x,y) \in E \setminus \mathbb{O}$. Then $(x \vee y, x \vee y) \in E^2$ because $(x,y), (y,x), (x,x)$ and (y,y) belong to E^2. Since every member of E^2 with equal coordinates is e_{E^2}-small, $(x \vee y, x \vee y) \in E_{U_{e_{E^2}}}$ and, consequently, $(x,y) \in E_{U_{e_{E^2}}}$.

(b) Let us verify that

$$\bigcup \{x \oplus x \mid x \text{ is } e_E\text{-small }\} \subseteq E^2$$

or, which is the same, that $(x,x) \in E^2$ whenever x is e_E-small. Consider $x \neq 0$, e_E-small. We have that $x \leq \bigvee\{z \in L \mid (z,z) \in E, z \wedge x \neq 0\}$. Since x is e_E-small, we get $x \leq e_E(z) = \bigvee\{y \in L \mid (y,y) \in E, y \wedge z \neq 0\}$, for any z such that $(z,z) \in E$ and $x \wedge z \neq 0$. For each y in this set we have that $(z, z \wedge y)$, $(z \wedge y, y) \in E$, which implies that $(z,y) \in E^2$. Therefore $(z,x) \in E^2$ and, consequently, $(x,x) \in E^2$. ■

Finally, for entourages we have:

Lemma 5.12. *Let f be a symmetric entourage of L. Then:*

(a) $e \leq e_{E_{U_{e^3}}}$;

(b) $e_{E_{U_e}} \leq e$.

Proof.

(a) By definition, for any $x \in L$, $e_{E_{U_{e^3}}}(x) = st(x, E_{U_{e^3}})$. On the other hand,

$$e(x) = e\left(\bigvee\{x \wedge y \mid y \text{ is } e\text{-small }\}\right) = \bigvee\{e(x \wedge y) \mid y \text{ is } e\text{-small and } x \wedge y \neq 0\}.$$

Consider any e-small element y such that $x \wedge y \neq 0$. Evidently $e(x \wedge y)$ is e^3-small so $\big(e(x \wedge y), e(x \wedge y)\big) \in E_{U_{e^3}}$. Since $e(x \wedge y) \wedge (x \wedge y) = x \wedge y \neq 0$, it follows that

$$e(x \wedge y) \leq e_{E_{U_{e^3}}}(x \wedge y) \leq e_{E_{U_{e^3}}}(x).$$

Hence $e(x) \leq e_{E_{U_{e^3}}}(x)$.

(b) By Remark 5.2, y is U_e-small whenever $(y,y) \in E_{U_e}$. Therefore

$$e_{E_{U_e}} = \bigvee\{y \in L \mid (y,y) \in E_{U_e}, y \wedge x \neq 0\} \leq st(x, U_e) \leq e(x). \qquad ■$$

Proposition 5.13. *Let $\mathcal{U}$, $\mathcal{E}$ and $\mathcal{M}$ denote, respectively, a uniformity, a Weil uniformity and an entourage uniformity on L. Then $\theta\phi\psi(\mathcal{U}) = \mathcal{U}$, $\psi\theta\phi(\mathcal{E}) = \mathcal{E}$ and $\phi\psi\theta(\mathcal{M}) = \mathcal{M}$.*

Proof. We first show that $\theta\phi\psi(\mathcal{U}) = \mathcal{U}$. The uniformity $\theta\phi\psi(\mathcal{U})$ has $\{U_e \mid e \in \phi\psi(\mathcal{U})\}$ as a basis. It suffices to prove that this is a basis for $\mathcal{E}$, which is a consequence of Lemma 5.10: by (a), $\{U_e \mid e \in \phi\psi(\mathcal{U})\} \subseteq \mathcal{U}$, and, by (b), for any $U \in \mathcal{U}$ there is some $V \in \mathcal{U}$ such that $U_{e_{E_V}} \subseteq U$.

Similarly, Lemma 5.11 implies that the basis $\{E_U \mid U \in \theta\phi(\mathcal{E})\}$ of $\psi\theta\phi(\mathcal{E})$ is also a basis for $\mathcal{E}$, which proves the second equality, and Lemma 5.12 implies that the basis $\{e_E \mid E \in \psi\theta(\mathcal{M})\}$ of $\phi\psi\theta(\mathcal{M})$ is also a basis for $\mathcal{M}$, which proves the equality $\phi\psi\theta(\mathcal{M}) = \mathcal{M}$. ■

In summary, it follows from Propositions 5.4, 5.7, 5.9 and 5.13 that:

Theorem 5.14. *The categories* UFrm, WUFrm *and* EUFrm *are isomorphic.* ■

6. An application: a theorem of Efremovič for uniform spaces in pointfree context

As it is the case for the several ways of endowing a set with a topological structure, sometimes one of the notions of cover, entourage or Weil entourage is more suitable than another for a particular use. We end the chapter with an illustration of this. We take the liberty of using either (Weil entourage or covering) approach to the problem at hand. This free movement between uniformities allows greater flexibility.

In [20] (cf. also Lemma 12.17 and Theorem 12.18 of [56]) Efremovič proved that, in the realm of uniform spaces,

> *if two uniformities of countable type on the same set have the same Samuel compactification, they are equal.*

Our purpose in this section is to prove the following theorem:

Theorem 6.1. *If two uniformities of countable type on the same frame have the same totally bounded coreflection, they are equal.*

Recall that a uniform frame is *totally bounded* if its uniformity has a basis of finite covers. Totally bounded uniform frames are coreflective in UFrm; the totally bounded coreflection of a uniform frame $(L,\mathcal{U})$ is constructed as follows: let $\mathcal{U}_\#$ be the filter of $(Cov(L),\leq)$ generated by the finite covers of $\mathcal{U}$. The pair $(L,\mathcal{U}_\#)$ is a uniform frame and it is the totally bounded coreflection of $(L,\mathcal{U})$, the coreflector map $(L,\mathcal{U}_\#) \longrightarrow (L,\mathcal{U})$ being the identity map of the underlying frames(cf. [10]).

The Samuel compactification of a uniform frame $(L,\mathcal{U})$ — i.e., its compact regular coreflection (which was firstly constructed by Banaschewski and Pultr in [10]) — can be described in the following way: take the frame $\mathcal{R}(L,\mathcal{U})$ of all regular ideals of L (an ideal I of L is *regular* whenever $x \in I$ implies that $x \overset{\mathcal{U}}{\triangleleft} y$ for some $y \in I$). Since $\mathcal{R}(L,\mathcal{U})$ is a compact regular frame, it has a unique uniformity $\mathcal{U}_{\mathcal{R}(L,\mathcal{U})}$ generated by all its finite covers. Moreover, the join map $\bigvee : \mathcal{R}(L,\mathcal{U}) \longrightarrow L$ taking each regular ideal to its join is a uniform homomorphism from $(\mathcal{R}(L,\mathcal{U}),\mathcal{U}_{\mathcal{R}(L,\mathcal{U})})$ to $(L,\mathcal{U})$. The pair $(\mathcal{R}(L,\mathcal{U}),\mathcal{U}_{\mathcal{R}(L,\mathcal{U})})$ is the Samuel compactification of $(L,\mathcal{U})$ and the join map is the coreflector map (for the details consult [10]).

But, as it is well-known:

Proposition 6.2. *Two uniform frames with the same underlying frame have the same totally bounded coreflection whenever they have the same Samuel compactification.*

Then, immediately, one gets from Theorem 6.1 the frame version of that result of Efremovič:

Corollary 6.3. *If two uniformities of countable type on the same frame have the same Samuel compactification, they are equal.* ■

The proofs

First, let us prove Proposition 6.2:

Assume that $(L,\mathcal{U})$ and $(L,\mathcal{V})$ are two uniform frames with the same Samuel compactification and consider a cover U in $\mathcal{U}_\#$. Then there is a finite refinement $V \in \mathcal{U}_\#$ of U. For each $x \in V$,

$$\Downarrow x := \{y \in L \mid y \overset{\mathcal{U}}{\triangleleft} x\} \in \mathcal{R}(L,\mathcal{U})$$

and, by the finiteness of V, $\bigvee\{\Downarrow x \mid x \in V\} = L$ (see [10]), that is, $\{\Downarrow x \mid x \in V\}$ is a finite cover of $\mathcal{R}(L,\mathcal{U})$. Therefore

$$\{\Downarrow x \mid x \in V\} \in \mathcal{U}_{\mathcal{R}(L,\mathcal{U})} = \mathcal{U}_{\mathcal{R}(L,\mathcal{V})}.$$

But

$$\bigvee : (\mathcal{R}(L,\mathcal{V}), \mathcal{U}_{\mathcal{R}(L,\mathcal{V})}) \longrightarrow (L, \mathcal{V}_\#)$$

is uniform thus $\{\bigvee \Downarrow x \mid x \in V\} \in \mathcal{V}_\#$, that is, $V \in \mathcal{V}_\#$.

In conclusion, U also belongs to $\mathcal{V}_\#$ and, therefore, $\mathcal{U}_\# \subseteq \mathcal{V}_\#$. Similarly, $\mathcal{V}_\# \subseteq \mathcal{U}_\#$.

■

Now, let us prove Theorem 6.1:

Consider two uniform frames, $(L,\mathcal{U})$ and $(L,\mathcal{V})$, defined in covering terms, with countable bases and the same totally bounded coreflection, and let $(U_n)_{n\in\mathbb{N}}$ be a descending basis for $\mathcal{U}$, i.e., such that $U_{n+1} \subseteq U_n$ for every natural n.

Suppose $\mathcal{V} \not\subseteq \mathcal{U}$. Then $\psi(\mathcal{V}) \not\subseteq \psi(\mathcal{U})$ (otherwise $\mathcal{V} = \theta\phi\psi(\mathcal{V}) \subseteq \theta\phi\psi(\mathcal{U}) = \mathcal{U}$).

Let $V \in \mathcal{V}$ such that $E_V \notin \psi(\mathcal{U})$, that is, $E_{U_n} \not\subseteq E_V$ for all n. Then, take $(a_n, b_n) \in E_{U_n} \setminus E_V$, for each n; further, take covers W_1, W_2 and Y in $\mathcal{V}$ such that $W_1^{**} \leq V$, $W_2^{**} \leq W_1$ and $Y^{**} \leq W_2$. Note that a_n and b_n are non-zero since $(a_n, b_n) \notin E_V$. For each n,

$$a_n = a_n \wedge 1 = \bigvee\{a_n \wedge y \mid y \in Y, a_n \wedge y \neq 0\};$$

by definition of C-ideal, $(a_n, b_n) \notin E_V$ implies that there is $y_n^1 \in Y$ such that $a_n \wedge y_n^1 \neq 0$ and $(a_n \wedge y_n^1, b_n) \notin E_V$. Put

$$c_n := a_n \wedge y_n^1.$$

Similarly there is, for each n, some $y_n^2 \in Y$ such that $(c_n, b_n \wedge y_n^2) \notin E_V$. Put

$$d_n := b_n \wedge y_n^2.$$

So, for every $n \in \mathbb{N}$, $(c_n, d_n) \in E_{U_n} \setminus E_V$ and $c_n, d_n \in {\downarrow} Y$.

Let

$$S := \{c_n, d_n \mid n \in \mathbb{N}\}.$$

The following auxiliary result will be useful in the sequel.

Lemma 6.4. *There exists an infinite subset I of $\mathbb{N}$ such that*

$$\left(\bigvee_{i\in I} st(c_i, W_2)\right) \wedge \left(\bigvee_{i\in I} st(d_i, W_2)\right) = 0.$$

Proof.

Case I: For some $x \in S$, $T := S \cap \{y \in L \mid (y, x) \in E_{W_1}\}$ is infinite.

Then, either $\{i \in \mathbb{N} \mid c_i \in T\}$ or $\{i \in \mathbb{N} \mid d_i \in T\}$ is infinite and this provides the desired set I. In fact, assume that $I = \{i \in \mathbb{N} \mid c_i \in T\}$ is infinite (the case of $\{i \in \mathbb{N} \mid d_i \in T\}$ being infinite can be proved in a similar way, by symmetry). We must show that, for any $i, j \in I$,

$$st(c_i, W_2) \wedge st(d_j, W_2) = 0.$$

To see this consider i and j in I and suppose (for a contradiction) that there is a pair $(c, d) \in W_2 \times W_2$ satisfying $c \wedge c_i \neq 0$, $d \wedge d_j \neq 0$ and $c \wedge d \neq 0$. Then (c, c) and (d, d) belong to E_{W_2} and, consequently, $(d, c) \in E_{W_2} \circ E_{W_2}$. Further, as i and j are supposed to be in I, (c_i, x) and (c_j, x) belong to E_{W_1}. Since x is non-zero and the Weil entourage E_{W_1} is symmetric, $(c_i, c_j) \in E_{W_1} \circ E_{W_1}$. On the other hand, from the facts that $Y \leq W_2$, $c_i \leq y_i^1 \in Y$ and $d_j \leq y_j^2 \in Y$, it follows that (c_i, c_i) and (d_j, d_j) are both in E_{W_2} and, therefore, that (c, c_i) and (d_j, d) belong to $E_{W_2} \circ E_{W_2}$. Thus $(d_j, c_i) \in E_{W_2}^4 \subseteq E_{W_1}$. But $(c_i, c_j) \in E_{W_1}^2$ so $(d_j, c_j) \in E_{W_1}^3 \subseteq E_V$, which is a contradiction.

Case II: Each $S \cap \{y \in L \mid (y, x) \in E_{W_1}\}$ is finite.

The pair (c_1, d_1) does not belong to E_V so c_1 and d_1 are not E_{W_1}-*near*, that is, $(c_1, d_1) \notin E_{W_1}$. Define $i_1 := 1$. The hypotheses that $S \cap \{y \in L \mid (y, c_1) \in E_{W_1}\}$ and $S \cap \{y \in L \mid (y, d_1) \in E_{W_1}\}$ are finite imply the existence of a natural i such that $c_i, d_i \notin S \cap \{y \in L \mid (y, c_1) \in E_{W_1}\}$ and $c_i, d_i \notin S \cap \{y \in L \mid (y, d_1) \in E_{W_1}\}$, i.e., that none of c_1, d_1, c_i, d_i are E_{W_1}-near. Define i_2 as the first natural in that conditions. Repeating inductively this reasoning, we obtain a sequence $(i_n)_{n\in\mathbb{N}}$ where i_{n+1} is the first natural k such that none of $c_{i_1}, d_{i_1}, \ldots, c_{i_n}, d_{i_n}, c_k, d_k$ are E_{W_1}-near. This determines the set I:

For every $i, j \in I$,

$$st(c_i, W_2) \wedge st(d_j, W_2) = \bigvee\{c \wedge d \mid c, d \in W_2, c \wedge c_i \neq 0, d \wedge d_j \neq 0\}.$$

But, as we observed in the previous case, if there is a pair $(c, d) \in W_2 \times W_2$ satisfying $c \wedge c_i \neq 0$, $d \wedge d_j \neq 0$ and $c \wedge d \neq 0$, then $(d_j, c_i) \in E_{W_1}$, i.e., d_j and c_i are E_{W_1}-near, which is contradictory with the definition of I. ∎

Now, resuming the proof of Theorem 6.1, let W_0 be the set defined by the following three elements:

$$\bigvee_{i\in I} st(c_i, W_2),$$

$$\bigvee_{i\in I} st(d_i, W_2)$$

and

$$\bigvee\Big\{st(x, Y) \mid x \in L \setminus \Big\{z \in L : z \wedge \bigvee_{i\in I}(st(c_i, Y) \vee st(d_i, Y)) \neq 0\Big\}\Big\}.$$

Let us show that $Y \leq W_0$:

Consider a non-zero $y \in Y$. If

$$y \in L \setminus \Big\{z \in L : z \wedge \bigvee_{i\in I}(st(c_i, Y) \vee st(d_i, Y)) \neq 0\Big\}$$

then

$$\begin{aligned} y &\leq st(y, Y) \\ &\leq \bigvee\Big\{st(x, Y) \mid x \in L \setminus \Big\{z \in L : z \wedge \bigvee_{i\in I}(st(c_i, Y) \vee st(d_i, Y)) \neq 0\Big\}\Big\} \in W_0. \end{aligned}$$

Otherwise,

$$y \wedge \bigvee_{i\in I}\Big(st(c_i, Y) \vee st(d_i, Y)\Big) \neq 0,$$

i.e., there is some $i \in I$ such that $y \wedge st(c_i, Y) \neq 0$ or $y \wedge st(d_i, Y) \neq 0$. Let us assume that $y \wedge st(c_i, Y) \neq 0$ (the other case can be treated in a similar way). Then there is $\overline{y}$ in Y satisfying $\overline{y} \wedge c_i \neq 0$ and $\overline{y} \wedge y \neq 0$. Let Y' be the cover

$$Y \cup \{y_1 \vee y_2 \mid y_1, y_2 \in Y, y_1 \wedge y_2 \neq 0\}.$$

Then $\overline{y} \vee y \in Y'$ and $(\overline{y} \vee y) \wedge c_i \neq 0$ so

$$y \leq \overline{y} \vee y \leq st(c_i, Y') \leq st(y_i^1, Y').$$

But, as one can easily observe, for any cover Y, $Y \leq Y' \leq Y^*$, so, in this case, we may conclude that $Y'^* \leq Y^{**} \leq W_2$. Thus, there is $w \in W_2$ such that $y \leq w$. Hence

$$y \leq w \leq st(c_i, W_2) \leq \bigvee_{i \in I} st(c_i, W_2) \in W_0.$$

In conclusion, W_0 is a finite cover of $\mathcal{V}$.

Moreover
$$\left(\bigvee_{i \in I} st(c_i, W_0)\right) \wedge \left(\bigvee_{i \in I} st(d_i, W_0)\right) = 0 :$$

For any $i, j \in I$,

$$st(c_i, W_0) \wedge st(d_j, W_0) = \bigvee\{u \wedge v \mid u, v \in W_0, u \wedge c_i \neq 0, v \wedge d_j \neq 0\}.$$

On the other hand, if $u \in W_0$ and $u \wedge c_i \neq 0$, then u must be equal to $\bigvee_{i \in I} st(c_i, W_2)$ because

- $u = \bigvee_{j \in I} st(d_j, W_2)$ would imply, due to the definition of the set I in Lemma 6.4, that $c_i \wedge u = 0$, which would be a contradiction;
- $u = \bigvee\Big\{st(x, Y) \mid x \in L \setminus \Big\{z \in L : z \wedge \bigvee_{i \in I}(st(c_i, Y) \vee st(d_i, Y)) \neq 0\Big\}\Big\}$ would imply the existence of x in

  $$L \setminus \Big\{z \in L : z \wedge \bigvee_{i \in I}(st(c_i, Y) \vee st(d_i, Y)) \neq 0\Big\}$$

  satisfying $c_i \wedge st(x, Y) \neq 0$ or, equivalently, $st(c_i, Y) \wedge x \neq 0$, which would imply

  $$x \wedge \bigvee_{i \in I}(st(c_i, Y) \vee st(d_i, Y)) \neq 0,$$

  a contradiction.

Similarly, if $v \in W_0$ and $v \wedge d_j \neq 0$ then $v = \bigvee_{i\in I} st(d_i, W_2)$. So, $u \wedge v = 0$ for any $u, v \in W_0$ such that $u \wedge c_i \neq 0$ and $v \wedge d_j \neq 0$, and the proof of the desired equality is done.

Finally, let us conclude the proof of the theorem. The cover W_0 is finite so it belongs to the totally bounded coreflection $\mathcal{V}_\#$ of $\mathcal{V}$. By hypothesis, $\mathcal{V}_\# = \mathcal{U}_\# \subseteq \mathcal{U}$. Hence $W_0 \in \mathcal{U}$. Consequently, there is some $n \in \mathbb{N}$ with $U_n \leq W_0$ and

$$\left(\bigvee_{i\in I} st(c_i, W_0)\right) \wedge \left(\bigvee_{i\in I} st(d_i, W_0)\right) \geq \left(\bigvee_{i\in I} st(c_i, U_n)\right) \wedge \left(\bigvee_{i\in I} st(d_i, U_n)\right),$$

which is a contradiction because, for any $n \in \mathbb{N}$,

$$\left(\bigvee_{i\in I} st(c_i, U_n)\right) \wedge \left(\bigvee_{i\in I} st(d_i, U_n)\right) \neq 0 :$$

In fact, for every $n \in \mathbb{N}$ there is some $i \in I$ with $i \geq n$. Consider $m \in \mathbb{N}$ such that $U_m^{**} \leq U_i$ and $j \in I$ with $j \geq m$. Then, since $(c_j, d_j) \in E_{U_j}$, $(c_j \vee d_j, c_j \vee d_j) \in E_{U_j} \circ E_{U_j}$. But $E_{U_j} \circ E_{U_j} \leq E_{U_j^*}$. Therefore $(c_j \vee d_j, c_j \vee d_j) \in E_{U_j^*} \leq E_{U_m^*}$, which implies that $c_j \vee d_j \in U_{E_{U_m^*}} \leq U_m^{**}$. Consequently, $c_j \vee d_j \in U_i \leq U_n$. Hence

$$\left(\bigvee_{i\in I} st(c_i, U_n)\right) \wedge \left(\bigvee_{i\in I} st(d_i, U_n)\right) \geq c_j \vee d_j \neq 0. \qquad \blacksquare$$

Notes on Chapter I:

(1) We point out that viewing the categories UFrm, WUFrm and EUFrm as concrete categories over the category of sets, the isomorphisms of Theorem 5.14 are concrete.

The same happens with all categorical isomorphisms that we shall present in the remaining chapters.

(2) Section 6 is an illustration of the way our language of Weil entourages is very convenient and manageable when trying to mimic in frames spatial results. So, in spite of, as for spaces, the covering approach revealed to have many advantages and to be easier to handle with, there are circumstances where the approach via Weil entourages is useful. Undoubtedly, the claim of Frith

[29] that "*covers constitute the only tool that works for frames*" is not true. Perhaps the claim of Isbell [39] that, for uniform spaces, Tukey's approach is most convenient "*nine-tenths of the time*" is also the right one for frames, and Section 6 is part of the remaining one-tenth.

I would like to stress the decisive influence in the demonstration of Theorem 6.1 of a proof for the corresponding result for spaces that was presented to me by Professor Bernhard Banaschewski, whom I am indebted for the suggestion which is in the origin of Section 6.

(3) Another characterization of uniform spaces exists; such a description was given by Bourbaki in [14] in terms of pseudometrics. In the next chapter we shall pursue this approach from the pointless point of view and conclude that, as it happens with the other approaches, this one is also feasible in frames.

(4) After concluding the axiomatization of the category of Weil uniform frames, it is natural to search for the right generalizations of "Weil quasi-uniformity" and "Weil nearness" and to establish their links to the corresponding notions in the literature. We investigate these problems in Chapters III and IV.

CHAPTER II

UNIFORM FRAMES IN THE SENSE OF BOURBAKI

One of the most important results of the theory of uniform spaces states that any uniform cover in a uniform space can be "aproximated" by a pseudometric (see e.g. [77], Lemma 38.1). Thus every uniformity on a set X gives rise to a family of pseudometrics on X, and, moreover, this family of pseudometrics can be used to recover the original uniformity. From this, it follows that uniform spaces can be described in terms of those families of pseudometrics which are usually called "gauge structures". Such a description was given by Bourbaki in [14]. The efficiency of this tool can be observed in [32].

The aim of this chapter is to extend gauge structures to frames. The classical pseudometric is here replaced by the notion of metric diameter of Pultr [64]. We characterize gauge structures for frames as families of metric diameters which completely describe frame uniformities, showing that this alternative approach to uniform frames also works for frames. As an application of this new description for uniform frames we show that there exists a final completion ([2], [35]) of the category of metric frames which contains an isomorphic copy of the category of uniform frames, providing a categorical link between metric and uniform frames, and extending the corresponding result of Adámek and Reiterman [2] for spaces.

1. Gauge spaces

Recall that a *pseudometric* on a set X is a function ρ from $X \times X$ into $\mathbb{R}$, satisfying, for all $x, y, z \in X$:

(a) $\rho(x, y) \geq 0$;

(b) $\rho(x, x) = 0$;

(c) $\rho(x, y) = \rho(y, x)$;

(d) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$.

So, a pseudometric differs from a metric only in that x is not necessarily equal to y whenever $\rho(x, y) = 0$.

Uniformities on a set X can be completely described by collections $\mathcal{G}$ of pseudometrics on X [14], called *gauge structures*, satisfying:

(UP1) if $\rho_1, \rho_2 \in \mathcal{G}$ then

$$\begin{array}{rccl} \rho_1 \vee \rho_2 : & X \times X & \longrightarrow & \mathbb{R} \\ & (x, y) & \longmapsto & \max(\rho_1(x, y), \rho_2(x, y)) \end{array}$$

also belongs to $\mathcal{G}$;

(UP2) if ρ is a pseudometric and

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \exists \rho' \in \mathcal{G} : \rho'(x, y) < \delta \Rightarrow \rho(x, y) < \epsilon,$$

then $\rho \in \mathcal{G}$.

In fact, defining a *gauge space* as a pair $(X, \mathcal{G})$ for X a set and $\mathcal{G}$ a gauge structure on X, and a *gauge homomorphism* $f : (X, \mathcal{G}) \longrightarrow (X', \mathcal{G}')$ between gauge spaces as a map $f : X \longrightarrow X'$ such that, for every $\rho \in \mathcal{G}'$, the pseudometric σ on X given by $\sigma(x, y) = \rho(f(x), f(y))$ belongs to $\mathcal{G}$, the category of gauge spaces and gauge homomorphisms is isomorphic to Unif ([14], [22]).

With the help of some results of Pultr ([64], [67]) one can present a similar approach to uniform frames. This approach will enable us to conclude that the category of uniform frames is fully embeddable in a (universal) final completion of the category of metric frames (Section 4).

2. Metric frames

The way of extending the metric structure to frames, due to Pultr [64], uses the notion of diameter — an extension of the classical distance function — instead of the metric, that we recall briefly in the sequel (for more details, see [9], [67] and [68]).

Let L be a frame. A *prediameter* on L is a map $d : L \longrightarrow [0, +\infty]$ such that

(D1) $d(0) = 0$,

(D2) $d(x) \leq d(y)$ if $x \leq y$, and

(D3) $d(x \vee y) \leq d(x) + d(y)$ if $x \wedge y \neq 0$.

Recall the notation

$$st(x, U) := \bigvee \{y \in U \mid y \wedge x \neq 0\},$$

for subsets U of L and $x \in L$. Obviously

$$st\Big(\bigvee_{i \in I} x_i, U\Big) = \bigvee_{i \in I} st(x_i, U),$$

and hence $st(\_\_, U) : L \longrightarrow L$ has a right adjoint α_U given by

$$\alpha_U(x) = \bigvee \{y \in L \mid st(y, U) \leq x\}.$$

For any prediameter d on L and any $\epsilon > 0$, let U_ϵ^d denote the set

$$\{x \in L \mid d(x) < \epsilon\}.$$

A prediameter d on L is said to be *compatible* if

(D4) for each $x \in L$, $\quad x \leq \bigvee \{y \in L \mid y \overset{d}{\lhd} x\}$,

where $y \overset{d}{\lhd} x$ means that $st(y, U_\epsilon^d) \leq x$ for some $\epsilon > 0$. It is easy to check that d is compatible if and only if, for each $x \in L$, we have $x \leq \bigvee \{\alpha_{U_\epsilon^d}(x) \mid \epsilon > 0\}$. In the sequel we shall denote $\alpha_{U_\epsilon^d}$ simply by α_ϵ^d.

A *star-prediameter* is a prediameter that satisfies

($\star$) if $S \subseteq L$ is *strongly connected* (i.e., $x \wedge y \neq 0$ for every $x, y \in S$), then

$$d(\bigvee S) \leq 2 \sup\{d(x) \mid x \in S\}.$$

A prediameter is *metric* if

(M) for any $\alpha < d(x)$ and $\epsilon > 0$, there exist $y, z \leq x$ such that $d(y), d(z) < \epsilon$ and $\alpha < d(y \vee z)$.

Condition (M) implies ($\star$). In fact, (M) even implies the property

($\star'$) for each $x \in L$ and each $S \subseteq L$ such that $x \wedge y \neq 0$ for all $y \in S$

$$d(x \vee \bigvee S) \leq d(x) + \sup\{d(y) + d(z) \mid y, z \in S, y \neq z\},$$

which is stronger than ($\star$).

A prediameter which, moreover, satisfies

(D5) for each $\epsilon > 0$, U^d_ϵ is a cover of L,

is called a *diameter*. The diameters naturally generalize the usual notion of the diameter of subsets of metric spaces. Note that when d is a compatible diameter the equality holds in (D4).

If L is the topology of some topological space $(X, \mathcal{T})$, the compatible metric diameters d on L correspond exactly to the metrizations ρ of $(X, \mathcal{T})$ via the relations

$$d(U) = \sup\{\rho(x, y) \mid x, y \in U\}$$

and

$$\rho(x, y) = \inf\{d(U) \mid x, y \in U\}.$$

A *prediametric frame* is a pair (L, d), where d is a compatible prediameter on the frame L. The pair (L, d) is a *star-diametric frame* if d is a compatible star-diameter. A *metric frame* is a pair (L, d), where d is a compatible metric diameter on L.

For prediametric frames (L_1, d_1) and (L_2, d_2), a frame homomorphism $f : L_1 \longrightarrow L_2$ is called *uniform* if, for each $\epsilon > 0$, there exists $\delta > 0$ such that $U^{d_2}_\delta \leq f[U^{d_1}_\epsilon]$. Thus we have the category of prediametric frames. The category of star-diametric frames and uniform homomorphisms will be denoted by $\star$-DFrm. Its full subcategory of metric frames will be denoted by MFrm.

The category $\star$-DFrm is epireflective in the category of star-prediametric frames. The reflection is described as follows:

Let (L, d) be a star-prediametric frame and consider the relation R generated by all pairs $(\bigvee U^d_\epsilon, 1)$ with $\epsilon > 0$. On L/R consider the diameter

$$\overline{d}(x) = \inf\{d(y) \mid x \leq \kappa(y)\}$$

induced by d. The projection $\kappa : (L, d) \longrightarrow (L/R, \overline{d})$ is the reflector morphism.

Given a star-diameter d on L, we denote by $\widetilde{d}$ the metric diameter given by

$$\widetilde{d}(x) = \inf_{\epsilon>0} \sup\{d(y \vee z) \mid y, z \leq x, d(y), d(z) \leq \epsilon\},$$

which is the unique metric diameter such that $\frac{1}{2}d \leq \widetilde{d} \leq d$ ([67], Proposition 3.4). Moreover, $st(x, U^{\widetilde{d}}_\epsilon) = st(x, U^d_\epsilon)$, for every $x \in L$ and $\epsilon > 0$ thus $\widetilde{d}$ is compatible whenever d is.

Let d be a compatible metric diameter on L. Since $\epsilon \leq \delta$ implies that $U^d_\epsilon \subseteq U^d_\delta$, (D4) and (D5) say that the family $\{U^d_\epsilon \mid \epsilon > 0\}$ satisfies axioms (U1) and (U3) of Definition I.2.1. This is actually a uniformity since $(U^d_\epsilon)^* \leq U^d_{3\epsilon}$. Conversely, any uniformity with a countable basis is obtained in this way from a compatible metric diameter, and hence a frame has a compatible metric diameter if and only if it has a uniformity with a countable basis.

Finally, let us recall the following way of constructing binary coproducts of metric frames [68]:

Let (L_1, d_1) and (L_2, d_2) be metric frames. On the frame $\mathcal{D}(L_1 \times L_2)$ of all down-sets of the cartesian product (with the usual order) $L_1 \times L_2$, consider the relation R' generated by all pairs

$$\left(\bigcup_{\epsilon>0} \downarrow(\alpha^{d_1}_\epsilon(x), \alpha^{d_2}_\epsilon(y)), \downarrow(x, y)\right),$$

$$\Big(\downarrow(S_1 \times \{y\}), \downarrow(\bigvee S_1, y)\Big)$$

and

$$\Big(\downarrow(\{x\} \times S_2), \downarrow(x, \bigvee S_2)\Big),$$

with $x \in L_1$, $y \in L_2$, $S_1 \subseteq L_1$ and $S_2 \subseteq L_2$, and denote $\kappa(\downarrow(x,y))$, which coincides with $\downarrow(x,y) \cup \mathbb{O}_{L_1 \times L_2}$, by $x \oplus y$.

Let $L_1 \oplus' L_2 := \mathcal{D}(L_1 \times L_2)/R'$. The elements $x \oplus y$ generate by joins $L_1 \oplus' L_2$. The maps $u'_{L_1} : L_1 \longrightarrow L_1 \oplus' L_2$ and $u'_{L_2} : L_2 \longrightarrow L_1 \oplus' L_2$ given by, respectively, $u'_{L_1}(x) = x \oplus 1$ and $u'_{L_2}(x) = 1 \oplus x$ are frame homomorphisms. The relation

$$d'_{12}(X) = \inf\{\max(d_1(x), d_2(y)) \mid X \subseteq x \oplus y\}$$

defines a compatible star-prediameter on $L_1 \oplus' L_2$. Then

$$(L_1, d_1) \xrightarrow{u'_{L_1}} (L_1 \oplus' L_2, d'_{12}) \xleftarrow{u'_{L_2}} (L_2, d_2)$$

is the coproduct of (L_1, d_1) and (L_2, d_2) in the category of star-prediametric frames; for any $f_{L_1} : (L_1, d_1) \longrightarrow (M, d)$ and $f_{L_2} : (L_2, d_2) \longrightarrow (M, d)$ the unique f from $(L_1 \oplus' L_2, d'_{12})$ into (M, d) such that $f \cdot u'_{L_1} = f_{L_1}$ and $f \cdot u'_{L_2} = f_{L_2}$ is given by

$$f(x \oplus y) = \bigvee_{\epsilon > 0} \left(f_{L_1}(\alpha_\epsilon^{d_1}(x)) \wedge f_{L_2}(\alpha_\epsilon^{d_2}(y)) \right).$$

Now taking the reflection $(L_1 \oplus L_2, d_{12})$ of $(L_1 \oplus' L_2, d'_{12})$ in $\star$-DFrm and the extensions $u_{L_1} : (L_1, d_1) \longrightarrow (L_1 \oplus L_2, d_{12})$ and $u_{L_2} : (L_2, d_2) \longrightarrow (L_1 \oplus L_2, d_{12})$ of u'_{L_1} and u'_{L_2} by the reflection morphism, we have the coproduct of (L_1, d_1) and (L_2, d_2) in $\star$-DFrm. Finally

$$(L_1, d_1) \xrightarrow{u_{L_1}} \left(L_1 \oplus L_2, \widetilde{d_{12}} \right) \xleftarrow{u_{L_2}} (L_2, d_2)$$

is the coproduct of (L_1, d_1) and (L_2, d_2) in MFrm.

3. Gauge frames

Lemma 3.1. *Let d_1 and d_2 be two star-diameters on a frame L. Then*

$$\begin{aligned} d_1 \vee d_2 : L &\longrightarrow [0, +\infty] \\ x &\longmapsto \max(d_1(x), d_2(x)) \end{aligned}$$

is a star-diameter.

Proof. Conditions (D1), (D2), (D3) and ($\star$) are trivially satisfied. To check condition (D5) just observe that, for any $\epsilon > 0$, $U_\epsilon^{d_1 \vee d_2} = U_\epsilon^{d_1} \cap U_\epsilon^{d_2} = U_\epsilon^{d_1} \wedge U_\epsilon^{d_2}$. ■

A word of warning: $d = d_1 \vee d_2$ is not necessarily metric, even if d_1 and d_2 are metric. Nevertheless, we can take the associated metric diameter $\widetilde{d}$ that we shall denote by $d_1 \sqcup d_2$.

Definition 3.2. We say that a non-empty collection $\mathcal{G}$ of metric diameters on L is a *gauge structure* if it satisfies the following conditions:

(UP1) for every $d_1, d_2 \in \mathcal{G}$, $d_1 \sqcup d_2 \in \mathcal{G}$;

(UP2) if d is a metric diameter and

$$\forall \epsilon > 0 \ \ \exists \delta > 0 \ \ \exists d' \in \mathcal{G} : U_\delta^{d'} \subseteq U_\epsilon^d,$$

then $d \in \mathcal{G}$;

(UP3) for every $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{G}}{\lhd} x\}$, where $y \overset{\mathcal{G}}{\lhd} x$ means that there are $d \in \mathcal{G}$ and $\epsilon > 0$ such that $st(y, U_\epsilon^d) \leq x$.

Proposition 3.3. *Let $\mathcal{U}$ be a uniformity on L. The family $\psi(\mathcal{U}) := \{d_\alpha \mid \alpha \in \Lambda\}$ of all metric diameters such that, for every $\alpha \in \Lambda$ and $\epsilon > 0$, $U_\epsilon^{d_\alpha} \in \mathcal{U}$, is a gauge structure on L.*

Proof. (UP1) Let $\alpha, \beta \in \Lambda$ and $\epsilon > 0$. Since $U_\epsilon^{d_\alpha} \wedge U_\epsilon^{d_\beta} \leq U_\epsilon^{d_\alpha \sqcup d_\beta}$, $U_\epsilon^{d_\alpha \sqcup d_\beta} \in \mathcal{U}$. Hence $d_\alpha \sqcup d_\beta \in \psi(\mathcal{U})$.

(UP2) Assume d is a metric diameter such that

$$\forall \epsilon > 0 \ \ \exists \delta > 0 \ \ \exists d_\alpha \in \psi(\mathcal{U}) : U_\delta^{d_\alpha} \subseteq U_\epsilon^d.$$

Then $U_\epsilon^d \in \mathcal{U}$ for any $\epsilon > 0$, i.e., $d \in \psi(\mathcal{U})$.

(UP3) Let $x \in L$. By hypothesis, $x = \bigvee\{y \in L \mid y \overset{\mathcal{U}}{\lhd} x\}$. So, it suffices to show that $y \overset{\mathcal{U}}{\lhd} x$ implies $y \overset{\psi(\mathcal{U})}{\lhd} x$. Consider $U \in \mathcal{U}$ such that $st(y, U) \leq x$ and take inductively $U_1, U_2, \ldots, U_n, \ldots$ in $\mathcal{U}$ such that $U_1 = U$ and $U_{n+1}^* \leq U_n$. The family $\{U_1, U_2, \ldots, U_n, \ldots\}$ generates a uniformity with a countable basis. So, according to Theorem 4.6 of [64], there is a metric diameter d in $\psi(\mathcal{U})$ such that $U_\epsilon^d \subseteq U$ for some $\epsilon > 0$. Hence $st(y, U_\epsilon^d) \leq st(y, U) \leq x$. ■

Note that the reverse

$$y \overset{\psi(\mathcal{U})}{\lhd} x \text{ implies } y \overset{\mathcal{U}}{\lhd} x$$

is obviously true.

Proposition 3.4. *Let $\mathcal{G}$ be a gauge structure on L. The family*

$$\mathcal{B}_{\mathcal{G}} := \{U^d_\epsilon \mid d \in \mathcal{G}, \epsilon > 0\}$$

is a basis for a uniformity $\phi(\mathcal{G})$ on L.

Proof. (U1) For $\epsilon, \delta > 0$ and $d_1, d_2 \in \mathcal{G}$ take $\gamma = \min(\frac{\epsilon}{2}, \frac{\delta}{2})$. Immediately,

$$U^{d_1 \sqcup d_2}_\gamma \subseteq U^{d_1}_\epsilon \cap U^{d_2}_\delta = U^{d_1}_\epsilon \wedge U^{d_2}_\delta$$

and $U^{d_1 \sqcup d_2}_\gamma \in \mathcal{B}_{\mathcal{G}}$ so $U^{d_1}_\epsilon \wedge U^{d_2}_\delta \in \phi(\mathcal{G})$.

(U2) Let us show that, for any $d \in \mathcal{G}$ and $\epsilon > 0$, U^d_ϵ has a star-refinement; we do this by proving that $\left(U^d_{\frac{\epsilon}{3}}\right)^* \leq U^d_\epsilon$. Consider $x \in U^d_{\frac{\epsilon}{3}}$ and choose $y_0 \in U^d_{\frac{\epsilon}{3}}$ such that $y_0 \wedge x \neq 0$. The set

$$S = \{y \vee y_0 \mid y \in U^d_{\frac{\epsilon}{3}}, y \wedge x \neq 0\}$$

is strongly connected and $st(x, U^d_{\frac{\epsilon}{3}}) = \bigvee S$. Thus

$$d(st(x, U^d_{\frac{\epsilon}{3}})) \leq 2 \sup\{d(x) \mid x \in S\} < \epsilon,$$

so $st(x, U^d_{\frac{\epsilon}{3}}) \in U^d_\epsilon$.

(U3) It is obvious, since $x \overset{\mathcal{G}}{\lhd} y$ if and only if $x \overset{\phi(\mathcal{G})}{\lhd} y$. ∎

In other words,

$$\phi(\mathcal{G}) = \{U \in \mathcal{P}(X) \mid \exists d \in \mathcal{G} \;\; \exists \epsilon > 0 \;:\; U^d_\epsilon \leq U\}$$

is a uniformity on L.

Theorem 3.5. *There is a one-to-one correspondence between the set of uniformities on L and the set of gauge structures on L.*

Proof. We first show that $\psi\phi(\mathcal{G}) \subseteq \mathcal{G}$, for any gauge structure $\mathcal{G}$. If $d \in \psi\phi(\mathcal{G})$ then $U_\epsilon^d \in \phi(\mathcal{G})$, for every $\epsilon > 0$, i.e., for every $\epsilon > 0$ there exist $\delta > 0$ and $d' \in \mathcal{G}$ such that $U_\delta^{d'} \subseteq U_\epsilon^d$. Hence $d \in \mathcal{G}$. The inclusion $\mathcal{G} \subseteq \psi\phi(\mathcal{G})$ is trivial.

On the other hand, for any uniformity $\mathcal{U}$, the inclusion $\phi\psi(\mathcal{U}) \subseteq \mathcal{U}$ is obvious. The reverse inclusion is a consequence of Theorem 4.6 of [64]: given $U \in \mathcal{U}$ take inductively $U_1, U_2, \ldots, U_n, \ldots$ in $\mathcal{U}$ such that $U_1 = U$ and $U_{n+1}^* \leq U_n$, and according to Theorem 4.6 of [64] the metric diameter d in $\psi(\mathcal{U})$ such that $U_\epsilon^d \subseteq U$ for some $\epsilon > 0$. Hence $U \in \phi\psi(\mathcal{U})$. ■

In conclusion, we can treat gauge structures as uniformities.

Naming the pairs $(L, \mathcal{G})$ for L a frame and $\mathcal{G}$ a gauge structure on L as *gauge frames* we have a bijection between uniform frames and gauge frames. With respect to this bijection, uniform homomorphisms correspond precisely to the *gauge homomorphisms*, i.e., to the frame maps $f : L \longrightarrow L'$ between gauge frames $(L, \mathcal{G})$ and $(L', \mathcal{G}')$ such that, for every $\epsilon > 0$ and $d \in \mathcal{G}$, there exist $\delta > 0$ and $d' \in \mathcal{G}'$ satisfying $U_\delta^{d'} \leq f[U_\epsilon^d]$. The category of gauge frames and gauge homomorphisms is therefore (concretely) isomorphic to Unif.

Gauge structures clearify the nature of the generalization from metric frames to uniform frames: a metric frame is a frame with a uniformity (gauge structure) generated by a single diameter.

It is worthwhile recording the following obvious fact:

Remark 3.6. For any gauge frame $(L, \mathcal{G})$, $(L, \bigsqcup_{d \in \mathcal{G}} d)$ is a metric frame.

4. An application: UFrm is fully embeddable in a final completion of MFrm

We now turn to the description of a universal final completion of MFrm which contains an isomorphic copy of UFrm.

We start by doing a brief digression about the categorical notions and results we shall need, with the aim of making this dissertation the most possible self-contained.

Let $(\mathcal{A}, |\cdot| : \mathcal{A} \longrightarrow \mathcal{X})$ be a concrete category over the base category $\mathcal{X}$. An $\mathcal{A}$-sink $(A_i \xrightarrow{f_i} A)_{i\in I}$ is called *final* provided that an $\mathcal{X}$-morphism $f : |A| \longrightarrow |B|$ is an $\mathcal{A}$-morphism whenever each composite $f \cdot |f_i| : |A_i| \longrightarrow |B|$ is an $\mathcal{A}$-morphism. An $\mathcal{A}$-morphism is *final* if it is final as a singleton sink.

Given a family Δ of sinks in the base category $\mathcal{X}$, the final sinks $(f_i)_{i\in I}$ of $\mathcal{A}$ such that $(|f_i|)_{i\in I} \in \Delta$ will be referred to as *final* Δ*-sinks*.

Definitions 4.1. (1) (Herrlich [35]) A concrete category $\mathcal{A}$ over $\mathcal{X}$ is called *finally complete* provided that, for any family $(A_i)_{i\in I}$ of $\mathcal{A}$-objects, indexed by some class I, and any sink $(|A_i| \xrightarrow{f_i} X)_{i\in I}$ in $\mathcal{X}$, there exists a final sink $(A_i \xrightarrow{g_i} A)_{i\in I}$ in $\mathcal{A}$ with $|A| = X$ and $|g_i| = f_i$ for every $i \in I$.

A concrete category $\mathcal{A}^-$ over $\mathcal{X}$ is a *final completion* of $\mathcal{A}$ if it is finally complete and there is a full embedding $\mathcal{A} \hookrightarrow \mathcal{A}^-$. If, furthermore,

(a) $\mathcal{A}$ *is closed under final sinks in* $\mathcal{A}^-$ (i.e., for each final sink $(A_i \xrightarrow{g_i} A)_{i\in I}$ in $\mathcal{A}^-$, A belongs to $\mathcal{A}$ whenever all A_i belong to $\mathcal{A}$) and

(b) for each finally complete category $\mathcal{B}$ and each concrete functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ preserving final sinks, there exists a unique (up-to natural isomorphism) functor $F^- : \mathcal{A}^- \longrightarrow \mathcal{B}$ preserving final sinks,

$\mathcal{A}^-$ is said to be a *universal final completion* of $\mathcal{A}$.

(2) (Ehresmann [21]) More generally, let Δ be a collection of sinks in the base category $\mathcal{X}$. A final completion $\mathcal{A}^-$ of $\mathcal{A}$ is a Δ*-universal final completion* of $\mathcal{A}$ provided that:

(a) $\mathcal{A}$ is closed under final Δ-sinks in $\mathcal{A}^-$;

(b) for each finally complete category $\mathcal{B}$ and each concrete functor $F : \mathcal{A} \longrightarrow \mathcal{B}$ preserving final Δ-sinks there exists a unique (up-to natural isomorphism) functor $F^- : \mathcal{A}^- \longrightarrow \mathcal{B}$ preserving final Δ-sinks.

Definitions 4.2. (Ehresmann [21]) Let Δ be a collection of sinks in the base category $\mathcal{X}$.

(1) An $\mathcal{X}$-sink $\sigma = (|A_i| \xrightarrow{f_i} X)_{i\in I}$ is called Δ*-complete* if:

(a) for every $\mathcal{A}$-morphism $f : A \longrightarrow A_i$, the maps $|A| \xrightarrow{|f|} |A_i| \xrightarrow{f_i} X$ belong to σ;

(b) for every $\mathcal{X}$-morphism $f : |B| \longrightarrow X$ and every final Δ-sink $(B_j \xrightarrow{g_j} B)_{j \in J}$ such that the maps $|B_j| \xrightarrow{|g_j|} |B| \xrightarrow{f} X$ belong to σ, the map f belongs to σ.

(2) An *homomorphism of* Δ*-complete sinks*

$$f : (|A_i| \xrightarrow{f_i} X)_{i \in I} \longrightarrow (|B_j| \xrightarrow{g_j} Y)_{j \in J}$$

is an $\mathcal{X}$-morphism $f : X \longrightarrow Y$ such that, for every $i \in I$, $f \cdot f_i \in (g_j)_{j \in J}$.

Theorem 4 of [21] states that the conglomerate of Δ-complete sinks of $\mathcal{X}$ and the conglomerate of Δ-complete sink homomorphisms form a category, that we shall denote by Δ-$\mathcal{CS}(\mathcal{A}, \mathcal{X})$. It also states that this category is a Δ-universal final completion of $\mathcal{A}$. However, Adámek and Reiterman in [2] show that Δ-$\mathcal{CS}(\mathcal{A}, \mathcal{X})$ is not always a legitimated category and present sufficient conditions for Δ-$\mathcal{CS}(\mathcal{A}, \mathcal{X})$ to be in fact a category. Evidently, under these conditions Δ-$\mathcal{CS}(\mathcal{A}, \mathcal{X})$ is also a concrete category over $\mathcal{X}$.

Recall from [2] that, assuming $\mathcal{X}$ has a factorization system $(\mathcal{E}, \mathcal{M})$, $\mathcal{A}$ is said to be *cohereditary* if every $\mathcal{E}$-morphism $e : |A| \longrightarrow X$ is a final morphism of $\mathcal{A}$.

Theorem 4.3. (Adámek and Reiterman [2]) *Let $\mathcal{X}$ be a category with a factorization system $(\mathcal{E}, \mathcal{M})$, $\mathcal{M}$-wellpowered, and let Δ be a collection of sinks containing all $\mathcal{E}$-morphisms (considered as singleton sinks). Then, for every fibre-small cohereditary concrete category $\mathcal{A}$ over $\mathcal{X}$, Δ-$\mathcal{CS}(\mathcal{A}, \mathcal{X})$ is a category.* ■

Remark 4.4. The proof of Theorem 4.3 is based in the following fact, which is a consequence of the coheredity of $\mathcal{A}$ and the inclusion $\mathcal{E} \subseteq \Delta$:

Let $\sigma = (|A_i| \xrightarrow{f_i} X)_{i \in I}$ be a Δ-complete sink of $\mathcal{X}$ and let $m_i \cdot e_i$ be the $(\mathcal{E}, \mathcal{M})$-factorization of f_i. Denoting by σ' the sink $(m_i)_{i \in I}$, σ is precisely the sink of all composites of $\mathcal{A}$-morphisms with the members of σ'.

Therefore, under the conditions of Theorem 4.3, the proof presented in [21] that Δ-$\mathcal{CS}(\mathcal{A}, \mathcal{X})$ is a Δ-universal final completion of $\mathcal{A}$ is valid.

Theorem 4.5. (Ehresmann [21], Adámek and Reiterman [2]) *Let $\mathcal{X}$ be a category with a factorization system $(\mathcal{E},\mathcal{M})$, $\mathcal{M}$-wellpowered, and let Δ be a collection of sinks containing all $\mathcal{E}$-morphisms (considered as singleton sinks). Then each fibre-small cohereditary concrete category $\mathcal{A}$ over $\mathcal{X}$ has a fibre-small Δ-universal final completion, which is the category Δ-$\mathcal{CS}(\mathcal{A},\mathcal{X})$.* ■

The category MFrm of metric frames is concrete over Frm and, since Frm is an algebraic category, the class $\mathcal{R}eg\mathcal{E}pi$ of regular epimorphisms (i.e., surjective maps) and the class $\mathcal{M}ono$ of monomorphisms (i.e., injective maps) form a factorization system $(\mathcal{R}eg\mathcal{E}pi,\mathcal{M}ono)$ (see e.g. [43]). Clearly Frm is a $\mathcal{M}ono$-wellpowered category. On the other hand, MFrm is fibre-small and cohereditary: any compatible metric diameter d on L induces, in any frame quotient M of L (i.e., any surjective homomorphism $e: L \longrightarrow M$), a compatible metric diameter $\overline{d}$ (cf. Propositions 2.11 and 2.12 of [68]); the proof that $e:(L,d)\longrightarrow(M,\overline{d})$ is a final morphism of MFrm is straightforward.

Thus, we can apply Theorem 4.5 to MFrm and conclude that, for every Δ containing the regular epimorphisms, the category Δ-$\mathcal{CS}(\mathsf{MFrm},\mathsf{Frm})$ is a Δ-universal final completion of MFrm. Our goal now is to prove that, for Δ the class of all finite episinks, this category contains an isomorphic copy of the category UFrm of uniform frames.

We start by presenting a different description of the category Δ-$\mathcal{CS}(\mathcal{A},\mathcal{X})$ of Theorem 4.3.

Proposition 4.6. *Under the conditions of Theorem 4.3, there is a one-to-one correspondence between the fiber of X on Δ-$\mathcal{CS}(\mathcal{A},\mathcal{X})$, for every $\mathcal{X}$-object X, and the class of all sinks $\sigma=(|A_i|\xrightarrow{m_i}X)_{i\in I}$ such that:*

(S1) *$m_i\in\mathcal{M}$, for every $i\in I$;*

(S2) *an $\mathcal{M}$-morphism $Y\xrightarrow{m}X$ belongs to σ whenever there exist $i\in I$ and $g\in\mathcal{A}$ with $m=m_i\cdot|g|$;*

(S3) *for each final Δ-sink $(A_i\xrightarrow{f_i}B)_{i\in I}$ and for each $f:|B|\longrightarrow X$ such that the $\mathcal{M}$-part of the $(\mathcal{E},\mathcal{M})$-factorization of $f\cdot|f_i|$ belongs to σ, the $\mathcal{M}$-part of the $(\mathcal{E},\mathcal{M})$-factorization of f belongs to σ.*

Proof. For $\sigma=(|A_i|\xrightarrow{f_i}X)_{i\in I}$ in Δ-$\mathcal{CS}(\mathcal{A},\mathcal{X})$ take $\sigma'=(|A_i'|\xrightarrow{m_i}X)_{i\in I}$, where m_i is the $\mathcal{M}$-part of the $(\mathcal{E}, \mathcal{M})$-factorization of f_i. By Remark 4.4, σ' satisfies conditions (S2) and (S3).

Conversely, for $\sigma' = (|A'_i| \xrightarrow{m_i} X)_{i \in I}$ satisfying (S1), (S2) and (S3), take the sink σ of all composites of morphisms of $\mathcal{A}$ with the members of σ'. This is a Δ-complete sink; indeed:

(a) For every $\mathcal{A}$-morphism $f : A \longrightarrow A_i$, if $f_i : |A_i| \longrightarrow X$ belongs to σ we may write $f_i = m_i \cdot |t_i|$ where $m_i \in \sigma'$ and $t_i \in \mathcal{A}$. Factorizing $f_i \cdot |f| = m'_i \cdot e'_i$, there is a (unique) morphism g such that the following diagram commutes:

$$\begin{tikzcd}
|A| \arrow[rr, "|f|"] \arrow[dd, "e'_i"'] & & |A_i| \arrow[dl, "t_i"'] \arrow[dd, "f_i"] \\
& |A'_i| \arrow[dr, "m_i"] & \\
|A'| \arrow[ur, dashed, "g"] \arrow[rr, "m'_i"'] & & X
\end{tikzcd}$$

Since $t_i \cdot f \in \mathcal{A}$ and e'_i is final, $g \in \mathcal{A}$. Therefore, by (S2), $m'_i \in \sigma'$, so $f_i \cdot |f| \in \sigma$.

(b) For every $\mathcal{X}$-morphism $f : |B| \longrightarrow X$ and every final Δ-sink $(B_j \xrightarrow{g_j} B)_{j \in J}$ such that every $f \cdot |g_j| \in \sigma$, i.e., $f \cdot |g_j| = m_j \cdot |t_j|$ for some $m_j \in \sigma'$ and $t_j \in \mathcal{A}$, if we factorize $f = m \cdot e$, then, applying (S3), $m \in \sigma'$ so $f \in \sigma$.

Remark 4.4 implies that these two correspondences are mutually inverse. ∎

The sinks defined in the previous proposition will be referred to as $\mathcal{M}$-Δ-*complete sinks*.

The following proposition characterizes the morphisms of Δ-$\mathcal{CS}(\mathcal{A}, \mathcal{X})$ in terms of $\mathcal{M}$-Δ-complete sinks and has a straightforward proof.

Proposition 4.7. *Let $\sigma_1 = (|A_i| \xrightarrow{f_i} X)_{i \in I}$ and $\sigma_2 = (|B_j| \xrightarrow{g_j} Y)_{j \in J}$ belong to Δ-$\mathcal{CS}(\mathcal{A}, \mathcal{X})$ and let $\sigma'_1 = (|A'_i| \xrightarrow{m_i} X)_{i \in I}$ and $\sigma'_2 = (|B'_j| \xrightarrow{u_j} Y)_{j \in J}$ be the two corresponding $\mathcal{M}$-Δ-complete sinks. An $\mathcal{X}$-morphism $f : X \longrightarrow Y$ is a morphism of Δ-$\mathcal{CS}(\mathcal{A}, \mathcal{X})$ if and only if, for every $i \in I$, the $\mathcal{M}$-part of $f \cdot m_i$ belongs to σ'_2.* ∎

In conclusion:

Proposition 4.8. *The category Δ-$\mathcal{CS}(\mathcal{A},\mathcal{X})$ is isomorphic to the category $\mathcal{M}$-Δ-$\mathcal{CS}(\mathcal{A},\mathcal{X})$ of $\mathcal{M}$-Δ-complete sinks and maps defined in Proposition 4.7.* ■

Next we describe how to induce, in any frame L, a gauge structure from a $\mathcal{M}ono$-Δ-complete sink.

Proposition 4.9. *Let M be a subframe of L. If d is a star diameter on M then*

$$\overset{\circ}{d}(x) = \inf\{d(y) \mid y \in M, x \leq y\}$$

defines a star-diameter on L.

Proof. The proof that $\overset{\circ}{d}$ is a diameter is straightforward. Let us verify the property $(\star)$. Let $S \subseteq L$ be strongly connected. For any $\epsilon > 0$ and $s \in S$ there is $y_s \in M$ with $d(y_s) < \overset{\circ}{d}(s) + \epsilon$ and $s \leq y_s$. Clearly $S_M = \{y_s \mid s \in S\}$ is strongly connected thus

$$\overset{\circ}{d}(\bigvee S) \leq d(\bigvee S_M) \leq 2\sup\{d(y_s) \mid s \in S\} < 2\sup\{\overset{\circ}{d}(s) \mid s \in S\} + \epsilon.$$

So, $\overset{\circ}{d}(\bigvee S) \leq 2\sup\{\overset{\circ}{d}(s) \mid s \in S\}$. ■

However $\overset{\circ}{d}$ is not necessarily metric, even if d is.

Remark 4.10. As before, there is a unique metric diameter $\tilde{d}$ on L such that $\frac{1}{2}\overset{\circ}{d} \leq \tilde{d} \leq \overset{\circ}{d}$ and $st(x, U_\epsilon^{\tilde{d}}) = st(x, U_\epsilon^{\overset{\circ}{d}})$ for every $x \in L$ and $\epsilon > 0$.

For any diameter d on L let L_d denote the subframe

$$\Big\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{d}{\triangleleft} x\}\Big\}$$

of L. Note that, for any $x \in L$, $\bigvee\{y \in L \mid y \overset{d}{\triangleleft} x\} \in L_d$. Indeed, denoting $\bigvee\{y \in L \mid y \overset{d}{\triangleleft} x\}$ by x_d we have that $x_d = \bigvee\{y \in L \mid y \overset{d}{\triangleleft} x_d\}$: if there is $\epsilon > 0$ with $st(y, U_\epsilon^d) \leq x$ then

$$st\Big(st(y, U_{\frac{\epsilon}{2}}^d), U_{\frac{\epsilon}{2}}^d\Big) \leq st(y, U_\epsilon^d) \leq x,$$

that is,

$$st(y, U^d_{\frac{\epsilon}{2}}) \overset{d}{\lhd} x$$

and so $st(y, U^d_{\frac{\epsilon}{2}}) \leq x_d$.

Therefore we may consider the map $f_d : L \longrightarrow L_d$ given by $f_d(x) = x_d$ which is a surjective frame homomorphism. By Propositions 2.9 and 2.11 of [68]

$$\overline{d}(x) = \inf\{d(y) \mid x \leq f_d(y)\}$$

defines a metric diameter on L_d whenever d is a metric diameter on L. Observe that, due to the particular definitions of L_d and f_d, $\overline{d}$ is the restriction of d to L_d. Moreover, we have:

Proposition 4.11. *Let d be a metric diameter on L. Then $(L_d, \overline{d})$ is a metric frame.*

Proof. It remains to be proved the compatibility of $\overline{d}$. Let $x \in L_d$. Then $x = \bigvee\{y \in L \mid y \overset{d}{\lhd} x\}$. Since $x = f_d(x)$ we have

$$x = \bigvee\{f_d(y) \mid y \in L, y \overset{d}{\lhd} x\} \leq \bigvee\{f_d(y) \mid y \in L, f_d(y) \overset{d}{\lhd} x\} = \bigvee\{z \in L_d \mid z \overset{d}{\lhd} x\}.$$

But, for any $z \in L_d$ and $\epsilon > 0$, $st(z, U^{\overline{d}}_\epsilon) \leq st(z, U^d_\epsilon)$ so

$$z \overset{d}{\lhd} x \text{ implies } z \overset{\overline{d}}{\lhd} x,$$

and, consequently, $x \leq \bigvee\{z \in L_d \mid z \overset{\overline{d}}{\lhd} x\} \leq x$. ■

Corollary 4.12. *Let d be a metric diameter on L. Then:*

(a) *for any $x \in L$ and $\epsilon > 0$, there exists $y \in L_d$ such that $x \leq y$ and $d(y) < d(x) + \epsilon$.*

(b) $\overset{\circ}{\overline{d}} = d$.

Proof.

(a) For any $x \in L$ and $\epsilon > 0$, since $\overline{d}$ is a diameter on L_d, we have that

$$\begin{aligned} x &= x \wedge \bigvee\{a \in L_d \mid \overline{d}(a) < \frac{\epsilon}{2}\} \\ &= \bigvee\{x \wedge a \mid a \in L_d, x \wedge a \neq 0, \overline{d}(a) < \frac{\epsilon}{2}\} \\ &\leq \bigvee\{a \in L_d \mid x \wedge a \neq 0, \overline{d}(a) < \frac{\epsilon}{2}\} \end{aligned}$$

Then

$$\bigvee\{a \in L_d \mid x \wedge a \neq 0, \overline{d}(a) < \frac{\epsilon}{2}\} \in L_d$$

is the desired element y. In fact, using property $(\star')$, $d(y) = d(x \vee y) \leq d(x) + \sup\{d(a_1) + d(a_2) \mid a_1, a_2 \in L_d, a_1 \neq a_2, a_1 \wedge x \neq 0, a_2 \wedge x \neq 0, \overline{d}(a_1), \overline{d}(a_2) < \frac{\epsilon}{2}\} < d(x) + \epsilon$.

(b) For any $x \in L$,

$$\begin{aligned} \mathring{\overline{d}}(x) &= \inf\{\overline{d}(y) \mid y \in L_d, x \leq y\} \\ &= \inf\{d(y) \mid y \in L_d, x \leq y\}, \end{aligned}$$

thus $\mathring{\overline{d}}(x) \geq d(x)$. The equality follows immediately from (a). ■

From (a) it also follows that the relation $\overset{\overline{d}}{\lhd}$ is the restriction of $\overset{d}{\lhd}$ to L_d.

Corollary 4.13. *Let $\mathcal{G}$ be a gauge structure on L. For any $d \in \mathcal{G}$, the inclusion $(L_d, \overline{d}) \hookrightarrow (L, \bigsqcup_{d\in\mathcal{G}} d)$ is a uniform homomorphism of metric frames.*

Proof. Let us show that, for any $\epsilon > 0$ and $d \in \mathcal{G}$,

$$U_{\frac{\epsilon}{4}}^{\bigsqcup_{d\in\mathcal{G}} d} \leq U_{\epsilon}^{\overline{d}}.$$

Assume $x \in L$ is such that $(\bigsqcup_{d\in\mathcal{G}} d)(x) < \frac{\epsilon}{4}$. Thus $d(x) < \frac{\epsilon}{2}$. This yields, via Corollary 4.12 (a), the existence of $y \in L_d$ satisfying $\overline{d}(y) = d(y) < d(x) + \frac{\epsilon}{2} < \epsilon$ and $x \leq y$. Hence $x \leq y \in U_{\epsilon}^{\overline{d}}$. ■

Let M be a subframe of L. For any metric diameter d on M it is obvious that $\mathring{\overline{d}} = d$. Moreover:

Proposition 4.14. *Let M be a subframe of L. For any compatible metric diameter d on M, $L_{\tilde{d}} = M$.*

Proof. Consider $x \in L_{\tilde{d}}$. Let us start by proving that, for any $y \in L$ such that $y \overset{\tilde{d}}{\lhd} x$, there exists $z \in M$ satisfying

$$y \leq z \overset{\tilde{d}}{\lhd} x :$$

By hypothesis, there is some $\epsilon > 0$ with $st(y, U_\epsilon^{\widetilde{d}}) \leq x$. Also

$$y = y \wedge \bigvee\{z' \in M \mid d(z') < \frac{\epsilon}{2}\} \leq \bigvee\{z' \in M \mid z' \wedge y \neq 0, d(z') < \frac{\epsilon}{2}\} \in M.$$

The element

$$z := \bigvee\{z' \in M \mid z' \wedge y \neq 0, d(z') \leq \frac{\epsilon}{2}\}$$

is the element in M we are looking for. In fact, $st(z, U_{\frac{\epsilon}{2}}^{\widetilde{d}}) \leq st(y, U_\epsilon^{\widetilde{d}})$: consider $w \in L$ with $w \wedge z \neq 0$ and $\widetilde{d}(w) < \frac{\epsilon}{2}$. Then there exists $z' \in M$ such that $z' \wedge w \neq 0$, $z' \wedge y \neq 0$ and $d(z') < \frac{\epsilon}{2}$. Therefore $w \leq z' \vee w$, $(z' \vee w) \wedge y \neq 0$ and

$$\widetilde{d}(z' \vee w) \leq \widetilde{d}(z') + \widetilde{d}(w) = d(z') + \widetilde{d}(w) < \epsilon.$$

Hence, in conclusion, $st(z, U_{\frac{\epsilon}{2}}^{\widetilde{d}}) \leq st(y, U_\epsilon^{\widetilde{d}}) \leq x$, i.e.,

$$y \leq z \overset{\widetilde{d}}{\lhd} x.$$

Now, the conclusion that $L_{\widetilde{d}} \subseteq M$ follows immediately; for any $x \in L_{\widetilde{d}}$,

$$x = \bigvee\{y \in L \mid y \overset{\widetilde{d}}{\lhd} x\} = \bigvee\{z \in M \mid z \overset{\widetilde{d}}{\lhd} x\} \in M.$$

Conversely, consider $x \in M$. Since $x = \bigvee\{y \in M \mid y \overset{d}{\lhd} x\}$, we only have to check that

$$y \overset{d}{\lhd} x \text{ implies } y \overset{\widetilde{d}}{\lhd} x$$

in order to conclude that $x \in L_{\widetilde{d}}$. So, suppose that $st(y, U_\epsilon^d) \leq x$ for some $\epsilon > 0$. Then $st(y, U_{\frac{\epsilon}{4}}^{\widetilde{d}}) \leq x$ because $st(y, U_{\frac{\epsilon}{4}}^{\widetilde{d}}) \leq st(y, U_\epsilon^d)$: for any $z \in L$ such that $z \wedge y \neq 0$ and $\widetilde{d}(z) < \frac{\epsilon}{4}$ there exists, by Corollary 4.12 (a), $w \in L_{\widetilde{d}}$ such that $z \leq w$ and $\widetilde{d}(w) < \widetilde{d}(z) + \frac{\epsilon}{4} < \frac{\epsilon}{2}$. We have already proved that $L_{\widetilde{d}} \subseteq M$ so $w \in M$. On the other hand, $w \wedge y \neq 0$ and $\frac{1}{2}d(w) = \frac{1}{2}\overset{\circ}{d}(w) \leq \widetilde{d}(w) < \frac{\epsilon}{2}$, i.e., $w \leq st(y, U_\epsilon^d)$. ∎

Note that from Remark 4.10 it follows that $L_{\overset{\circ}{d}} = L_{\widetilde{d}}$.

We say that a sink

$$(|(M_i, d_i)| \xrightarrow{m_i} L)_{i \in I}$$

of $\mathcal{M}ono$-Δ-$\mathcal{CS}(\mathsf{MFrm}, \mathsf{Frm})$ is a *weak gauge* $\mathcal{M}ono$-Δ-*complete sink* if L is generated by $\bigcup_{i\in I} m_i(M_i)$. We denote by d_i the compatible metric diameter on $m_i(M_i)$ — which is a subframe of L isomorphic to M_i — induced by d_i. Accordingly, $L_{\widetilde{d_i}} = m_i(M_i) \cong M_i$ by Proposition 4.14.

Lemma 4.15. *Let* $\sigma = (|(M_i, d_i)| \xrightarrow{m_i} L)_{i\in I} \in \mathcal{M}ono$-$\Delta$-$\mathcal{CS}(\mathsf{MFrm}, \mathsf{Frm})$. *For every* $i, j \in I$ *there is* $k \in I$ *such that*

$$\forall \epsilon > 0 \ \exists \delta > 0 \ : \ U_\delta^{\widetilde{d_k}} \subseteq U_\epsilon^{\widetilde{d_i} \sqcup \widetilde{d_j}}.$$

Proof. Denote $\widetilde{d_i} \sqcup \widetilde{d_j}$ by d. We start by proving that, for any $x \in m_i(M_i)$, $st(x, U_\epsilon^d) \leq st(x, U_{4\epsilon}^{d_i})$:

For any $y \in L$ such that $y \wedge x \neq 0$ and $d(y) < \epsilon$, we have that $\frac{1}{2}(\widetilde{d_i} \vee \widetilde{d_j})(y) < \epsilon$, which implies that $\widetilde{d_i}(y) < 2\epsilon$ and $\overset{\circ}{d_i} < 4\epsilon$. This means that there is $z \in m_i(M_i)$ with $y \leq z$ and $d_i(z) < 4\epsilon$.

From that inequality it follows that, for any $x, y \in m_i(M_i)$, $y \overset{d_i}{\lhd} x$ implies $y \overset{d}{\lhd} x$ and, consequently, that $m_i(M_i) \subseteq L_d$. Similarly $m_j(M_j) \subseteq L_d$. It is now very easy to see that $(M_i, d_i) \xrightarrow{m_i} (L_d, \overline{d})$ and $(M_j, d_j) \xrightarrow{m_j} (L_d, \overline{d})$ are uniform frame homomorphisms. Consider the coproduct

$$(M_i, d_i) \xrightarrow{u_i} (M_i \oplus M_j, \widetilde{d_{ij}}) \xleftarrow{u_j} (M_j, d_j)$$

of (M_i, d_i) and (M_j, d_j) in MFrm. Then there exists a unique uniform frame homomorphism f such that the diagram

$$\begin{tikzcd}
(M_i, d_i) \arrow[r, "u_i"] \arrow[dr, "m_i"'] & (M_i \oplus M_j, \widetilde{d_{ij}}) \arrow[d, dashed, "f"] & (M_j, d_j) \arrow[l, "u_j"'] \arrow[dl, "m_j"] \\
& (L_d, \overline{d}) &
\end{tikzcd}$$

is commutative.

Lastly we consider the $(RegEpi, Mono)$-factorization of f in MFrm ([68], Proposition 4.10)

$$(M_i \oplus M_j, \widetilde{d_{ij}}) \xrightarrow{\quad e \quad} (M, d_M) \xrightarrow{\quad u \quad} (L_d, \overline{d})$$

Since (u_i, u_j) is a finite final epi-sink (cf. Section 10 of [1]), we may conclude from condition (S3) that the frame map

$$m : |(M, d_M)| \xrightarrow{u} |(L_d, \overline{d})| \hookrightarrow L$$

belongs to σ and, therefore, that there is $k \in I$ such that $\widetilde{d_M} = \widetilde{d_k}$. This k fulfils the condition of the lemma. In fact,

$$\forall \epsilon > 0 \quad U^{\widetilde{d_M}}_{\frac{\epsilon}{4}} \subseteq U^{\widetilde{d_i} \sqcup \widetilde{d_j}}_{\epsilon} :$$

Consider $x \in L$ such that $\widetilde{d_M}(x) < \frac{\epsilon}{4}$. Since $\frac{1}{2}\overset{\circ}{d}_M \leq \widetilde{d_M}$ it suffices to show that $\frac{1}{2}(\overset{\circ}{d}_i \vee \overset{\circ}{d}_j)(x) \leq \overset{\circ}{d}_M(x)$. By definition

$$\overset{\circ}{d}_M(x) = \inf\{d_M([X]_M) \mid X \in M_i \oplus M_j, x \leq u([X]_M) = f(X)\}$$

so we only have to prove that

$$\text{if } X \in M_i \oplus M_j \text{ and } x \leq f(X) \text{ then } \frac{1}{2}(\overset{\circ}{d}_i \vee \overset{\circ}{d}_j)(x) \leq d_M([X]_M). \tag{4.15.1}$$

But

$$d_M([X]_M) = \inf\{\widetilde{d_{ij}}(Y) \mid Y \in M_i \oplus M_j, [X]_M \leq [Y]_M\}$$

and $\frac{1}{2}d_{ij} \leq \widetilde{d_{ij}}$, thus the condition 4.15.1 holds if, whenever $Y \in M_i \oplus M_j$ and $x \leq f(Y)$, $(\overset{\circ}{d}_i \vee \overset{\circ}{d}_j)(x) \leq d_{ij}(Y)$. Since

$$d_{ij}(Y) = \inf\{d'_{ij}(Z) \mid Z \in M_i \oplus' M_j, Y \leq [Z]_{M_i \oplus M_j}\},$$

this is a consequence of Lemma 4.16 below. ∎

Lemma 4.16. *Let (M_i, d_i) and (M_j, d_j) be metric frames and consider*

$$X = \bigvee_{\gamma \in \Gamma} (x_\gamma \oplus y_\gamma) \in M_i \oplus' M_j.$$

Then

$$d'_{ij}(X) \geq (\overset{\circ}{d}_i \vee \overset{\circ}{d}_j)(\bigvee_{\gamma \in \Gamma} (x_\gamma \wedge y_\gamma)).$$

Proof. We may assume without loss of generality that, for every $\gamma \in \Gamma$, $(x_\gamma, y_\gamma) \notin \mathbb{O}$. If $X \subseteq x \oplus y$ then, for every $\gamma \in \Gamma$, $(x_\gamma, y_\gamma) \leq (x, y)$ thus $\bigvee_{\gamma\in\Gamma} x_\gamma \leq x$ and $\bigvee_{\gamma\in\Gamma} y_\gamma \leq y$. Therefore

$$\max(d_i(x), d_j(y)) \geq \max(d_i(\bigvee_{\gamma\in\Gamma} x_\gamma), d_j(\bigvee_{\gamma\in\Gamma} y_\gamma))$$

for any $(x, y) \in M_i \times M_j$ such that $X \subseteq x \oplus y$. Hence

$$d'_{ij}(X) \geq \max\left(d_i\Big(\bigvee_{\gamma\in\Gamma} x_\gamma\Big), d_j\Big(\bigvee_{\gamma\in\Gamma} y_\gamma\Big)\right) \geq \left(\overset{\circ}{d_i} \vee \overset{\circ}{d_j}\right)\left(\bigvee_{\gamma\in\Gamma}(x_\gamma \wedge y_\gamma)\right). \quad \blacksquare$$

Theorem 4.17. *Given a weak gauge $\mathcal{M}ono$-Δ-complete sink*

$$\sigma = (|(M_i, d_i)| \xrightarrow{m_i} L)_{i\in I},$$

the family $\Gamma(\sigma)$ of all metric diameters d' on L such that, for every $\epsilon > 0$, there exist $i \in I$ and $\delta > 0$ satisfying $U_\delta^{\widetilde{d_i}} \subseteq U_\epsilon^{d'}$ is a gauge structure on L.

Proof. (UP1) Consider $d'_1, d'_2 \in \Gamma(\sigma)$. For each $\epsilon > 0$, there are $i_1, i_2 \in I$ and $\delta_1, \delta_2 > 0$ such that $U_{\delta_1}^{\widetilde{d_{i_1}}} \subseteq U_\epsilon^{d'_1}$ and $U_{\delta_2}^{\widetilde{d_{i_2}}} \subseteq U_\epsilon^{d'_2}$. Take $\delta = \min(\delta_1, \delta_2)$. Then

$$U_{\frac{\delta}{2}}^{\widetilde{d_{i_1}} \sqcup \widetilde{d_{i_2}}} \leq U_\delta^{\widetilde{d_{i_1}}} \wedge U_\delta^{\widetilde{d_{i_2}}} \leq U_\epsilon^{d'_1} \wedge U_\epsilon^{d'_2} \leq U_\epsilon^{d'_1 \sqcup d'_2}.$$

Now, using Lemma 4.15, it follows that $d'_1 \sqcup d'_2 \in \Gamma(\sigma)$.

(UP2) Let d be a metric diameter on L such that

$$\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \exists d' \in \Gamma(\sigma) : U_\delta^{d'} \subseteq U_\epsilon^d.$$

Since $d' \in \Gamma(\sigma)$, there exist $\gamma > 0$ and $i \in I$ such that $U_\gamma^{\overline{d_i}} \subseteq U_\delta^{d'}$ so $U_\gamma^{\overline{d_i}} \subseteq U_\epsilon^d$ and $d \in \Gamma(\sigma)$.

(UP3) This condition follows immediately from the fact that σ is a weak gauge $\mathcal{M}ono$-Δ-complete sink. $\blacksquare$

If we consider, for each frame L, the set of all gauge structures on L ordered by inclusion and the set of all $\mathcal{M}ono$-Δ-complete sinks also ordered by inclusion, Theorem 4.17 gives us an order-preserving map Γ from the partially ordered set of weak gauge $\mathcal{M}ono$-Δ-complete sinks to the partially ordered set of gauge structures on L.

For any metric diameter d on L let us denote by m_d the frame monomorphism $L_d \hookrightarrow L$.

Theorem 4.18. *Assume that Δ is the class of all finite episinks and let $(L, \mathcal{G})$ be a gauge frame. The sink*

$$\Upsilon(\mathcal{G}) \ := \ \Big\{ |(M,d)| \xrightarrow{m} L \mid m \in \mathcal{M}ono \text{ and there are } d' \in \mathcal{G} \text{ and } f : (M,d) \longrightarrow (L_{d'}, \overline{d'}) \text{ in } \mathsf{MFrm} \text{ such that } m_{d'} \cdot |f| = m \Big\}$$

is a weak gauge $\mathcal{M}ono$-Δ-complete sink.

Proof. Conditions (S1) and (S2) are obviously satisfied. Let us check condition (S3):

Consider the following commutative diagram

$$\begin{tikzcd}
|(M_i,d_i)| \arrow[r, "|f_i|"] \arrow[d, "e_i"'] & |(M,d)| \arrow[d, "f"] \arrow[dr, "e"] & \\
 & & |(M',d')| \arrow[dl, "m"] \\
|(M_i',d_i')| \arrow[r, "m_i"'] & L &
\end{tikzcd}$$

where $(f_i)_{i \in I}$ is a final Δ-sink and $m_i \cdot e_i$ and $m \cdot e$ are the $(\mathcal{R}eg\mathcal{E}pi, \mathcal{M}ono)$-factorizations of, respectively, $f \cdot |f_i|$ and f. We need to show that $m \in \Upsilon(\mathcal{G})$ whenever every m_i belongs to $\Upsilon(\mathcal{G})$. So, assume that each m_i belongs to $\Upsilon(\mathcal{G})$, i.e., that there are $d_i'' \in \mathcal{G}$ and $g_i : (M_i', d_i') \longrightarrow (L_{d_i''}, \overline{d_i''}) \in \mathsf{MFrm}$ such that $m_{d_i''} \cdot |g_i| = m_i$:

$$\begin{tikzcd}
|(M_i,d_i)| \arrow[r, "|f_i|"] \arrow[d, "e_i"'] & |(M,d)| \arrow[d, "f"] \arrow[dr, "e"] & \\
 & & |(M',d')| \arrow[dl, "m"] \\
|(M_i',d_i')| \arrow[r, "m_i"'] \arrow[dr, "|g_i|"'] & L & \\
 & |(L_{d_i''}, \overline{d_i''})| \arrow[u, "m_{d_i''}"'] &
\end{tikzcd}$$

Since $\mathcal{G}$ is a gauge structure and I is finite, $d'' := \bigsqcup_{i\in I} d''_i \in \mathcal{G}$. In order to show that $m \in \Upsilon(\mathcal{G})$, it suffices to prove that there exists $g : (M', d') \longrightarrow (L_{d''}, \overline{d''})$ in MFrm such that $m_{d''} \cdot |g| = m$:

$$\begin{tikzcd}
{|(M_i, d_i)|} \arrow[rr, "|f_i|"] \arrow[dd, "e_i"'] & & {|(M, d)|} \arrow[dr, "e"] \arrow[dd, "f"] & \\
& & & {|(M', d')|} \arrow[dl, "m"] \arrow[dd, "|g|"] \\
{|(M'_i, d'_i)|} \arrow[rr, "m_i"] \arrow[dr, "|g_i|"'] & & L & \\
& {|(L_{d''_i}, \overline{d''_i})|} \arrow[ur, "m_{d''_i}"'] & & {|(L_{d''}, \overline{d''})|} \arrow[ul, "m_{d''}"]
\end{tikzcd}$$

Let us see first that $M' \subseteq L_{d''}$:

Considering $x \in M'$, since $M' = f(M)$, we may write $x = f(y)$ for some $y \in M$. But $(f_i)_{i\in I}$ is an epi-sink because $(f_i)_{i\in I}$ belongs to Δ, so $M = \bigvee_{i\in I} f_i(M_i)$. Therefore, m can be written as $\bigvee_{i\in I} f_i(y_i)$ for some family $\{y_i \mid i \in I\}$ (where each y_i belongs to M_i). Hence

$$x = \bigvee_{i\in I}(f \cdot f_i(y_i)) = \bigvee_{i\in I}(m_i \cdot e_i(y_i)) = \bigvee_{i\in I}(m_{d_i} \cdot g_i \cdot e_i(y_i)) \in \bigvee_{i\in I} L_{d''_i} \subseteq L_d.$$

Now let us prove that the inclusion $g : M' \hookrightarrow L_d$ is a uniform homomorphism from (M', d') into $(L_{d''}, \overline{d''})$. By Corollary 4.13,

$$m_{d''_i} : (L_{d''_i}, \overline{d''_i}) \longrightarrow (L, \bigsqcup_{d\in\mathcal{G}} d)$$

is a uniform homomorphism. Therefore, for each $i \in I$,

$$m_{d''} \cdot g \cdot e \cdot f_i = f \cdot f_i = m_{d''_i} \cdot g_i \cdot e_i$$

is uniform. Then $g \cdot e \cdot f_i$ is uniform. Hence, by the finality of $(f_i)_{i\in I}$ and e, g is uniform.

Since every m_d belongs to $\Upsilon(\mathcal{G})$ and, $\mathcal{G}$ satisfying (UP3), $L = \bigvee_{d \in \mathcal{G}} L_d$, $\Upsilon(\mathcal{G})$ is weak gauge. ■

So, for any frame L we have an order-preserving map Υ from the partially ordered set of gauge structures on L to the partially ordered set of weak gauge $\mathcal{M}ono$-Δ-complete sinks on L.

We are at last ready to settle the embedding of UFrm in a Δ-universal final completion of MFrm.

Theorem 4.19. *Suppose that Δ is the class of all finite episinks in* Frm. *For every frame L, Γ and Υ define a Galois connection $\Gamma \dashv \Upsilon$ between the partially ordered set of gauge structures and the partially ordered set of weak gauge $\mathcal{M}ono$-Δ-complete sinks. Moreover $\Gamma\Upsilon = 1$.*

Proof. Let us prove first that $\Gamma\Upsilon(\mathcal{G}) = \mathcal{G}$ for any gauge $\mathcal{G}$ on L. In order to prove that $d \in \mathcal{G}$ whenever $d \in \Gamma\Upsilon(\mathcal{G})$ it suffices to show that, for any $\epsilon > 0$, there are $\gamma > 0$ and $d' \in \mathcal{G}$ such that $U_\gamma^{d'} \subseteq U_\epsilon^d$. So, consider $\epsilon > 0$. Then there exist $\delta > 0$ and

$$m_\epsilon : |(M_\epsilon, d_\epsilon)| \longrightarrow L$$

in $\Upsilon(\mathcal{G})$ satisfying $U_\delta^{\widetilde{d_\epsilon}} \subseteq U_\epsilon^d$. Since $m_\epsilon \in \Upsilon(\mathcal{G})$ there are d'_ϵ and

$$f_\epsilon : (M_\epsilon, d_\epsilon) \longrightarrow (L_{d'_\epsilon}, \overline{d'_\epsilon})$$

in MFrm for which the diagram

$$\begin{tikzcd} {|(M_\epsilon, d_\epsilon)|} \arrow[rr, "m_\epsilon"] \arrow[dr, "|f_\epsilon|"'] & & L \\ & {|(L_{d'_\epsilon}, \overline{d'_\epsilon})|} \arrow[ur, "m_{d'_\epsilon}"'] & \end{tikzcd}$$

is commutative. The uniformity of f_ϵ implies that, for any $\delta > 0$, there is $\delta' > 0$ satisfying $U_{\delta'}^{\overline{d'_\epsilon}} \subseteq f_\epsilon[U_\delta^{d_\epsilon}]$. Let $\gamma = \frac{\delta'}{2}$. We claim that $U_\gamma^{d'_\epsilon} \subseteq U_\delta^{\widetilde{d_\epsilon}}$: for any $x \in U_\gamma^{d'_\epsilon}$ there exists, by Corollary 4.12 (a), $y \in L_{d'_\epsilon}$ with $x \le y$ and $d'_\epsilon < d'_\epsilon(x) + \gamma < \delta'$. Consequently, there exists $z \in M_\epsilon$ such that $y \le f_\epsilon(z)$ and $d_\epsilon(z) < \delta$. Then $\widetilde{d_\epsilon}\,(x) \le \mathring{d_\epsilon}\,(x) \le d_\epsilon(z) < \delta$.

In conclusion, $U_\gamma^{d'_\epsilon} \subseteq U_\delta^{\tilde{d}_\epsilon} \subseteq U_\epsilon^d$ as required.

Conversely, if $d \in \mathcal{G}$ then $m_d \in \Upsilon(\mathcal{G})$ so $\overset{\sim}{\overline{d}} \in \Gamma\Upsilon(\mathcal{G})$. By Corollary 4.12 (b), $\overset{\sim}{\overline{d}} = \overset{\circ}{\overline{d}} = d$ so $d \in \Gamma\Upsilon(\mathcal{G})$.

Finally, let us prove that $\sigma \subseteq \Upsilon\Gamma(\sigma)$ for every weak gauge $\mathcal{M}ono$-Δ-complete sink σ on L. For each

$$|(M,d)| \xrightarrow{m} L$$

in σ, the diameter $\tilde{d}$ belongs to $\Gamma(\sigma)$. By Proposition 4.14, $L_{\tilde{d}} = m(M)$. Since

$$\frac{1}{2}d = \frac{1}{2}\overline{\overset{\circ}{d}} \leq \overline{\tilde{d}} \leq \overline{\overset{\circ}{d}} \leq d,$$

then

$$(M,d) \xrightarrow{m} (L_{\tilde{d}}, \overline{\tilde{d}}) \in \mathsf{MFrm},$$

which shows that $m \in \Upsilon\Gamma(\sigma)$. ∎

It follows that, for every frame L, $\Upsilon\Gamma$ is a closure operator on the partially ordered set of weak gauge $\mathcal{M}ono$-Δ-complete sinks on L, assigning to every weak gauge $\mathcal{M}ono$-Δ-complete sink σ on L what we shall call its *gauge closure* σ^-; then the restriction of Γ to the weak gauge $\mathcal{M}ono$-Δ-complete sinks σ on L for which $\sigma^- = \sigma$ is a bijection between these sinks and the gauge structures on L. We call these sinks the *gauge $\mathcal{M}ono$-Δ-complete sinks* and we consider the category of gauge $\mathcal{M}ono$-Δ-complete sinks as the full subcategory of $\mathcal{M}ono$-Δ-$\mathcal{CS}(\mathsf{MFrm}, \mathsf{Frm})$ whose objects are the gauge $\mathcal{M}ono$-Δ-complete sinks.

Since under this bijection the morphisms of $\mathcal{M}ono$-Δ-$\mathcal{CS}(\mathsf{MFrm}, \mathsf{Frm})$ between gauge $\mathcal{M}ono$-Δ-complete sinks correspond precisely to the gauge homomorphisms, we have:

Corollary 4.20. *The category* UFrm *is isomorphic to the category of gauge $\mathcal{M}ono$-Δ-complete sinks of* MFrm *(for Δ the class of finite episinks) and, therefore, the category* UFrm *is fully embeddable in a final completion of* MFrm*, universal with respect to the class Δ of finite episinks.* ∎

We end this chapter with some remarks.

Let us replace the condition

(UP2′) if d is a metric diameter and there exists $d' \in \mathcal{G}$ such that

$$\forall \epsilon > 0 \ \exists \delta > 0 : U_\delta^{d'} \subseteq U_\epsilon^d, \text{ then } d \in \mathcal{G},$$

for the condition (UP2) in Definition 3.2 and call the resulting structure a *weak gauge structure* on the frame L.

The image by Υ of a weak gauge structure on L is still a weak gauge $\mathcal{M}ono$-Δ-complete sink.

Conversely, we have that:

Proposition 4.21. *For any weak gauge $\mathcal{M}ono$-Δ-complete sink on L*

$$\sigma = (|(M_i, d_i)| \xrightarrow{m_i} L)_{i \in I},$$

the subfamily $\Gamma'(\sigma) := \left\{ \widetilde{d_i} | \ i \in I \right\}$ *of* $\Gamma(\sigma)$ *is a weak gauge structure on* L.

Proof. (UP2′) Let d be a metric diameter and suppose that there is $d' \in \Gamma'(\sigma)$ such that

$$\forall \epsilon > 0 \ \exists \delta > 0 \ : \ U_\delta^{d'} \subseteq U_\epsilon^d.$$

Then $y \overset{d'}{\triangleleft} x$ whenever $y \overset{d}{\triangleleft} x$ so $L_d \subseteq L_{d'}$ and the inclusion $(L_d, \overline{d}) \hookrightarrow (L_{d'}, \overline{d'})$ is uniform. But $d' \in \Gamma'(\sigma)$ thus there is $|(M_i, d_i)| \xrightarrow{m_i} L$ in σ such that $d' = \widetilde{d_i}$. By Proposition 4.14, $(M_i, d_i) \cong (L_{d'}, \overline{d'})$ so, applying (S2), $|(L_d, \overline{d})| \longrightarrow L$ belongs to σ and, consequently,

$$d = \mathring{\overline{d}} = \widetilde{\overline{d}} \in \Gamma'(\sigma).$$

(UP1) Consider $\widetilde{d_i}, \widetilde{d_j} \in \Gamma'(\sigma)$. Lemma 4.15 guarantees the existence of $k \in I$ such that

$$\forall \epsilon > 0 \ \exists \delta > 0 \ : \ U_\delta^{\widetilde{d_k}} \subseteq U_\epsilon^{\widetilde{d_i} \sqcup \widetilde{d_j}}.$$

Then, by (UP2′) above, $\widetilde{d_i} \sqcup \widetilde{d_j} \in \Gamma'(\sigma)$.

The proof of (UP3) is obvious. ∎

Adapting the proof of Theorem 4.19 to this setting, we can conclude that $\Gamma'\Upsilon = 1$ and $\Upsilon\Gamma' = 1$. So, we have an isomorphism between the category of *weak gauge frames* (whose morphisms are the gauge homomorphisms) and the full subcategory of $\mathcal{M}ono$-Δ-$\mathcal{CS}(\mathsf{MFrm}, \mathsf{Frm})$ of weak gauge $\mathcal{M}ono$-Δ-complete sinks.

Note that, in the particular case of σ being a gauge $\mathcal{M}ono$-Δ-complete sink, $\Gamma'(\sigma)$ and $\Gamma(\sigma)$ coincide. Indeed, whenever $\sigma = \sigma^-$,

$$\Gamma'(\sigma) = \Gamma'(\sigma^-) = \Gamma'\Upsilon\Gamma(\sigma) = \Gamma(\sigma).$$

The following diagram summarizes our conclusions:

Δ =finite episinks

$$\begin{tikzcd}
 & & \boxed{\mathcal{M}ono\text{-}\Delta\text{-}CS(\mathsf{MFrm}, \mathsf{Frm})} \\
 & \boxed{\text{weak gauge frames}} \arrow[r, shift left, "\Upsilon"] & \boxed{\text{weak gauge } \mathcal{M}ono\text{-}\Delta\text{-complete sinks}} \arrow[l, shift left, "\Gamma'"', "\text{(isomorphism)}"] \arrow[u, hook, "E"'] \\
\mathsf{UFrm} \cong & \boxed{\text{gauge frames}} \arrow[u, hook] \arrow[r, shift left, "\Upsilon"] & \boxed{\text{gauge } \mathcal{M}ono\text{-}\Delta\text{-complete sinks}} \arrow[l, shift left, "\Gamma"', "\text{(isomorphism)}"] \arrow[u, hook]
\end{tikzcd}$$

Notes on Chapter II:

(1) An interesting question is whether the notion of weak gauge $\mathcal{M}ono$-Δ-complete sink is stronger than the notion of $\mathcal{M}ono$-Δ-complete sink, that is, whether the full embedding E of the diagram above is really strict.

(2) Gauge structures could be defined as collections of star-diameters satisfying axioms (UP2), (UP3) and

$$\text{(UP1}'\text{)}\ d_1 \vee d_2 \in \mathcal{G} \text{ whenever } d_1, d_2 \in \mathcal{G}.$$

In fact, it can be proved — similarly to the proof of Theorem 3.5 — that the partially ordered set of gauge structures on a frame L (in this sense) is also isomorphic to the partially ordered set of uniformities on L.

(3) The knowledge of some facts about coproducts of frames and about C-ideals generated by down-sets revealed to be the crucial point in the proof that Weil entourages do work for uniform frames.

Here, the conclusion that gauge structures may be also viewed from the "pointless point of view" relies on the remarkable work of Pultr on metric frames ([64], [65], [67], [68]).

CHAPTER III

WEIL QUASI-UNIFORM FRAMES

The classical theory of quasi-uniform spaces — as opposed to the theory of uniform spaces — is usually presented in the literature in terms of entourages [28]: a quasi-uniformity on a set is defined by dropping the symmetry axiom from the set of axioms of Weil for a uniform space. This is a theory which has achieved much success (as a justification for this statement see, for example, [28] and [50]).

The goal of this chapter is to present an easily applicable theory of frame quasi-uniformities that emerges from the theory of frame uniformities we introduced in Chapter I. Further, we show that this theory of frame quasi-uniformities is equivalent to the one established (via covers) by Frith [29]. Consequently, our category of quasi-uniform frames is also isomorphic to the category QEUFrm of quasi-uniform frames of Fletcher, Hunsaker and Lindgren [27].

1. Quasi-uniform spaces

Quasi-uniform spaces were first defined by Nachbin in terms of entourages by dropping the symmetry requirement of uniform spaces.

Definition 1.1. (Nachbin [55]) Let X be a set and let $\mathcal{E}$ be a family of entourages of X. The pair $(X, \mathcal{E})$ is a *quasi-uniform space* provided that:

(QUW1) $\mathcal{E}$ is a filter with respect to $\subseteq$;

(QUW2) for each $E \in \mathcal{E}$ there is $F \in \mathcal{E}$ such that $F \circ F \subseteq E$.

A map f from a quasi-uniform space $(X, \mathcal{E})$ to a quasi-uniform space $(X', \mathcal{E}')$ is called *uniformly continuous* if $(f \times f)^{-1}(E) \in \mathcal{E}$ whenever $E \in \mathcal{E}'$. We denote the category of quasi-uniform spaces and uniformly continuous maps by QUnif.

A cover-like approach for quasi-uniform spaces was given by Gantner and Steinlage [30]. A *conjugate pair of covers* (briefly, *conjugate cover*) of a set X is a subset $\mathcal{U}$ of $\mathcal{P}(X) \times \mathcal{P}(X)$ satisfying

$$\bigcup\{U_1 \cap U_2 \mid (U_1, U_2) \in \mathcal{U}\} = X.$$

The conjugate cover $\mathcal{U}$ is called *strong* if, in addition,

$$U_1 \cap U_2 \neq \emptyset \text{ whenever } (U_1, U_2) \in \mathcal{U} \text{ and either } U_1 \neq \emptyset \text{ or } U_2 \neq \emptyset.$$

Let $\mathcal{U}$ and $\mathcal{V}$ be conjugate covers. Usually, one writes $\mathcal{U} \leq \mathcal{V}$ if for each $(U_1, U_2) \in \mathcal{U}$, there is $(V_1, V_2) \in \mathcal{V}$ with $U_1 \subseteq V_1$ and $U_2 \subseteq V_2$.

Definition 1.2. (Gantner and Steinlage [30]) A non-empty family μ of conjugate covers of a set X is a *covering quasi-uniformity* on X provided that:

(QU1) if $\mathcal{U} \in \mu$, $\mathcal{V}$ is a conjugate cover of X and $\mathcal{U} \leq \mathcal{V}$ then $\mathcal{V} \in \mu$;

(QU2) for every $\mathcal{U}, \mathcal{V} \in \mu$ there exists a strong conjugate cover $\mathcal{W} \in \mu$ such that

$$\mathcal{W} \leq \{(U_1 \cap V_1, U_2 \cap V_2) \mid (U_1, U_2) \in \mathcal{U}, (V_1, V_2) \in \mathcal{V}\};$$

(QU3) for each $\mathcal{U} \in \mu$ there is $\mathcal{V} \in \mu$ such that

$$\mathcal{V}^* := \{(st_1(V_1, \mathcal{V}), st_2(V_2, \mathcal{V})) \mid (V_1, V_2) \in \mathcal{V}\} \leq \mathcal{U}$$

where, for each $V \subseteq X$,

$$st_1(V, \mathcal{V}) := \bigcup\{V_1' \mid (V_1', V_2') \in \mathcal{V} \text{ and } V_2' \cap V \neq \emptyset\}$$

and

$$st_2(V, \mathcal{V}) := \bigcup\{V_2' \mid (V_1', V_2') \in \mathcal{V} \text{ and } V_1' \cap V \neq \emptyset\}.$$

In [30], the authors proved that these two approaches are equivalent.

We refer the reader to Fletcher and Lindgren [28] for a general reference on quasi-uniform spaces.

2. Covering quasi-uniform frames

In the same way that bitopological spaces play a natural role in the study of quasi-uniform spaces ([51]), quasi-uniform frames lend themselves naturally to the consideration of biframes.

The following definitions and notations are transcribed from [29].

Let $B = (B_0, B_1, B_2)$ be a biframe. A subset U of $B_1 \times B_2$ is called a *conjugate cover* of B if

$$\bigvee\{u_1 \wedge u_2 \mid (u_1, u_2) \in U\} = 1.$$

A conjugate cover U is *strong* if, for every $(u_1, u_2) \in U$, $u_1 \wedge u_2 \neq 0$ whenever $u_1 \vee u_2 \neq 0$.

We recall some additional definitions. If U and V are conjugate covers of B, the relation $U \leq V$ means that, for every $(u_1, u_2) \in U$, there is $(v_1, v_2) \in V$ such that $u_1 \leq v_1$ and $u_2 \leq v_2$. With relation to this preorder,

$$\{(u_1 \wedge v_1, u_2 \wedge v_2) \mid (u_1, u_2) \in U, (v_1, v_2) \in V\},$$

which is clearly a conjugate cover, is the meet $U \wedge V$ of the conjugate covers U and V. For any conjugate cover U of B and any $x \in B_0$,

$$st_1(x, U) := \bigvee\{u_1 \mid (u_1, u_2) \in U, u_2 \wedge x \neq 0\}$$

and

$$st_2(x, U) := \bigvee\{u_2 \mid (u_1, u_2) \in U, u_1 \wedge x \neq 0\}.$$

Finally, U^* stands for the conjugate cover

$$\Big\{\Big(st_1(u_1, U), st_2(u_2, U)\Big) \mid (u_1, u_2) \in U\Big\}.$$

Note that, for every $x \in B_0$ and $i \in \{1, 2\}$ [29]:

- $x \leq st_i(x, U)$;
- $st_i(st_i(x, U), U) \leq st_i(x, U^*)$.

Definition 2.1. (Frith [29]) A family $\mathcal{U}$ of conjugate covers of a biframe B is a *quasi-uniformity* on B provided that:

(QU1) $\mathcal{U}$ is a filter with respect to the preorder $\leq$ and the family of strong members of $\mathcal{U}$ is a basis for this filter;

(QU2) for every $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $V^* \leq U$;

(QU3) for each $x \in B_i$ ($i \in \{1,2\}$), $x = \bigvee\{y \in B_i \mid y \overset{\mathcal{U}}{\lhd}_i x\}$, where $y \overset{\mathcal{U}}{\lhd}_i x$ means that there is $U \in \mathcal{U}$ such that $st_i(y,U) \leq x$.

A *quasi-uniform frame* is a pair $(B,\mathcal{U})$ where B is a biframe and $\mathcal{U}$ is a quasi-uniformity on B. Let $(B,\mathcal{U})$ and $(B',\mathcal{U}')$ be quasi-uniform frames. A *uniform homomorphism* $f : (B,\mathcal{U}) \longrightarrow (B',\mathcal{U}')$ is a biframe map $f : B \longrightarrow B'$ such that, for every $U \in \mathcal{U}$,

$$f[U] := \{(f(u_1), f(u_2)) \mid (u_1,u_2) \in U\} \in \mathcal{U}'.$$

The category of quasi-uniform frames and uniform homomorphisms will be denoted by QUFrm.

3. Weil quasi-uniform frames

In the spatial setting, by dropping the symmetry axiom one gets the notion of quasi-uniformity. Here, in the pointfree context, after dropping the symmetry axiom one must observe the following: the equivalence between conditions (i) and (ii) of Proposition I.4.6 is no longer valid and so, in the place of $\overset{\mathcal{E}}{\lhd}$, we have two order relations,

$$x \overset{\mathcal{E}}{\lhd}_1 y \equiv E \circ (x \oplus x) \subseteq y \oplus y, \text{ for some } E \in \mathcal{E},$$

and

$$x \overset{\mathcal{E}}{\lhd}_2 y \equiv (x \oplus x) \circ E \subseteq y \oplus y, \text{ for some } E \in \mathcal{E},$$

which in turn give rise, as we shall see, to two subframes of L,

$$L^1 := \Big\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd}_1 x\}\Big\}$$

and

$$L^2 := \Big\{x \in L \mid x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd}_2 x\}\Big\},$$

that correspond, in the spatial case, to the two topologies defined by the quasi-uniformity.

Note that $\overset{\mathcal{E}}{\lhd}_2$ is the order $\overset{\mathcal{E}^{-1}}{\lhd}_1$ where $\mathcal{E}^{-1}$ is the *conjugate* $\{E^{-1} \mid E \in \mathcal{E}\}$ of $\mathcal{E}$.

For every $x \in L$ and every C-ideal E, we denote the elements

$$\bigvee\{y \in L \mid (y, z) \in E, z \wedge x \neq 0\},$$

$$\bigvee\{y \in L \mid (z, y) \in E, z \wedge x \neq 0\}$$

and

$$\bigvee\{y \in L \mid (y, y) \in E, y \wedge x \neq 0\}$$

by, respectively, $st_1(x, E)$, $st_2(x, E)$ and $st(x, E)$. For every $x, y \in L$ and every $E, F \in L \oplus L$, we have:

- $st_1(x, E) \wedge y = 0$ if and only if $x \wedge st_2(y, E) = 0$;
- $st_1(st_1(x, E), F) \leq st_1(x, F \circ E)$ and $st_2(st_2(x, E), F) \leq st_2(x, E \circ F)$.

It is also easy to conclude that, for every $x \in L$ and every Weil entourage E of L,

$$x \leq st(x, E) \leq st_1(x, E) \wedge st_2(x, E).$$

Proposition 3.1. *Assume that $\mathcal{E}$ is a basis for a filter of $(WEnt(L), \subseteq)$. Then, for $i \in \{1, 2\}$, the relations $\overset{\mathcal{E}}{\lhd}_i$ are sublattices of $L \times L$ satisfying the following properties:*

(a) *for any x, y, z, w in L, $x \leq y \overset{\mathcal{E}}{\lhd}_i z \leq w$ implies $x \overset{\mathcal{E}}{\lhd}_i w$;*

(b) *$x \overset{\mathcal{E}}{\lhd}_i y$ if and only if, for some $E \in \mathcal{E}$, $st_i(x, E) \leq y$;*

(c) *$x \overset{\mathcal{E}}{\lhd}_i y$ implies $x \prec y$;*

(d) *L^i is a subframe of L.*

Proof. The proof that each $\overset{\mathcal{E}}{\lhd}_i$ defines a sublattice of $L \times L$ is similar to the proof of the correlated property of $\overset{\mathcal{E}}{\lhd}$ in Proposition 4.8 of Chapter I.

(a) It is obvious.

(b) If $E \circ (x \oplus x) \subseteq y \oplus y$ and $(a, b) \in E$ with $b \wedge x \neq 0$, then, since $(a, b \wedge x) \in E$ and $(b \wedge x, b \wedge x) \in x \oplus x$, $(a, b \wedge x) \in y \oplus y$, and, consequently, $a \leq y$. Therefore $st_1(x, E) \leq y$.

Conversely, consider $(a, b) \in E$ and $(b, c) \leq (x, x)$ with $b \neq 0$. Then $(a, c) \in y \oplus y$; indeed, $a \leq st_1(x, E) \leq y$ and $c \leq x \leq st_1(x, E) \leq y$.

(c) From (b) it follows that $st(x, E) \leq y$ whenever $x \overset{\mathcal{E}}{\triangleleft}_i y$. It suffices now to recall the proof of Proposition I.4.8.

(d) Since each $\overset{\mathcal{E}}{\triangleleft}_i$ is a sublattice of $L \times L$ and, by (c), $x \overset{\mathcal{E}}{\triangleleft}_i y$ implies $x \leq y$, it is an immediate corollary of property (a). ■

Remark 3.2. A straightforward calculation (analogous to the proof of (b) above) shows that:

- $x \overset{\mathcal{E}}{\triangleleft}_1 y$ is also equivalent to the existence of some $E \in \mathcal{E}$ such that $E \circ (x \oplus 1) \subseteq y \oplus 1$;
- similarly, $x \overset{\mathcal{E}}{\triangleleft}_2 y$ if and only if $(1 \oplus x) \circ E \subseteq 1 \oplus y$ for some $E \in \mathcal{E}$.

In order to state the appropriate definition of Weil quasi-uniformity, we have to replace the condition that (L, L^1, L^2) is a biframe for the admissibility condition (UW4) of Definition I.4.5:

Definition 3.3. A family $\mathcal{E}$ of Weil entourages of a frame L is a *Weil quasi-uniformity* if it satisfies the following conditions:

(QUW1) $\mathcal{E}$ is a filter of $(WEnt(L), \subseteq)$;

(QUW2) for each $E \in \mathcal{E}$ there exists $F \in \mathcal{E}$ such that $F \circ F \subseteq E$;

(QUW3) (L, L^1, L^2) is a biframe.

A *Weil quasi-uniformity basis* is a set $\mathcal{E}$ of Weil entourages such that $\uparrow\mathcal{E}$ is a Weil quasi-uniformity. Therefore, $\mathcal{E}$ is a Weil quasi-uniformity basis if and only if it is a basis of a filter of $(WEnt(L), \subseteq)$ satisfying conditions (QUW2) and (QUW3).

A *Weil quasi-uniform frame* is just a pair $(L, \mathcal{E})$ constituted of a frame L and a Weil quasi-uniformity $\mathcal{E}$ on L. Let $(L, \mathcal{E})$ and $(L', \mathcal{E}')$ be Weil quasi-uniform frames. A

Weil uniform homomorphism $f : (L, \mathcal{E}) \longrightarrow (L', \mathcal{E}')$ is a frame map $f : L \longrightarrow L'$ such that $(f \oplus f)(E) \in \mathcal{E}'$ whenever $E \in \mathcal{E}$. These are the objects and the morphisms of the category QWUFrm.

Remarks 3.4. (a) In case the Weil quasi-uniformity $\mathcal{E}$ is symmetric, i.e., has a basis of symmetric Weil entourages, the relations $\overset{\mathcal{E}}{\triangleleft}_1$ and $\overset{\mathcal{E}}{\triangleleft}_2$ are the same and coincide with the relation $\overset{\mathcal{E}}{\triangleleft}$ of Chapter I. Thus $L^1 = L^2$ and, therefore, the condition (QUW3) means that $L = L^1 = L^2$ or, which is the same, that, for every $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\triangleleft} x\}$. In conclusion, it happens here the same as for spaces: a family of Weil entourages is a Weil uniformity if and only if it is a symmetric Weil quasi-uniformity.

(b) If $\mathcal{E}$ is a Weil quasi-uniformity on L, then its conjugate $\mathcal{E}^{-1}$ is also a Weil quasi-uniformity on L.

(c) Because of the risk of boring the reader, we dispense with the description of the obvious adaptation of the dual adjunction of Theorem I.4.14 to the categories QUnif and QWUFrm.

4. The isomorphism between the categories QUFrm and WQUFrm

The following lemma, of the same flavour of Lemma 4.2 of Chapter I, will be essencial in the sequel.

Lemma 4.1. *Let $A \in \mathcal{D}(L \times L)$ and let $x \in L$. For each $i \in \{1, 2\}$, $st_i(x, k(A)) = st_i(x, A)$.*

Proof. We only prove the lemma for $i = 1$ since the proof for $i = 2$ is similar.

Let $A \in \mathcal{D}(L \times L)$ and consider the non-empty set

$$\mathbf{E} = \{E \in \mathcal{D}(L \times L) \mid A \subseteq E \subseteq k(A), st_1(x, E) = st_1(x, A)\}.$$

If $E \in \mathbf{E}$ then also $k_0(E) \in \mathbf{E}$:

Clearly, it suffices to check that $st_1(x, k_0(E)) \leq st_1(x, E)$. Let $(a, b) \in k_0(E)$ with $b \wedge x \neq 0$. If $(a, b) = (a, \bigvee S)$ for some S with $\{a\} \times S \subseteq E$, there is a non-zero

$s \in S$ such that $s \wedge x \neq 0$ and $(a,s) \in E$, and therefore $a \leq st_1(x,E)$. Otherwise, if $(a,b) = (\bigvee S, b)$ for some S with $S \times \{b\} \subseteq E$, then, immediately, $a = \bigvee S \leq st_1(x,E)$.

Besides, for any non-empty $\mathbf{F} \subseteq \mathbf{E}$, $\bigcup_{F\in\mathbf{F}} F \in \mathbf{E}$ because

$$st_1(x, \bigcup_{F\in\mathbf{F}} F) = \bigvee_{F\in\mathbf{F}} st_1(x,F).$$

Therefore $T := \bigcup_{E\in\mathbf{E}} E$ belongs to $\mathbf{E}$, i.e., $\mathbf{E}$ has a largest element T. But then, as we proved above, $k_0(T) \in \mathbf{E}$ so $T = k_0(T)$, i.e., T is a C-ideal. Hence $k(A) = T \in \mathbf{E}$ and, consequently, $st_1(x, k(A)) = st_1(x, A)$. ■

Proposition 4.2. *Let $\mathcal{U}$ be a quasi-uniformity on a biframe (B_0, B_1, B_2). For each conjugate cover U define $E_U := k(U)$. Then $\mathcal{E}_{\mathcal{U}} = \{E_U \mid U \in \mathcal{U}\}$ is a Weil quasi-uniformity basis on B_0.*

Proof. It is obvious that every element of $\mathcal{E}_{\mathcal{U}}$ is a Weil entourage since

$$\bigvee\{x \mid (x,x) \in k(U)\} \geq \bigvee\{u_1 \wedge u_2 \mid (u_1,u_2) \in U\}.$$

Let $E_{U_1}, E_{U_2} \in \mathcal{E}_{\mathcal{U}}$ and take $U \in \mathcal{U}$ such that $U \leq U_1 \wedge U_2$. Clearly $E_U \subseteq E_{U_1} \cap E_{U_2}$, thus $\mathcal{E}_{\mathcal{U}}$ is a filter basis of Weil entourages of B_0.

Consider $E_U \in E_{\mathcal{U}}$ and take $V \in \mathcal{U}$ such that $V^* \leq U$. Then $E_V \circ E_V \subseteq E_U$:

An application of Lemma I.4.2 yields $E_V \circ E_V = \downarrow V \circ \downarrow V$. Let $(x,y) \leq (v_1,v_2) \in V$ and $(y,z) \leq (v_1', v_2') \in V$ with $x,y,z \neq 0$. As $x \leq v_1 \leq st_1(v_1,V)$, $z \leq v_2' \leq st_2(v_2,V)$ and $(st_1(v_1,V), st_2(v_2,V)) \in V^*$, this says that there is a pair $(u_1,u_2) \in U$ such that $(x,z) \leq (u_1,u_2)$, and, consequently, that $(x,z) \in E_U$.

Finally, (B_0, B_0^1, B_0^2) is a biframe:

By hypothesis

$$B_i \subseteq \Big\{x \in B_0 \mid x = \bigvee\{y \in B_i \mid y \overset{\mathcal{U}}{\lhd}_i x\}\Big\}.$$

Moreover, by Lemma 4.1, $y \overset{\mathcal{U}}{\lhd}_i x$ means that there is $U \in \mathcal{U}$ such that $st_i(y, k(U)) \leq x$, which, as we already observed, is equivalent to $y \overset{\mathcal{E}_{\mathcal{U}}}{\lhd}_i x$. Therefore

$$B_i \subseteq \Big\{x \in B_0 \mid x = \bigvee\{y \in B_i \mid y \overset{\mathcal{E}_{\mathcal{U}}}{\lhd}_i x\}\Big\}.$$

Now, since (B_0, B_1, B_2) is a biframe and, for $i \in \{1,2\}$,

$$B_0^i = \{x \in B_0 \mid x = \bigvee\{y \in B_i \mid y \overset{\mathcal{E}_{\mathcal{U}}}{\triangleleft}_i x\}\}$$

is a subframe of B_0 (Proposition 3.1 (d)), it easily follows that (B_0, B_0^1, B_0^2) is a biframe.

■

Let $\mathcal{E}$ be a Weil quasi-uniformity on L. Since, for any $E \in \mathcal{E}$ and $x \in L$, $st_i(x, E)$ may not belong to L^i, let us introduce in $L \oplus L$ the relations $\overset{\mathcal{E}}{\sqsubseteq}_i$ for $i \in \{1,2\}$:

For every $I, J \in L \oplus L$,

$$I \overset{\mathcal{E}}{\sqsubseteq}_1 J \equiv E \circ I \subseteq J \text{ for some } E \in \mathcal{E}$$

and

$$I \overset{\mathcal{E}}{\sqsubseteq}_2 J \equiv I \circ E \subseteq J \text{ for some } E \in \mathcal{E}.$$

These relations are stronger than the inclusion $\subseteq$. In fact, for every $E \in \mathcal{E}$ and $I \in L \oplus L$, $I \subseteq (E \circ I) \cap (I \circ E)$ as we now prove; for any $(x, y) \in I$ we may write $x = \bigvee\{x \wedge z \mid (z, z) \in E, x \wedge z \neq 0\}$, and, moreover, for any z such that $(z, z) \in E$ and $x \wedge z \neq 0$, $(x \wedge z, x \wedge z) \in E$ and $(x \wedge z, y) \in I$. Thus $(x, y) \in E \circ I$. Similarly $I \subseteq I \circ E$. Hence $I \subseteq (E \circ I) \cap (I \circ E)$, as we claimed.

Further, for each $i \in \{1,2\}$, we also need the following operator on $\mathcal{E}$:

$$int_i(E) := \bigvee\{I \in L \oplus L \mid I \overset{\mathcal{E}}{\sqsubseteq}_i E\}.$$

Proposition 4.3. *Let $\mathcal{E}$ be a Weil quasi-uniformity on L. For $i \in \{1,2\}$ and for any $E \in \mathcal{E}$, we have that:*

(a) $int_i(E) \subseteq E \subseteq int_i(E^2)$;

(b) *for every $x \in L$, $st_i(x, int_i(E)) \in L^i$.*

Proof. (a) It is trivial.

(b) In order to show that

$$st_1(x, int_1(E)) \leq \bigvee\Big\{y \in L \mid y \overset{\mathcal{E}}{\triangleleft}_1 st_1(x, int_1(E))\Big\}$$

it suffices to prove that

$$int_1(E) \circ (x \oplus 1) \subseteq \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\triangleleft}_1 st_1(x, int_1(E))\} \oplus 1 \qquad \text{(cf. Remark 3.2).}$$

By Lemma I.4.2,

$$int_1(E) \circ (x \oplus 1) = \bigcup\{I \in L \oplus L \mid I \overset{\mathcal{E}}{\sqsubseteq}_1 E\} \circ \downarrow(x, 1).$$

If $(b, c) \leq (x, 1)$, $b \neq 0$, $(a, b) \in I$ and there is $F \in \mathcal{E}$ such that $F \circ I \subseteq E$, then for any $G \in \mathcal{E}$ such that $G^2 \subseteq F$ we have that $G \circ I \subseteq int_1(E)$, since $G \circ I \overset{\mathcal{E}}{\sqsubseteq}_1 E$. Then, a straightforward calculation shows that

$$G \circ (a \oplus a) \subseteq st_1(x, int_1(E)) \oplus st_1(x, int_1(E)),$$

i.e., that $a \overset{\mathcal{E}}{\triangleleft}_1 st_1(x, int_1(E))$. Hence

$$(a, c) \in \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\triangleleft}_1 st_1(x, int_1(E))\} \oplus 1. \qquad \blacksquare$$

By (a) each $int_i(E)$ is a Weil entourage. Note that, for $i = 1$ and $i = 2$, $\{int_i(E) \mid E \in \mathcal{E}\}$ is a basis for $\mathcal{E}$.

Remark 4.4. Proposition 4.3 also enables us to conclude that condition (QUW3) of Definition 3.3 can be stated as a condition of admissibility of a uniformity on L; in fact, it is equivalent to saying that the filter $\overline{\mathcal{E}}$ generated by $\{E \cap E^{-1} \mid E \in \mathcal{E}\}$ is admissible, i.e., that, for every $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\overline{\mathcal{E}}}{\triangleleft} x\}$:

If $\overline{\mathcal{E}}$ is admissible and $x \in L$, then $x = \bigvee S$ where $S = \{y \in L \mid y \overset{\overline{\mathcal{E}}}{\triangleleft} x\}$. For any $y \in S$ there exist $\overline{E}_y \in \overline{\mathcal{E}}$ and $E_y \in \mathcal{E}$ such that $\overline{E}_y \circ (y \oplus y) \subseteq x \oplus x$ and $E_y \cap E_y^{-1} \subseteq \overline{E}_y$. Consider $F_y \in \mathcal{E}$ such that $F_y^2 \subseteq \overline{E}_y$ and denote the intersection $F_y \cap F_y^{-1} \in \overline{\mathcal{E}}$ by $\overline{F}_y$. An easy computation shows that $\overline{F}_y^2 \subseteq \overline{E}_y$ so $int_i(\overline{F}_y^2) \subseteq int_i(\overline{E}_y)$ ($i \in \{1, 2\}$). Thus we have

$$\begin{aligned} x = \bigvee S &\leq \bigvee_{y \in S} (st_1(y, \overline{F}_y) \wedge st_2(y, \overline{F}_y)) \\ &\leq \bigvee_{y \in S} (st_1(y, int_1(\overline{F}_y^2)) \wedge st_2(y, int_2(\overline{F}_y^2))) \\ &\leq \bigvee_{y \in S} (st_1(y, int_1(\overline{E}_y)) \wedge st_2(y, int_2(\overline{E}_y))). \end{aligned}$$

Now

$$\bigvee_{y\in S}(st_1(y,int_1(\overline{E}_y))\wedge st_2(y,int_2(\overline{E}_y)))\leq x$$

because $\overline{E}_y\circ(y\oplus y)\subseteq x\oplus x$. Then, since $st_i(y,int_i(\overline{E}_y))\in L^i$ $(i\in\{1,2\})$, (L,L^1,L^2) is a biframe. Conversely, for any $x\in L$, we may write $x=\bigvee_{\gamma\in\Gamma}(x^1_\gamma\wedge x^2_\gamma)$ for $\{x^1_\gamma\mid\gamma\in\Gamma\}\subseteq L^1$ and $\{x^2_\gamma\mid\gamma\in\Gamma\}\subseteq L^2$. But, for any $\gamma\in\Gamma$,

$$x^1_\gamma=\bigvee\{y\in L\mid y\overset{\mathcal{E}}{\lhd}_1 x^1_\gamma\}$$

and

$$x^2_\gamma=\bigvee\{y\in L\mid y\overset{\mathcal{E}}{\lhd}_2 x^2_\gamma\},$$

so it suffices to prove that $y_1\wedge y_2\overset{\overline{\mathcal{E}}}{\lhd}x_1\wedge x_2$ whenever $y_1\overset{\mathcal{E}}{\lhd}_1 x_1$ and $y_2\overset{\mathcal{E}}{\lhd}_2 x_2$ which is an easy-to-prove statement: if $E\circ(y_1\oplus y_1)\subseteq x_1\oplus x_1$ and $(y_2\oplus y_2)\circ F\subseteq x_2\oplus x_2$ then $F^{-1}\circ(y_2\oplus y_2)\subseteq x_2\oplus x_2$ so

$$(E\cap F)^{-1}\circ(y_1\wedge y_2\oplus y_1\wedge y_2)\subseteq x_1\wedge x_2\oplus x_1\wedge x_2.$$

In summary, as it happens with uniform spaces, if $(L,\mathcal{E})$ is a Weil quasi-uniform frame, the coarsest Weil quasi-uniformity $\overline{\mathcal{E}}$ on L that contains $\mathcal{E}\cup\mathcal{E}^{-1}$ is a uniformity and has the family $\{E\cap E^{-1}\mid E\in\mathcal{E}\}$ as basis.

Definition 4.5. Suppose that E is a Weil entourage of L. We say that an element x of L is E-*small* if $x\leq st(y,E)$ whenever $x\wedge y\neq 0$.

For each Weil entourage E define

$$U_E:=\Big\{\Big(st_1(x,int_1(E)),st_2(x,int_2(E))\Big)\mid x\text{ is an }E\text{-small member of }L\Big\}.$$

Proposition 4.6. *Let $(L,\mathcal{E})$ be a Weil quasi-uniform frame. For each $x\in L$ and each $E\in\mathcal{E}$ we have that:*

(a) $st_i(x,U_E)\leq st_i(x,E^3)$ $(i\in\{1,2\})$;

(b) $st_i(x,E)\leq st_i(x,U_{E^2})$ $(i\in\{1,2\})$.

Proof. (a) Fix $i \in \{1,2\}$. Let z be an E-small element such that $st_j(z, int_j(E)) \wedge x \neq 0$, $(j \in \{1,2\}; j \neq i)$. Then $z \wedge st_i(x, int_j(E)) \neq 0$ and, due to the E-smallness,

$$z \leq st(st_i(x, int_j(E)), E) \leq st_i(st_i(x, E), E) \leq st_i(x, E^2).$$

Hence $st_i(z, int_i(E)) \leq st_i(x, E^3)$, and, consequently,

$$\bigvee\left\{st_i(z, int_i E) \mid z \text{ is } E\text{-small},\ st_j(z, int_j E) \wedge x \neq 0,\ (j \in \{1,2\}; j \neq i)\right\} \leq st_i(x, E^3),$$

that is, $st_i(x, U_E) \leq st_i(x, E^3)$.

(b) Let $(a, b) \in E$ auch that $b \wedge x \neq 0$. We may write

$$a = \bigvee\{a \wedge z \mid (z, z) \in E^2, a \wedge z \neq 0\}.$$

Observe that

$$\Big(st_1(a \wedge z, int_1(E^2)), st_2(a \wedge z, int_2(E^2))\Big) \in U_{E^2}$$

whenever $(z, z) \in E^2$ and $a \wedge z \neq 0$. But

$$\begin{aligned} x \wedge st_2(a \wedge z, int_2(E^2)) &= \bigvee\{x \wedge d \mid (c, d) \in int_2(E^2), c \wedge a \wedge z \neq 0\} \\ &\geq \bigvee\{x \wedge d \mid (c, d) \in E, c \wedge a \wedge z \neq 0\} \\ &\geq x \wedge b \neq 0 \end{aligned}$$

so the desired inequality is proved. ∎

Given a Weil quasi-uniformity $\mathcal{E}$ on L, let us denote the set $\{U_E \mid E \in \mathcal{E}\}$ by $\mathcal{U}_{\mathcal{E}}$.

Proposition 4.7. *$\mathcal{U}_{\mathcal{E}}$ is a basis of strong conjugate covers for a quasi-uniformity on (L, L^1, L^2).*

Proof. By Proposition 4.3 (b) each U_E is a subset of $L^1 \times L^2$. Moreover, it is a strong conjugate cover:

- $\bigvee\{st_1(x, int_1(E)) \wedge st_2(x, int_2(E)) \mid x \text{ is } E\text{-small}\} \geq \bigvee\{x \in L \mid x \text{ is } E\text{-small}\}$ $\geq \bigvee\{x \in L \mid (x, x) \in E\} = 1$;

- if

$$st_1(x, int_1(E)) \vee st_2(x, int_2(E)) \neq 0$$

then $x \neq 0$ so

$$st_1(x, int_1(E)) \wedge st_2(x, int_2(E)) \neq 0.$$

For $E, F \in \mathcal{E}$, we have that $U_G \leq U_E \wedge U_F$ for any $G \in \mathcal{E}$ such that $G \subseteq E \cap F$ (clearly, x is E-small and F-small whenever it is G-small).

Now let us check conditions (QU2) and (QU3) of Definition 2.1:

(QU2) Consider $U_E \in \mathcal{U}_{\mathcal{E}}$ and take $F \in \mathcal{E}$ such that $F^8 \subseteq E$. Then $U_F^* \leq U_E$:

Let x be an F-small member of L. We have that $st_1(st_1(x, int_1(F)), U_F)$ is equal to

$$\bigvee\{st_1(z, int_1(F)) \mid z \text{ is } F\text{-small, } st_2(z, int_2(F)) \wedge st_1(x, int_1(F)) \neq 0\}.$$

Since

$$st_2(z, int_2(F)) \wedge st_1(x, int_1(F)) \neq 0$$

is equivalent to

$$z \wedge st_1(st_1(x, int_1(F)), int_2(F)) \neq 0$$

and z is F-small,

$$z \leq st(st_1(st_1(x, int_1(F)), int_2(F)), F) \leq st_1(st_1(st_1(x, F), F), F).$$

It follows that

$$st_1(z, int_1(F)) \leq st_1(z, F) \leq st_1(x, F^4) \leq st_1(x, int_1(E)).$$

In summary, we have just proved that $st_1(st_1(x, int_1(F)), U_F) \leq st_1(x, int_1(E))$.

Similarly, one proves that $st_2(st_2(x, int_2(F)), U_F) \leq st_2(x, int_2(E))$.

(QU3) Let $x \in L^1$. Then $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\triangleleft}_1 x\}$. We check (QU3) by showing that, for any $y \in L$ satisfying $y \overset{\mathcal{E}}{\triangleleft}_1 x$, there is $y' \in L^1$ such that $y \leq y' \overset{\mathcal{U}_{\mathcal{E}}}{\triangleleft}_1 x$. So, consider $y \in L$ with $y \overset{\mathcal{E}}{\triangleleft}_1 x$ and take $F, G \in \mathcal{E}$ such that $G^{16} \subseteq F$ and $F^3 \subseteq E$ (where $E \in \mathcal{E}$ satisfies $st_1(y, E) \leq x$). Since $y \leq st_1(y, int_1(G)) \in L^1$, we only need to check that $st_1(st_1(y, int_1(G)), U_{G^2}) \leq x$:

Since

$$\begin{aligned} st_1\left(st_1(y, int_1(G)), U_{G^2}\right) &\leq st_1\left(st_1(y, G), U_{G^2}\right) \\ &\leq st_1\left(st_1(y, U_{G^2}), U_{G^2}\right) \\ &\leq st_1(y, U_{G^2}{}^*) \end{aligned}$$

and, by the above proof of (QU2), $U_{G^2}{}^* \leq U_{G^{16}}$, we have

$$st_1\left(st_1(y, int_1(G)), U_{G^2}\right) \leq st_1(y, U_{G^2}{}^*) \leq st_1(y, U_F) \leq st_1(y, F^3) \leq st_1(y, E) \leq x,$$

as we claimed. ■

Lemma 4.8. *Let $(L, \mathcal{U})$ be a quasi-uniform frame. For any strong conjugate cover V in $\mathcal{U}$, we have that:*

(a) $V \leq U_{E_{V^*}}$;

(b) $U_{E_V} \leq V^{**}$.

Proof. (a) Let $(v_1, v_2) \in V$ with $v_1 \vee v_2 \neq 0$. Of course, $v_1 \wedge v_2 \neq 0$ and $(v_1 \wedge v_2, v_1 \wedge v_2) \in E_V \subseteq E_{V^*}$, so $v_1 \wedge v_2$ is E_{V^*}-small and

$$\Big(st_1(v_1 \wedge v_2, int_1(E_{V^*})), st_2(v_1 \wedge v_2, int_2(E_{V^*}))\Big) \in U_{E_{V^*}}.$$

Easily one can deduce that $E_V \circ E_V \subseteq E_{V^*}$, so $E_V \subseteq int_i(E_{V^*})$ ($i \in \{1, 2\}$). Consequently,

$$\begin{aligned} (v_1, v_2) &\leq \Big(st_1(v_1 \wedge v_2, E_V), st_2(v_1 \wedge v_2, E_V)\Big) \\ &\leq \Big(st_1(v_1 \wedge v_2, int_1(E_{V^*})), st_2(v_1 \wedge v_2, int_2(E_{V^*}))\Big). \end{aligned}$$

(b) For any non-zero E_V-small element x of L, there exists a pair $(v_1, v_2) \in V$ such that $x \wedge v_1 \wedge v_2 \neq 0$, by definition of conjugate cover. Obviously

$$x \leq st(v_1 \wedge v_2, E_V) \leq st_1(v_1, E_V) \wedge st_2(v_2, E_V) = st_1(v_1, V) \wedge st_2(v_2, V).$$

Thus

$$st_i(x, V) \leq st_i(st_i(v_i, V), V) \leq st_i(st_i(v_i, V), V^*) \quad (i \in \{1, 2\}),$$

so

$$\begin{aligned} & \Big(st_1(x, int_1(E_V)), st_2(x, int_2(E_V))\Big) \\ \leq\ & \Big(st_1(x, V), st_2(x, V)\Big) \\ \leq\ & \Big(st_1(st_1(v_1, V), V^*), st_2(st_2(v_2, V), V^*)\Big) \in V^{**}. \end{aligned}$$

■

Lemma 4.9. *Let $(L, \mathcal{E})$ be a Weil quasi-uniform frame. For any $F \in \mathcal{E}$, we have that:*

(a) $F \subseteq E_{U_{F^2}}$;

(b) $E_{U_F} \subseteq F^3$.

Proof. (a) Let $(x, y) \in F$. Writing y in the form

$$\bigvee \{y \wedge z \mid (z, z) \in F^2, y \wedge z \neq 0\},$$

it suffices to verify that

$$(x, y \wedge z) \in k(U_{F^2}) \text{ whenever } (z, z) \in F^2 \text{ and } y \wedge z \neq 0. \tag{4.9.1}$$

But as $x \leq st_1(z, F) \leq st_1(z, int_1(F^2))$, $y \wedge z \leq z \leq st_2(z, int_2(F^2))$ and z is F^2-small, then 4.9.1 holds.

(b) Let us verify that $U_F \subseteq F^3$ or, which is the same, that $(x, w) \in F^3$ whenever $(x, y) \in int_1(F), (z, w) \in int_2(F), y \wedge s \neq 0, z \wedge s \neq 0$, and s is F-small. The F-smallness of s implies that $s \leq st(z, F) \leq st_1(z, F)$, so $y \wedge st_1(z, F) \neq 0$, i.e., $st_2(y, F) \wedge z \neq 0$. But $(x, st_2(y, F)) \in F^2$ and $(z, w) \in F$, thus $(x, w) \in F^3$. ■

Let $\mathcal{U}$ and $\mathcal{E}$ be, respectively, a quasi-uniformity on L and a Weil quasi-uniformity on L. In the sequel, $\psi(\mathcal{U})$ denotes the Weil quasi-uniformity generated by $\mathcal{E}_{\mathcal{U}}$ and $\psi'(\mathcal{E})$ denotes the quasi-uniformity for which $\mathcal{U}_{\mathcal{E}}$ is a basis.

Proposition 4.10. $\psi'\psi(\mathcal{U}) = \mathcal{U}$ *and* $\psi\psi'(\mathcal{E}) = \mathcal{E}$.

Proof. We first show that $\psi'\psi(\mathcal{U}) = \mathcal{U}$. The set $\{U_E \mid E \in \psi(\mathcal{U})\}$ is a basis for the quasi-uniformity $\psi'\psi(\mathcal{U})$. It suffices now to observe that it is a basis for $\mathcal{U}$, which is a consequence of Lemma 4.9: by (a), $\{U_E \mid E \in \psi(\mathcal{U})\} \subseteq \mathcal{U}$, and, by (b), for any $V \in \mathcal{U}$ there is some $V' \in \mathcal{U}$ such that $U_{E_{V'}} \subseteq V$.

Similarly, Lemma 4.8 implies that the basis $\{E_U \mid U \in \psi'(\mathcal{E})\}$ of $\psi\psi'(\mathcal{E})$ is also a basis for $\mathcal{E}$, which proves the second equality. ∎

The correspondence $(L,\mathcal{U}) \longmapsto (L,\psi(\mathcal{U}))$ is functorial. Indeed, it is the function on objects of a functor from QUFrm into WQUFrm whose function on morphisms is described in the following proposition:

Proposition 4.11. *Let $(L,\mathcal{U})$ and $(L',\mathcal{U}')$ be quasi-uniform frames and let $f : (L,\mathcal{U}) \longrightarrow (L',\mathcal{U}')$ be a uniform homomorphism. Then $f : (L,\psi(\mathcal{U})) \longrightarrow (L',\psi(\mathcal{U}'))$ is a Weil uniform homomorphism.*

Proof. Let us show that $(f \oplus f)(E) \in \psi(\mathcal{U}')$, for every $E \in \psi(\mathcal{U})$. So, let E be a C-ideal containing $k(U)$, for some $U \in \mathcal{U}$. By hypothesis, $f[U] \in \mathcal{U}'$. It suffices now to show that $k(f[U]) \subseteq (f \oplus f)(E)$, which is trivial since $(f(u_1), f(u_2)) \in (f \oplus f)(E)$ for every $(u_1,u_2) \in U$. ∎

On the other hand, the correspondence $(L,\mathcal{E}) \longmapsto (L,\psi'(\mathcal{E}))$ defines a functor from WQUFrm to QUFrm as we shall see in the sequel.

Lemma 4.12. *Let $f : (L,\mathcal{E}) \longrightarrow (L',\mathcal{E}')$ be a Weil uniform homomorphism. Then, for any $x, y \in L$ and $i \in \{1,2\}$, $f(x) \overset{\mathcal{E}'}{\lhd}_i f(y)$ whenever $x \overset{\mathcal{E}}{\lhd}_i y$.*

Proof. The case $x = 0$ is trivial. If x is non-zero, since $(f \oplus f)(E) \in \mathcal{E}'$ for every $E \in \mathcal{E}$, it suffices to check that

$$(f \oplus f)(E) \circ (f(x) \oplus f(x)) \subseteq f(y) \oplus f(y)$$

whenever $E \in \mathcal{E}$ is such that $E \circ (x \oplus x) \subseteq y \oplus y$. But

$$(f \oplus f)(E) = \bigvee\{f(a) \oplus f(b) \mid (a,b) \in E\}.$$

Apply Lemma 4.2 of Chapter I; it follows that

$$(f \oplus f)(E) \circ (f(x) \oplus f(x)) = \left(\bigcup_{(a,b)\in E} (f(a) \oplus f(b))\right) \circ \downarrow(f(x), f(x)).$$

If $(c,d) \in f(a) \oplus f(b)$ for some $(a,b) \in E$, $(d,e) \in \downarrow (f(x), f(x))$ and $d \neq 0$, then, in case $c \neq 0$ (the case $c = 0$ is obvious), we have that $b \wedge x \neq 0$, and, consequently, that $(a,x) \in E \circ (x \oplus x) \subseteq y \oplus y$. Hence $a \leq y$ and $x \leq y$, which imply that $c \leq f(y)$ and $e \leq f(y)$, i.e., that $(c,e) \in f(y) \oplus f(y)$. ■

Lemma 4.13. *Suppose that $f : L \longrightarrow L'$ is a frame homomorphism. Then, for every $x \in L$ and $E \in L \oplus L$, $st_i(f(x), (f \oplus f)(E)) \leq f(st_i(x, E))$ for $i \in \{1, 2\}$.*

Proof. Let

$$E = \bigvee_{(a,b)\in E} (a \oplus b).$$

We have that

$$\begin{aligned} st_1(f(x), (f \oplus f)(E)) &= st_1\Big(f(x), \bigvee_{(a,b)\in E} (f(a) \oplus f(b))\Big) \\ &= st_1\Big(f(x), \bigcup_{(a,b)\in E} (f(a) \oplus f(b))\Big) \qquad \text{(by Lemma 4.1)} \\ &= \bigvee \{f(a) \mid (a,b) \in E, f(b) \wedge f(x) \neq 0\} \\ &\leq f\Big(\bigvee \{a \mid (a,b) \in E, b \wedge x \neq 0\}\Big) = f(st_1(x, E)). \end{aligned}$$

The proof for $i = 2$ is similar. ■

Proposition 4.14. *Let $(L, \mathcal{E})$ and $(L', \mathcal{E}')$ be Weil quasi-uniform frames and let $f : (L, \mathcal{E}) \longrightarrow (L', \mathcal{E}')$ be a Weil uniform homomorphism. Then $f : (L, \psi'(\mathcal{E})) \longrightarrow (L', \psi'(\mathcal{E}'))$ is a uniform homomorphism.*

Proof. By Lemma 4.12, f is a biframe map. Indeed, for any $a \in L^i$,

$$f(a) = \bigvee \{f(b) \mid b \in L, b \overset{\mathcal{E}}{\triangleleft}_i a\}$$

thus

$$f(a) \leq \bigvee \Big\{f(b) \mid f(b) \overset{\mathcal{E}'}{\triangleleft}_i f(a)\Big\} \leq \bigvee \Big\{x \in L' \mid x \overset{\mathcal{E}'}{\triangleleft}_i f(a)\Big\},$$

that is, $f(a) \in M^i$.

Our goal is to show that, for any $F \in \mathcal{E}$, $f[U_F]$ belongs to $\psi'(\mathcal{E}')$:

Consider $G \in \mathcal{E}$ such that $G^6 \subseteq F$. Then $E_{U_G} \in \psi\psi'(\mathcal{E}) = \mathcal{E}$, so $(f \oplus f)(E_{U_G}) \in \mathcal{E}'$ by hypothesis. Since $\mathcal{E}' = \psi\psi'(\mathcal{E}')$, we may take $V \in \psi'(\mathcal{E}')$ such that $E_V \subseteq (f \oplus f)(E_{U_G})$. We may assume, without loss of generality, that V is strong. In order to show that $f[U_F] \in \psi'(\mathcal{E}')$ we prove that $V \leq f[U_F]$. Assume that $(v_1, v_2) \in V$ with $v_1 \wedge v_2 \neq 0$. Then there is a pair $(x, y) \in U_G$ such that $v_1 \wedge v_2 \wedge f(x) \wedge f(y) \neq 0$. Consequently,

$$v_1 \leq st_1(f(x \wedge y), V) = st_1(f(x \wedge y), E_V) \leq st_1\Big(f(x \wedge y), (f \oplus f)(E_{U_G})\Big),$$

and then, from Lemma 4.13, we deduce that

$$v_1 \leq f(st_1(x \wedge y, E_{U_G})).$$

Whence, due to Lemma 4.9 (b),

$$v_1 \leq f(st_1(x \wedge y, G^3)) \leq f(st_1(x \wedge y, int_1(F))).$$

Analogously,

$$v_2 \leq f(st_2(x \wedge y, int_2(F))).$$

On the other hand, since

$$(x \wedge y, x \wedge y) \in E_{U_G} \subseteq G^3 \subseteq F,$$

the element $x \wedge y$ is F-small, and, in conclusion, we have

$$(v_1, v_2) \leq \Big(f(st_1(x \wedge y, int_1(F))), f(st_2(x \wedge y, int_2(F)))\Big) \in f[U_F]. \qquad \blacksquare$$

In summary, it follows from Propositions 4.10, 4.11 and 4.14 that:

Theorem 4.15. *The categories* WQUFrm *and* QUFrm *are isomorphic.* $\blacksquare$

Notes on Chapter III:

(1) In [25], [26] and [27] Fletcher, Hunsaker and Lindgren studied the theory of frame quasi-uniformities that arises from the theory of entourage uniform frames of Fletcher and Hunsaker [23]. The corresponding category QEUFrm is also isomorphic to QWUFrm, as we pointed out in the introduction to this chapter.

(2) Besides covers and entourages, a quasi-uniformity on a set may be described in terms of "quasi-pseudometrics" (non-symmetric pseudometrics) [30]. So there are characterizations of quasi-uniformities that are analogous to those of uniformities which are given in terms of uniform covers, entourages and pseudometrics. To complete the similar picture for frames it seems that one needs to know what may be used in frames in analogy with quasi-pseudometrics in the spatial setting, that is, one needs a theory of "non-symmetric diameters". This question is under consideration.

CHAPTER IV

WEIL NEARNESS SPACES AND WEIL NEARNESS FRAMES

The purpose of this chapter is to present the notion of Weil nearness frame and its links with nearness frames and Weil nearness spaces (i.e., the spaces that arise as the natural notion of spatial nearness via entourages). These spaces, although distinct from the classical nearness spaces of Herrlich, form a nice topological category. It is shown that, in the realm of Weil nearness spaces, it is possible to consider several spaces of topological nature such as, for example, the symmetric topological spaces, the proximal spaces or the uniform spaces. Weil's concept of entourage is, therefore, a basic topological concept by means of which several topological notions or ideas can be expressed.

We also study some aspects of proximal frames using Weil entourages.

1. Nearness spaces

Topological spaces are the result of the axiomatization of the concept of nearness between a point x and a set A (expressed by the relation $x \in cl(A)$). On the other hand, proximal spaces are obtained by an axiomatization of the concept of nearness between two sets A and B (usually denoted by $A\delta B$, i.e., "A is near B" [56]) and contigual spaces express the concept of nearness between the elements of a finite family of sets

$\mathcal{A}$ (usually denoted by $\sigma(\mathcal{A})$ [41]).

The concept of nearness space was introduced by Herrlich [33] as an axiomatization of the concept of "nearness of an arbitrary collection of sets" $\mathcal{A}$ (usually denoted by $\mathcal{A} \in \xi$, i.e., "$\mathcal{A}$ is near" [33]), with the goal of unifying several types of topological structures; as the author says in [34]:

> "*The aim of this approach is to find a basic topological concept — if possible intuitively accessible — by means of which any topological concept or idea can be expressed*".

There are other (equivalent) axiomatizations of the category of nearness spaces and nearness maps [34]. Here, we prefer to use the one given in terms of covers.

Definitions 1.1. (Herrlich [34]) Let X be a set and let μ be a non-empty family of covers of X. Consider the following axioms:

(N1) if $\mathcal{U} \in \mu$, $\mathcal{V} \in \mathcal{P}(X)$ and $\mathcal{U} \leq \mathcal{V}$ then $\mathcal{V} \in \mu$;

(N2) if $\mathcal{U}, \mathcal{V} \in \mu$ then $\mathcal{U} \wedge \mathcal{V} \in \mu$;

(N3) if $\mathcal{U} \in \mu$ then

$$int_\mu \mathcal{U} := \{int_\mu U \mid U \in \mathcal{U}\} \in \mu,$$

where, for every $U \subseteq X$,

$$int_\mu U := \{x \in X \mid \{U, X \setminus x\} \in \mu\}.$$

The family μ is called a *prenearness structure* on X if it satisfies (N1); μ is called a *seminearness structure* on X if it satisfies (N1) and (N2) and it is called a *nearness structure* on X if it fulfils (N1), (N2) and (N3). The pair (X, μ) is called a *prenearness space* (respectively, *seminearness space*, *nearness space*) if μ is a prenearness (respectively, seminearness, nearness) structure on X.

If (X, μ) and (X', μ') are prenearness spaces, a function $f : X \longrightarrow X'$ is called a *nearness map* $f : (X, \mu) \longrightarrow (X', \mu')$ from (X, μ) to (X', μ') if f^{-1} preserves nearness covers, that is, if $\mathcal{U} \in \mu'$ implies $f^{-1}[\mathcal{U}] \in \mu$.

Let us denote by PNear the category of prenearness spaces and nearness maps and by SNear and Near the full subcategories of PNear of, respectively, seminearness spaces and nearness spaces.

Remarks 1.2.

(a) If μ is a prenearness on X then int_μ is an operator on $\mathcal{P}(X)$ satisfying:

(T0) $x \in int_\mu(X \setminus \{y\})$ if and only if $y \in int_\mu(X \setminus \{x\})$;

(T1) $int_\mu(X) = X$;

(T2) $int_\mu(A) \subseteq A$;

(T3) $int_\mu(A) \subseteq int_\mu(B)$ whenever $A \subseteq B$.

In case μ is a seminearness then int_μ also satisfies the axiom

(T4) $int_\mu(A \cap B) = int_\mu(A) \cap int_\mu(B)$.

Finally, if μ is a nearness, int_μ satisfies in addition the axiom

(T5) $int_\mu(int_\mu(A)) = int_\mu(A)$.

Thus any nearness structure on X induces on X a topology satisfying axiom (T0), that is, a *symmetric topology.* These topological spaces are usually known as *symmetric topological spaces* or R_0*-spaces.* We shall denote the full subcategory of Top formed by these spaces by $\mathsf{R_0Top}$.

When (X, μ) is only a seminearness space then int_μ is not necessarily an interior operator but it only defines a closure space in the sense of Čech [16]. The seminearness spaces are the "quasi-uniform spaces" of Isbell [38]. The category SNear is isomorphic to the category of "merotopic spaces" and "merotopic maps" of Katětov [46].

(b) Since (N3) is implied by the star-refinement condition (U2) of Definition I.1.3, the category of uniform spaces is a full subcategory of the category of nearness spaces. Herrlich [34] proved that Near is a topological category — as well as PNear and SNear — and that the categories of symmetric topological spaces, uniform spaces, proximal spaces and contigual spaces are nicely embedded in Near as bireflective (the case of uniform spaces, proximal spaces and contigual spaces) or bicoreflective (the case of symmetric topological spaces) subcategories.

2. Covering nearness frames

The concept of nearness frame was introduced by Banaschewski and Pultr [12] as a generalization of uniform frames: nearness frames are uniform frames without the star-refinement property (U2). Their motivation is that some situations in uniform frames naturally lead to the consideration of nearnesses.

Definition 2.1. (Banaschewski and Pultr [12]) Let L be a frame and let $\mathcal{U} \subseteq Cov(L)$. The pair $(L,\mathcal{U})$ is a *nearness frame* if:

(N1) $\mathcal{U}$ is a filter of $(Cov(L), \leq)$;

(N2) for any $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{U}}{\lhd} x\}$.

These are the objects of the category NFrm, whose morphisms — the *uniform homomorphisms* — are the frame maps $f : (L,\mathcal{U}) \longrightarrow (L',\mathcal{U}')$ such that $f[U] \in \mathcal{U}'$ whenever $U \in \mathcal{U}$.

The corresponding spatial notion is the following:

(2.1.1) The topological spaces $(X, \mathcal{T})$ supplied with a filter μ of $\mathcal{T}$-open covers of X such that, for any $U \in \mathcal{T}$ and $x \in U$, there exist $V \in \mathcal{T}$ and $\mathcal{U} \in \mu$ with $x \in V$ and $st(V,\mathcal{U}) \subseteq U$.

In fact, adapting here the open and spectrum functors in the obvious way we have a dual adjunction (the open functor assigning to each space $(X, \mathcal{T}, \mu)$ of 2.1.1, the nearness frame $(\mathcal{T}, \mu)$ and the spectrum functor assigning to each nearness frame $(L,\mathcal{U})$ the space $(ptL, \mathcal{T}_{ptL}, \mu_{ptL})$.

The spaces of 2.1.1 are not Herrlich's nearness spaces; any nearness μ for a space $(X, \mathcal{T}, \mu)$ of 2.1.1 is a nearness in the sense of Herrlich but the converse is not true.

A characterization of the nearness spaces of Herrlich that correspond to those spaces was given recently by Hong and Kim:

Definition 2.2. (Hong and Kim [37]) Given subsets A and B of a nearness space (X, μ), $A <_\mu B$ means that $\{X \setminus A, B\} \in \mu$. A nearness space (X, μ) is said to be *framed* if, for every $x \in X$ and $A \subseteq X$ such that $\{x\} <_\mu A$, there exists a subset B of X such that $\{x\} <_\mu B <_\mu A$.

Since, for every space $(X, \mathcal{T}, \mu)$ defined in 2.1.1, $\mathcal{T}_\mu = \mathcal{T}$, it is easy to prove that the category of these spaces — where the morphisms are the maps $f : (X, \mathcal{T}, \mu) \longrightarrow (X', \mathcal{T}', \mu')$ for which $f^{-1}[\mathcal{U}] \in \mu$ whenever $\mathcal{U} \in \mu'$ — is isomorphic to the category FrNear of framed nearness spaces and nearness maps. Moreover:

Theorem 2.3. (Hong and Kim [37]) *The category* FrNear *is a bireflective subcategory of* Near *and it is dually equivalent with the full subcategory* SpNFrm *of spatial nearness frames of* NFrm. ■

In summary, we have:

$$\begin{tikzcd}
\mathsf{Near} & & \\
\mathsf{FrNear} \arrow[u, hook, "\text{bireflective full embedding}"] \arrow[rr, shift left, "\Omega"] \arrow[d, equal] & & \mathsf{NFrm} \arrow[ll, shift left, "pt", "\text{dual adjunction}"'] \\
\mathsf{FrNear} \arrow[rr, shift left, "\Omega"] & & \mathsf{SpNFrm} \arrow[u, hook, "\text{monoreflective full embedding}"'] \arrow[ll, shift left, "pt", "\text{dual equivalence}"']
\end{tikzcd}$$

3. Weil nearness frames

We now embark on the study of the natural notion of nearness that arises as a generalization of our notion of Weil uniformity.

Definitions 3.1.

(1) A *Weil nearness* on L is a family $\mathcal{E}$ of Weil entourages such that:

(NW0) $E^{-1} \in \mathcal{E}$ for every $E \in \mathcal{E}$;

(NW1) $\mathcal{E}$ is a filter of $(WEnt(L), \subseteq)$;

(NW2) for every $x \in L$, $x = \bigvee\{y \in L \mid y \overset{\mathcal{E}}{\lhd} x\}$.

(2) A *Weil nearness frame* is just a pair $(L, \mathcal{E})$ where L is a frame and $\mathcal{E}$ is a Weil nearness on L. The morphisms of the category WNFrm of Weil nearness frames — the *Weil uniform homomorphisms* — are the frame maps $f : (L, \mathcal{E}) \longrightarrow (L', \mathcal{E}')$ such that $f \oplus f$ preserves nearness entourages, i.e., $(f \oplus f)(E) \in \mathcal{E}'$ whenever $E \in \mathcal{E}$.

Remark 3.2. As the refinement condition (UW2) is no longer valid in a Weil nearness $\mathcal{E}$, $x \overset{\mathcal{E}}{\lhd} y$ is not equivalent to $st(x, E) \leq y$ for some $E \in \mathcal{E}$. Now we only have:

$$\begin{aligned} x \overset{\mathcal{E}}{\lhd} y \quad &\Leftrightarrow \quad \exists E \in \mathcal{E} : st_1(x, E) \leq y \\ &\Leftrightarrow \quad \exists E \in \mathcal{E} : st_2(x, E) \leq y \\ &\Rightarrow \quad \exists E \in \mathcal{E} : st(x, E) \leq y. \end{aligned}$$

The latter condition still implies $x \prec y$. Thus L is regular whenever it has a Weil nearness. The reverse implication is also true: take $\mathcal{E} = WEnt(L)$; if $x \prec y$ then, using Lemma 4.2 of Chapter I, one can easily prove that

$$\Big((x^* \oplus x^*) \vee (y \oplus y)\Big) \circ (x \oplus x) \subseteq y \oplus y$$

and then, as $(x^* \oplus x^*) \vee (y \oplus y) \in \mathcal{E}$, $x \overset{\mathcal{E}}{\lhd} y$.

In conclusion, as for nearnesses, a frame L has a Weil nearness if and only if it is regular.

The correspondence Ψ, that we introduced in Chapter I in order to establish an isomorphism between the categories of uniform frames and Weil uniform frames, and its inverse Ψ^{-1} also work for the corresponding categories of nearness structures (the nearness frames of Banaschewski and Pultr and our Weil nearness frames):

$$(L,\mathcal{U}) \in \mathsf{NFrm} \quad \stackrel{\Psi}{\longmapsto} \quad \Big\{\bigvee_{x\in U}(x\oplus x) \mid U\in\mathcal{U}\Big\} \text{ forms a basis for a Weil nearness on } L,$$

$$(L,\mathcal{E}) \in \mathsf{WNFrm} \quad \stackrel{\Psi^{-1}}{\longmapsto} \quad \Big\{\{x\in L \mid x\oplus x\subseteq E\} \mid E\in\mathcal{E}\Big\} \text{ forms a basis for a nearness on } L.$$

These correspondences constitute a Galois connection between the partially ordered sets (by inclusion) of, respectively, nearnesses and Weil nearnesses on L: $1 \le \Psi\Psi^{-1}$ and $\Psi^{-1}\Psi \le 1$. This Galois connection induces a "maximal" isomorphism between the partially ordered set of Weil nearnesses $\mathcal{E}$ on L satisfying the condition

$$\forall\, E\in\mathcal{E}\ \exists\, F\in\mathcal{E} : F \subseteq \bigvee_{(x,x)\in E}(x\oplus x)$$

and the partially ordered set of nearnesses $\mathcal{U}$ on L satisfying the condition

$$\forall\, U\in\mathcal{U}\ \exists\, V\in\mathcal{U} : \Bigg(x\oplus x\subseteq \bigvee_{x\in V}(x\oplus x) \ \Rightarrow\ \exists\, u\in U : x\le u\Bigg).$$

Therefore, our bijection between Weil uniform frames and uniform frames can not be lifted up to nearnesses. For that reason we do not know whether WNFrm is isomorphic to NFrm.

4. Weil nearness spaces

We proceed to discuss the following problems:

> Which is the right spatial concept in analogy with the choosen notion of Weil nearness frame? May this concept be expressed in terms of Weil's entourages for sets?

Definitions 4.1. Let $(X, \mathcal{T})$ be a topological space.

(1) For any entourage E of X and any $x \in X$ let

$$E[x] = \{y \in X \mid (x, y) \in E\}.$$

We denote by $int_{\mathcal{T}}(E)$ (or, briefly, by $int(E)$ whenever no ambiguity arises) the subset

$$\Big\{(x, y) \in X \times X \mid x \in int_{\mathcal{T}}(E^{-1}[y]), y \in int_{\mathcal{T}}(E[x])\Big\}$$

of E. We say that E is an *interior entourage* if $int(E)$ is still an entourage of X.

(2) An entourage E of X is *open* if $int(E) = E$.

The following proposition is obvious.

Proposition 4.2. *Let E be an entourage of a topological space $(X, \mathcal{T})$.*

(a) *The following assertions are equivalent:*

(i) *E is interior;*

(ii) *E contains some open entourage;*

(iii) *for every $x \in X$, $x \in int_{\mathcal{T}}(E[x]) \cap int_{\mathcal{T}}(E^{-1}[x])$.*

(b) *The following assertions are equivalent:*

(i) *E is open;*

(ii) *for every $x \in X$, $E[x]$ and $E^{-1}[x]$ are open.* ∎

Any Weil nearness on the frame $\mathcal{T}$ of open sets of a topological space $(X, \mathcal{T})$ is a set $\mathcal{E}$ of open entourages of $(X, \mathcal{T})$ such that

(Fr0) $E^{-1} \in \mathcal{E}$ for every $E \in \mathcal{E}$,

(Fr1) $\mathcal{E}$ is a filter (with respect to $\subseteq$),

(Fr2) for every $U \in \mathcal{T}$ and for every $x \in U$ there exist $V \in \mathcal{T}$ and $E \in \mathcal{E}$ such that $x \in V$ and $E \circ (V \times V) \subseteq U \times U$, since condition (Fr2) is just the pointwise formulation of the admissibility condition (NW2) for Weil nearness frames. We call the topological spaces $(X, \mathcal{T})$ endowed with a symmetric filter of open entourages of $(X, \mathcal{T})$ satisfying condition (Fr2) *framed Weil nearness spaces*. The morphisms of the category FrWNear of framed Weil nearness spaces are the maps

$$f : \Big((X, \mathcal{T}), \mathcal{E}\Big) \longrightarrow \Big((X', \mathcal{T}'), \mathcal{E}'\Big)$$

for which $(f \times f)^{-1}(E) \in \mathcal{E}$ for any $E \in \mathcal{E}'$.

On the reverse direction, for the same reason, any framed Weil nearness on $(X, \mathcal{T})$ is a Weil nearness on the frame $\mathcal{T}$, so that the spatial and frame notions coincide in this context. Therefore, the notion of framed Weil nearness space is the right spatial analogue of the frame concept of Weil nearness.

The open and spectrum functors of Section I.4 determine contravariant functors between FrWNear and WNFrm, but in order to get a dual adjunction we have to extend the classes of morphisms to, respectively, all continuous maps and all frame homomorphisms, because the proof in Theorem 4.14 of Chapter I that $\overline{f}$ is uniform depends on the refinement condition (UW2).

Let us see how framed Weil nearness spaces can be equated within the framework of a generalization of Weil's uniform spaces.

Definitions 4.3. Let X be a set and let $\mathcal{E}$ be a non-empty set of entourages of X. Consider the following axioms:

(NW0) $E^{-1} \in \mathcal{E}$ for every $E \in \mathcal{E}$;

(NW1) $E \subseteq F$, $E \in \mathcal{E} \;\Rightarrow\; F \in \mathcal{E}$;

(NW2) $E \cap F \in \mathcal{E}$ for every $E, F \in \mathcal{E}$;

(NW3) for every $E \in \mathcal{E}$,

$$\Big\{(x, y) \in X \times X \mid x \in int_{\mathcal{E}}(E^{-1}[y]), y \in int_{\mathcal{E}}(E[x])\Big\} \in \mathcal{E},$$

where, for any $A \subseteq X$,

$$int_{\mathcal{E}}(A) = \{x \in X \mid \exists E \in \mathcal{E} : E[x] \subseteq A\}.$$

$\mathcal{E}$ is called a *Weil prenearness* on X if it satisfies (NW0) and (NW1); $\mathcal{E}$ is called a *Weil seminearness* on X if it satisfies (NW0), (NW1) and (NW2) and it is called a *Weil nearness* on X if it fulfils (NW0), (NW1), (NW2) and (NW3). The pair $(X, \mathcal{E})$ is called a *Weil prenearness space* (respectively, *Weil seminearness space*, *Weil nearness space*) if $\mathcal{E}$ is a Weil prenearness (respectively, Weil seminearness, Weil nearness) on X.

A *Weil nearness map* is just a map $f : (X, \mathcal{E}) \longrightarrow (X', \mathcal{E}')$ between Weil prenearness spaces for which $(f \times f)^{-1}(E) \in \mathcal{E}$ for every $E \in \mathcal{E}'$.

We denote by PWNear the category of Weil prenearness spaces and Weil nearness maps and by SWNear and WNear its full subcategories of, respectively, Weil seminearness spaces and Weil nearness spaces.

Remarks 4.4. (a) If $(X, \mathcal{E})$ is a Weil prenearness space then $int_{\mathcal{E}}$ is an operator on $\mathcal{P}(X)$ satisfying the following axioms:

(T1) $int_{\mathcal{E}}(X) = X$,

(T2) $int_{\mathcal{E}}(A) \subseteq A$ for every $A \subseteq X$,

(T3) $A \subseteq B \Rightarrow int_{\mathcal{E}}(A) \subseteq int_{\mathcal{E}}(B)$.

If $(X, \mathcal{E})$ is a Weil seminearness space then, in addition, $int_{\mathcal{E}}$ satisfies the following axioms

(T4) $int_{\mathcal{E}}(A \cap B) = int_{\mathcal{E}}(A) \cap int_{\mathcal{E}}(B)$;

(T0) $x \in int_{\mathcal{E}}(X \setminus \{y\}) \Leftrightarrow y \in int_{\mathcal{E}}(X \setminus \{x\})$.

Finally, if $(X, \mathcal{E})$ is a Weil nearness space, then $int_{\mathcal{E}}$ also satisfies the axiom

(T5) $int_{\mathcal{E}}(int_{\mathcal{E}}(A)) = int_{\mathcal{E}}(A)$.

Thus any Weil nearness structure $\mathcal{E}$ on X induces on X a symmetric topology $\mathcal{T}_{\mathcal{E}}$. Axiom (NW3) says that $int_{\mathcal{T}_{\mathcal{E}}}(E) \in \mathcal{E}$ whenever $E \in \mathcal{E}$.

(b) The Weil seminearness spaces are the "semiuniform spaces" of Čech [16]. In this case $int_{\mathcal{E}}$ may not be an interior operator; it only defines a closure operator in the sense of Čech [16].

(c) In the presence of (NW0), (NW1) and (NW2), condition (NW3) is equivalent to the following one:

$$\text{(NW3}'\text{)} \qquad \text{for every } E \in \mathcal{E}, \Big\{(x,y) \in X \times X \mid y \in int_{\mathcal{E}}(E[x])\Big\} \in \mathcal{E}.$$

(d) Every uniform space is a Weil nearness space since the refinement condition

$$\forall E \in \mathcal{E}\ \exists F \in \mathcal{E}\ :\ F \circ F \subseteq E$$

implies condition (NW3).

The category SWNear is bicoreflective in PWNear. If $(X, \mathcal{E})$ is a Weil prenearness space and $\mathcal{E}_S$ is the set of all entourages of X which contain the intersection of a finite number of elements of $\mathcal{E}$, then $1_X : (X, \mathcal{E}_S) \longrightarrow (X, \mathcal{E})$ is the bicoreflection of $(X, \mathcal{E})$ with respect to SWNear.

The category WNear is bireflective in SWNear. For $(X, \mathcal{E}) \in \mathsf{SWNear}$ define, for every ordinal α, the operator int^α on $\mathcal{P}(X)$ by

- $int^0(A) = A$,
- $int^\alpha(A) = int^\beta(A) \setminus \{x \in int^\beta(A) \mid \forall E \in \mathcal{E}\ \ E[x] \cap (X \setminus A) \neq \emptyset\}$ if $\alpha = \beta + 1$,
- $int^\alpha(A) = \bigcap_{\beta<\alpha} int^\beta(A)$ if α is a limit ordinal.

Then

$$int(A) := \bigcap_{\alpha \in Ord} int^\alpha(A)$$

is the "largest" operator on $\mathcal{P}(X)$ satisfying axioms (T0), (T1), (T2), (T3), (T4) and (T5) and so it defines a symmetric topology $\mathcal{T}$ on X. Putting

$$\mathcal{E}_N := \{E \subseteq X \times X \mid int_{\mathcal{T}}(E) \in \mathcal{E}\},$$

$1_X : (X, \mathcal{E}) \longrightarrow (X, \mathcal{E}_N)$ is the bireflection of $(X, \mathcal{E})$ with respect to WNear.

From now on, for subsets A and B of a set X, we denote the set

$$(X \setminus A \times X \setminus A) \cup (B \times B)$$

by $E^X_{A,B}$ (or, briefly, by $E_{A,B}$ whenever there is no ambiguity). The set $E^X_{\{x\},B}$ will be denoted by $E^X_{x,B}$ (or $E_{x,B}$).

Given subsets A and B of a Weil nearness space $(X, \mathcal{E})$, we write $A <_{\mathcal{E}} B$ whenever $E_{A,B} \in \mathcal{E}$.

Lemma 4.5. *Let $(X, \mathcal{E}) \in \mathsf{WNear}$. For any $x \in X$ and $A, B \subseteq X$, the following hold:*

(a) *$x <_{\mathcal{E}} A$ if and only if $x \in int_{\mathcal{T}_{\mathcal{E}}}(A)$;*

(b) *if $B <_{\mathcal{E}} A$ then $B \subseteq int_{\mathcal{T}_{\mathcal{E}}}(A)$.*

Proof. (a) If $x <_{\mathcal{E}} A$ just take $E_{x,A} \in \mathcal{E}$. Since $E_{x,A}[x] \subseteq A$ then $x \in int_{\mathcal{T}_{\mathcal{E}}}(A)$. Conversely, if $E[x] \subseteq A$ and $E \in \mathcal{E}$ consider $F = E \cap E^{-1}$. It is clear that $F \subseteq E_{x,A}$. Hence $E_{x,A} \in \mathcal{E}$.

(b) It is obvious, because $E_{B,A}[x] \subseteq A$ for any $x \in B$. ■

Theorem 4.6. *Suppose that $\big((X, \mathcal{T}), \mathcal{E}\big)$ is a framed Weil nearness space and let $\overline{\mathcal{E}}$ be the filter of $(WEnt(X), \subseteq)$ generated by $\mathcal{E}$. Then:*

(a) *$\mathcal{T}_{\overline{\mathcal{E}}} = \mathcal{T}_{\mathcal{E}} = \mathcal{T}$;*

(b) *$(X, \overline{\mathcal{E}})$ is a Weil nearness space satisfying the condition*

$$x <_{\overline{\mathcal{E}}} A \Rightarrow \exists B \subseteq X : x <_{\overline{\mathcal{E}}} B <_{\overline{\mathcal{E}}} A.$$

Proof. (a) Let $A \in \mathcal{T}$ and $x \in A$. By assumption, there are $V \in \mathcal{T}$ and $E \in \mathcal{E}$ such that $x \in V$ and $E \circ (V \times V) \subseteq A \times A$. Then $E^{-1}[x] \subseteq A$, so $x \in int_{\mathcal{T}_{\mathcal{E}}}(A)$ and $A \in \mathcal{T}_{\mathcal{E}}$. Conversely, if $A \in \mathcal{T}_{\mathcal{E}}$, there is, for each $x \in A$, $E^x \in \mathcal{E}$ with $E^x[x] \subseteq A$. Therefore

$$A = \bigcup \{E^x[x] \mid x \in A\} \in \mathcal{T}.$$

Finally, let us prove the equality $\mathcal{T} = \mathcal{T}_{\overline{\mathcal{E}}}$. The inclusion $\mathcal{T} \subseteq \mathcal{T}_{\overline{\mathcal{E}}}$ is obvious because $\mathcal{E} \subseteq \overline{\mathcal{E}}$. The reverse inclusion $\mathcal{T}_{\overline{\mathcal{E}}} \subseteq \mathcal{T}$ is also evident: for any $x \in A$, where $A \in \mathcal{T}_{\overline{\mathcal{E}}}$, there is $E^x \in \overline{\mathcal{E}}$ such that $E^x[x] \subseteq A$. But each E^x contains some F^x in $\mathcal{E}$, thus

$$A = \bigcup\{F^x[x] \mid x \in A\} \in \mathcal{T}.$$

(b) The proof that $(X, \overline{\mathcal{E}})$ is a Weil nearness space is trivial.

Assume $x <_{\overline{\mathcal{E}}} A$, i.e., $x \in int_{\mathcal{T}_{\overline{\mathcal{E}}}}(A)$. Then $x \in int_{\mathcal{T}}(A)$. By hypothesis, there are $B \in \mathcal{T}$ and $E \in \mathcal{E}$ such that $x \in B$ and $E \circ (B \times B) \subseteq int_{\mathcal{T}}(A) \times int_{\mathcal{T}}(A)$. Since V is open, $x <_{\overline{\mathcal{E}}} B$. In order to prove that $B <_{\overline{\mathcal{E}}} A$ it suffices to check that $E_{B,A} \in \overline{\mathcal{E}}$. So, consider $F = E \cap E^{-1} \in \mathcal{E} \subseteq \overline{\mathcal{E}}$. We have that $F \circ (B \times B) \subseteq A \times A$. Since F is a symmetric entourage of X,

$$F \circ (B \times B) \subseteq A \times A \Leftrightarrow F \subseteq (X \setminus B \times X \setminus B) \cup (A \times A).$$

Hence $E_{B,A} \in \overline{\mathcal{E}}$. ■

Theorem 4.7. *Let $(X, \mathcal{E})$ be a Weil nearness space satisfying axiom*

$$\text{(NW4)} \quad x <_{\mathcal{E}} A \Rightarrow \exists B \subseteq X : x <_{\mathcal{E}} B <_{\mathcal{E}} A,$$

and let $\overset{\circ}{\mathcal{E}} = \{int_{\mathcal{T}_{\mathcal{E}}}(E) \mid E \in \mathcal{E}\}$ be the set of open entourages in $\mathcal{E}$. Then $\left((X, \mathcal{T}_{\mathcal{E}}), \overset{\circ}{\mathcal{E}}\right)$ is a framed Weil nearness space.

Proof. Since $(int(E))^{-1} = int(E^{-1})$, (Fr0) is satisfied.

Axiom (Fr1) is a consequence of the fact that $int(E_1) \cap int(E_2) = int(E_1 \cap E_2)$.

Let us check axiom (Fr2): consider $U \in \mathcal{T}_{\mathcal{E}}$ and $x \in U$. Then, by Lemma 4.5, $x <_{\mathcal{E}} U$ so, by assumption, there is some $B \subseteq X$ such that $x <_{\mathcal{E}} B <_{\mathcal{E}} U$. Take $V = int_{\mathcal{T}_{\mathcal{E}}}(B)$. The fact $x <_{\mathcal{E}} B$ means that $x \in V$. On the other hand, $B <_{\mathcal{E}} U$ means that $E_{B,U} \in \mathcal{E}$. Let $E = int_{\mathcal{T}_{\mathcal{E}}}(E_{B,U}) \in \overset{\circ}{\mathcal{E}}$. We have

$$\begin{aligned} E \circ (V \times V) &\subseteq (X \setminus B \times X \setminus B \cup U \times U) \circ (B \times B) \\ &\subseteq U \times U. \end{aligned}$$

■

Corollary 4.8. *The category* $\mathsf{FrWNear}$ *is isomorphic to the full subcategory* $\mathsf{WNear}_{(\mathsf{NW4})}$ *of* WNear *of the Weil nearness spaces satisfying* (NW4).

Proof. For any morphism $f : (X, \mathcal{E}) \longrightarrow (X', \mathcal{E}')$ of $\mathsf{WNear}_{(\mathsf{NW4})}$,

$$(f \times f)^{-1}(int(E)) \subseteq int((f \times f)^{-1}(E)).$$

Therefore $f : \left((X, \mathcal{T}_{\mathcal{E}}), \overset{\circ}{\mathcal{E}}\right) \longrightarrow \left((X', \mathcal{T}_{\mathcal{E}'}), \overset{\circ}{\mathcal{E}'}\right)$ belongs to $\mathsf{FrWNear}$. The converse is also true.

Now the existence of the isomorphism is an immediate corollary of Theorems 4.6 and 4.7 and of the following obvious facts:

- $\overset{\circ}{\overline{\mathcal{E}}} = \mathcal{E}$ for any framed Weil nearness $\mathcal{E}$;
- $\overline{\overset{\circ}{\mathcal{E}}} = \mathcal{E}$ for any Weil nearness $\mathcal{E}$ satisfying (NW4). ■

In the sequel we shall identify $\mathsf{WNear}_{(\mathsf{NW4})}$ as $\mathsf{FrWNear}$.

5. The category WNear as a unified theory of (symmetric) topology and uniformity

Proceeding with the study of Weil nearness spaces, we are now naturally led to ask:

> How do Weil nearness spaces relate with the classical nearness spaces of Herrlich ([33], [34])?

The classical correspondence between uniform covers and uniform entourages still works for these spaces:

$$
\begin{array}{ccc}
\boxed{\text{Nearness spaces}} & & \boxed{\text{Weil nearness spaces}} \\[2ex]
(X,\mu) & \xrightarrow{\Psi} & \left\{\bigcup_{U\in\mathcal{U}}(U\times U) \mid \mathcal{U}\in\mu\right\} \\
 & & \text{forms a basis for} \\
 & & \text{a Weil nearness on } X \\[2ex]
\Big\{\{E[x] : x\in X\} \mid E\in\mathcal{E}\Big\} & \xleftarrow{\Psi^{-1}} & (X,\mathcal{E}) \\
\text{forms a basis for} & & \\
\text{a nearness on } X & &
\end{array}
$$

It also works for morphisms. These functors define a Galois correspondence ($\Psi\Psi^{-1} \leq 1$ and $\Psi^{-1}\Psi \leq 1$) which is an isomorphism precisely when restricted to uniformities.

In spite of not having an equivalence between the two notions, our category of Weil nearness spaces still have the nice categorical properties that Herrlich was looking for when searching for a good axiomatization of nearness ([33], [34]).

For example:

Proposition 5.1. *The category* WNear *is a well-fibred topological category over the category* Set *of sets.*

Proof. The well-fibreness is obvious: for any set X, the class of all Weil nearness spaces $(X,\mathcal{E})$ with underlying set X is a set and there exists exactly one Weil nearness space with underlying set X whenever X is of cardinality at most one.

It remains to show that the forgetful functor $\mathsf{WNear} \xrightarrow{|\cdot|} \mathsf{Set}$ is topological, i.e., that every $|\cdot|$-structured source $\Big(X \xrightarrow{f_i} |X_i,\mathcal{E}_i|\Big)_{i\in I}$ has a unique $|\cdot|$-initial lift

$$\Big((X,\mathcal{E}) \xrightarrow{f_i} (X_i,\mathcal{E}_i)\Big)_{i\in I}.$$

Consider

$$\mathcal{B} = \left\{\bigcap_{j=1}^{n}(f_{i_j}\times f_{i_j})^{-1}(E_j) \mid n\in\mathbb{N}, i_j\in I, E_j\in\mathcal{E}_{i_j}\right\} \cup \{X\times X\}.$$

This is a basis for a Weil nearness $\mathcal{E}$ on X. We only verify axiom (NW3) — by checking the equivalent condition (NW3′) of Remark 4.4 (c) — since the others are obviously satisfied. So, let us check that

$$\bigcap_{j=1}^{n}(f_{i_j}\times f_{i_j})^{-1}\Big(\{(x,y)\in X_j\times X_j \mid y\in int_{\mathcal{E}_j}(E_j[x])\}\Big)$$

is included in

$$\Big\{(x,y)\in X\times X \mid y\in int_{\mathcal{E}}\Big((\bigcap_{j=1}^{n}(f_{i_j}\times f_{i_j})^{-1}(E_j))[x]\Big)\Big\}$$

for $n\in\mathbb{N}$, $i_j\in I$ and $E_j\in\mathcal{E}_{i_j}$. If

$$(x,y)\in\bigcap_{j=1}^{n}(f_{i_j}\times f_{i_j})^{-1}\Big(\{(a,b)\in X_j\times X_j \mid b\in int_{\mathcal{E}_j}(E_j[a])\}\Big)$$

then, for every $j\in\{1,\ldots,n\}$, $f_{i_j}(y)\in int_{\mathcal{E}_j}(E_j[f_{i_j}(x)])$, that is, for every $j\in\{1,\ldots,n\}$ there is $F_j\in\mathcal{E}_j$ such that $F_j[f_{i_j}(y)]\subseteq E_j[f_{i_j}(x)]$. On the other hand,

$$(x,y)\in\Bigg\{(a,b)\in X\times X \mid b\in int_{\mathcal{E}}\Big(\Big(\bigcap_{j=1}^{n}(f_{i_j}\times f_{i_j})^{-1}(E_j)\Big)[a]\Big)\Bigg\}$$

if and only if there is some $E\in\mathcal{E}$ for which $E[y]\subseteq\Big(\bigcap_{j=1}^{n}(f_{i_j}\times f_{i_j})^{-1}(E_j)\Big)[x]$, i.e., if and only if there are $k_1,\ldots,k_m\in I$ and $E_l\in\mathcal{E}_{k_l}$ $(l\in\{1,\ldots,m\})$ such that

$$\Big(\bigcap_{l=1}^{m}(f_{k_l}\times f_{k_l})^{-1}(E_l)\Big)[y]\subseteq\Big(\bigcap_{j=1}^{n}(f_{i_j}\times f_{i_j})^{-1}(E_j)\Big)[x].$$

Putting $m:=n$, $k_1:=i_1,\ldots,k_m:=i_n$ and $E_l:=F_l$, for $l\in\{1,\ldots,n\}$, we have, for every $z\in\Big(\bigcap_{j=1}^{n}(f_{i_j}\times f_{i_j})^{-1}(F_j)\Big)[y]$, $f_{i_j}(z)\in F_j[f_{i_j}(y)]$ for any $j\in\{1,\ldots,n\}$. Since $F_j[f_{i_j}(y)]\subseteq E_j[f_{i_j}(x)]$, we conclude that, for every $j\in\{1,\ldots,n\}$, $(f_{i_j}(x),f_{i_j}(z))\in E_j$, i.e., that

$$z\in\Big(\bigcap_{j=1}^{n}(f_{i_j}\times f_{i_j})^{-1}(E_j)\Big)[x].$$

In conclusion, $\mathcal{B}$ is a basis for a Weil nearness on X. It is clear that this is the least nearness $\mathcal{E}$ on X for which every $(X,\mathcal{E})\xrightarrow{f_i}(X_i,\mathcal{E}_i)$ is a Weil nearness map. ■

As we shall see in the sequel, the category WNear also unifies several types of topological structures such as symmetric topological spaces, proximal spaces and uniform spaces.

Symmetric topological spaces

Lemma 5.2. *In a symmetric topological space* $(X, \mathcal{T})$ *the following assertions are equivalent:*

(i) $x \in int_{\mathcal{T}}(A)$;

(ii) $E_{x,A}$ *is an interior entourage of* $(X, \mathcal{T})$.

Proof. (i)⇒(ii): Of course, if $x \in int_{\mathcal{T}}(A)$, $E_{x,A}$ is a symmetric entourage of X. Since

$$E_{x,A}[y] = \begin{cases} A & \text{if } y = x \\ X & \text{if } y \in A \setminus \{x\} \\ X \setminus \{x\} & \text{if } y \in X \setminus A \end{cases}$$

we have

$$int_{\mathcal{T}}(E_{x,A}[y]) = \begin{cases} int_{\mathcal{T}}(A) & \text{if } y = x \\ X & \text{if } y \in A \setminus \{x\} \\ int_{\mathcal{T}}(X \setminus \{x\}) & \text{if } y \in X \setminus A \end{cases}$$

so it remains to show that $y \in int_{\mathcal{T}}(X \setminus \{x\})$ whenever $y \in X \setminus A$. This is an immediate consequence of the symmetry of $(X, \mathcal{T})$:

$$\begin{aligned} y \in X \setminus A &\Rightarrow A \subseteq X \setminus \{y\} \\ &\Rightarrow x \in int_{\mathcal{T}}(X \setminus \{y\}) \\ &\Rightarrow y \in int_{\mathcal{T}}(X \setminus \{x\}). \end{aligned}$$

(ii)⇒(i): It is obvious. ■

The implication (ii)⇒(i) can be generalized in the following way:

Lemma 5.3. *Given two subsets* A *and* B *of a symmetric topological space* $(X, \mathcal{T})$, *if* $E_{B,A}$ *is an interior entourage of* $(X, \mathcal{T})$, *then* $B \subseteq int_{\mathcal{T}}(A)$.

Proof. For every $b \in B$, $b \in int_{\mathcal{T}}(E_{B,A}[b])$. Since $B \subseteq A$, $E_{B,A}[b] = A$. Hence $b \in int_{\mathcal{T}}(A)$. ■

Proposition 5.4. *The set $\mathcal{E}$ of all interior entourages of a symmetric topological space $(X, \mathcal{T})$ is a Weil nearness on X satisfying in addition the axiom*

$$\text{(NW5)} \ \ E \in \mathcal{E} \textit{ whenever } int_{\mathcal{T}}(E) \textit{ is an entourage of } X,$$

and the topology induced by $\mathcal{E}$ coincides with $\mathcal{T}$.

Proof. The fact that $\mathcal{E}$ is a Weil nearness on X satisfying (NW5) is obvious. Let us prove that $\mathcal{T}$ coincides with the topology induced by $\mathcal{E}$, i.e., that for any subset A of X,

$$int_{\mathcal{T}}(A) = \{x \in X \mid \exists E \in \mathcal{E} : E[x] \subseteq A\}.$$

For $x \in int_{\mathcal{T}}(A)$, consider the entourage $E_{x,A}$, which, by Lemma 5.2, belongs to $\mathcal{E}$. Of course, $E_{x,A}[x] = A$. Conversely, if there is some $E \in \mathcal{E}$ with $E[x] \subseteq A$, then $x \in int_{\mathcal{T}}(E[x]) \subseteq int_{\mathcal{T}}(A)$. ■

Proposition 5.5. *If $\mathcal{E}$ is a Weil nearness on a set X satisfying* (NW5)*, there exists precisely one symmetric topology $\mathcal{T}$ on X such that $\mathcal{E}$ is the set of all interior entourages of $(X, \mathcal{T})$.*

Proof. Take for $\mathcal{T}$ the topology $\mathcal{T}_{\mathcal{E}}$ induced by $\mathcal{E}$. We already observed that $\mathcal{T}_{\mathcal{E}}$ is a symmetric topology on X. By (NW5), $\mathcal{E}$ contains all interior entourages of $(X, \mathcal{T}_{\mathcal{E}})$. The reverse inclusion follows from (NW3): take $E \in \mathcal{E}$; then $int(E)$ belongs to $\mathcal{E}$ and, in particular, it is an entourage. Hence E is an interior entourage.

The uniqueness of $\mathcal{T}$ is a corollary of the previous proposition. ■

The preceding propositions show that symmetric topological spaces can be always identified as Weil nearness spaces satisfying axiom (NW5).

Proposition 5.6. *Suppose $f : (X, \mathcal{T}) \longrightarrow (X', \mathcal{T}')$ is a map between symmetric topological spaces and let $\mathcal{E}_{(X,\mathcal{T})}$ (respectively, $\mathcal{E}_{(X',\mathcal{T}')}$) denote the set of all interior entourages of $(X, \mathcal{T})$ (respectively, $(X', \mathcal{T}')$). The following conditions are equivalent:*

(i) *f is continuous;*

(ii) $E \in \mathcal{E}_{(X',\mathcal{T}')}$ *implies* $(f \times f)^{-1}(E) \in \mathcal{E}_{(X,\mathcal{T})}$.

Proof. (i)⇒(ii): Let $E \in \mathcal{E}_{(X',\mathcal{T}')}$ and let $F = (f \times f)^{-1}(E)$. We have

$$int_{\mathcal{T}}(F[x]) = int_{\mathcal{T}}(f^{-1}(E[f(x)])).$$

Besides

$$f^{-1}(int_{\mathcal{T}'}(E[f(x)])) \subseteq int_{\mathcal{T}}(f^{-1}(E[f(x)]))$$

because

$$f^{-1}(int_{\mathcal{T}'}(E[f(x)])) \subseteq f^{-1}(E[f(x)])$$

and, by hypothesis, $f^{-1}(int_{\mathcal{T}'}(E[f(x)])) \in \mathcal{T}$. But E is interior. In particular, for every $x \in X$, $f(x) \in int_{\mathcal{T}'}(E[f(x)])$, that is, $x \in f^{-1}(int_{\mathcal{T}'}(E[f(x)]))$. Therefore $x \in int_{\mathcal{T}}(F[x])$. Similarly, $x \in int_{\mathcal{T}}(F^{-1}[x])$. Hence $F \in \mathcal{E}_{(X,\mathcal{T})}$.

(ii)⇒(i): Suppose $V \in \mathcal{T}'$ and let $v \in V$. Then, by Lemma 5.2, $E^{X'}_{v,V} \in \mathcal{E}_{(X',\mathcal{T}')}$. Thus $(f \times f)^{-1}(E^{X'}_{v,V}) \in \mathcal{E}_{(X,\mathcal{T})}$, that is, $E^{X}_{f^{-1}(v),f^{-1}(V)} \in \mathcal{E}_{(X,\mathcal{T})}$. By Lemma 5.3, $f^{-1}(v) \subseteq int_{\mathcal{T}}(f^{-1}(V))$ for every $v \in V$. Consequently,

$$f^{-1}(V) = \bigcup_{v \in V} f^{-1}(v) \subseteq int_{\mathcal{T}}(f^{-1}(V)),$$

i.e., $f^{-1}(V) \in \mathcal{T}$. ■

It follows from Propositions 5.4, 5.5 and 5.6 that the category $\mathsf{WNear}_{(\mathsf{NW5})}$ of Weil nearness spaces satisfying (NW5) is isomorphic to the category of symmetric topological spaces. We have now an alternative way of equipping a set with the structure of a symmetric topological space: by prescribing the set of interior entourages. Moreover:

Proposition 5.7. *The category* $\mathsf{WNear}_{(\mathsf{NW5})}$ *is a bicoreflective subcategory of* WNear.

Proof. Given a Weil nearness space $(X, \mathcal{E})$, let $\mathcal{E}_T$ denote the set of all interior entourages of $(X, \mathcal{T}_{\mathcal{E}})$. We already know that $(X, \mathcal{E}_T)$ is a Weil nearness space satisfying (NW5). Furthermore, for any morphism $f : (X, \mathcal{E}) \longrightarrow (X', \mathcal{E}')$ in WNear, f is continuous from $(X, \mathcal{T}_{\mathcal{E}})$ to $(X', \mathcal{T}_{\mathcal{E}'})$ so, by Proposition 5.6, $f : (X, \mathcal{E}_T) \longrightarrow (X', \mathcal{E}'_T)$ is also in WNear. We get this way a functor

$$\begin{array}{ccc} T : \mathsf{WNear} & \longrightarrow & \mathsf{WNear}_{(\mathsf{N5})} \\ (X,\mathcal{E}) & \longmapsto & (X,\mathcal{E}_T) \\ \Big((X,\mathcal{E}) \xrightarrow{f} (X',\mathcal{E}')\Big) & \longmapsto & \Big((X,\mathcal{E}_T) \xrightarrow{f} (X',\mathcal{E}'_T)\Big). \end{array}$$

This is the coreflector functor. Since $\mathcal{E} \subseteq \mathcal{E}_T$, $id : (X,\mathcal{E}_T) \longrightarrow (X,\mathcal{E})$ is in WNear. This is the coreflection map for $(X,\mathcal{E})$. ∎

Uniform spaces

We observed in 4.4 that the category Unif is a full subcategory of WNear. Furthermore, the following holds:

Proposition 5.8. *The category* Unif *is bireflective in* WNear*.*

Proof. For any $(X,\mathcal{E}) \in \mathsf{WNear}$ let

$$\mathcal{E}_U = \{E \in \mathcal{E} \mid \exists (E_n)_{n\in\mathbb{N}} \text{ in } \mathcal{E} \text{ such that } E_1 = E \text{ and } E_{n+1}^2 \subseteq E_n \text{ for each } n \in \mathbb{N}\}.$$

Obviously, $(X,\mathcal{E}_U)$ is a uniform space and $1_X : (X,\mathcal{E}) \longrightarrow (X,\mathcal{E}_U)$ is in WNear. This is the bireflection map. In fact, for any $f : (X,\mathcal{E}) \longrightarrow (X',\mathcal{E}')$ in WNear with $(X',\mathcal{E}') \in \mathsf{Unif}$, $f : (X,\mathcal{E}_U) \longrightarrow (X',\mathcal{E}')$ is uniformly continuous: for any $E \in \mathcal{E}'$, as $(X',\mathcal{E}')$ is uniform, there is a family $(E_n)_{n\in\mathbb{N}}$ in $\mathcal{E}'$ with $E_n = E$ and $E_{n+1}^2 \subseteq E_n$ for every $n \in \mathbb{N}$. Take the family $\Big((f\times f)^{-1}(E_n)\Big)_{n\in\mathbb{N}}$ which is in $\mathcal{E}$. This shows that $(f\times f)^{-1}(E) \in \mathcal{E}_U$. ∎

Proximal spaces

Historically, the first axiomatization of proximal spaces was given by Efremovič in [19], in terms of the proximal (or infinitesimal) relation "A is near B" (usually denoted by $A\delta B$ [56]) for subsets A and B of any set:

Definitions 5.9. (Efremovič [19], [20]; cf. Naimpally and Warrack [56])

(1) Let X be a set and let δ be a binary relation on $\mathcal{P}(X)$. The pair (X,δ) is a *proximal space* provided that δ is an *infinitesimal relation* on X, i.e.:

(I1) $A\delta B$ implies $B\delta A$;

(I2) $A\delta(B\cup C)$ if and only if $A\delta B$ or $A\delta C$;

(I3) $A\delta B$ implies $A\neq\emptyset$ and $B\neq\emptyset$;

(I4) $A\not\delta B$ implies the existence of a subset C of X such that $A\not\delta C$ and $(X\setminus C)\not\delta B$;

(I5) $A\delta B$ whenever $A\cap B\neq\emptyset$.

(2) Let (X_1,δ_1) and (X_2,δ_2) be two proximal spaces. A function $f: X_1 \longrightarrow X_2$ is an *infinitesimal map* if, for every $A, B\subseteq X$, $A\delta_1 B$ implies $f(A)\delta_2 f(B)$.

(3) Proximal spaces and infinitesimal maps are the objects and morphisms of the category Prox.

It is well-known that the category Prox is isomorphic to the category TBUnif of totally bounded uniform spaces and uniformly continuous maps and that TBUnif is bireflective in Unif. Thus, since the considered categories are topological, we have by Proposition 5.8:

Proposition 5.10. Prox *is, up to isomorphism, a bireflective subcategory of* WNear.

∎

Let us present, in the sequel, another way of concluding Proposition 5.10 which yields as a corollary a characterization of proximal spaces in terms of Weil nearnesses. We shall use the following alternative axiomatization of proximal spaces (cf. Naimpally and Warrack [56] or Smirnov [70]):

Theorem 5.11. (Efremovič [20]) *The category* Prox *is isomorphic to the category whose objects are the pairs $(X,\ll)$, where X is a set and $\ll$ is a binary relation on $\mathcal{P}(X)$ satisfying*

(P1) $X\ll X$ *and* $\emptyset\ll\emptyset$,

(P2) $A\ll B$ *implies* $A\subseteq B$,

(P3) $A\subseteq B\ll C\subseteq D$ *implies* $A\ll D$, (P4) *$A \ll C$ and $B \ll C$ imply $A \cup B \ll C$,*

(P5) *$A \ll B$ and $A \ll C$ imply $A \ll B \cap C$,*

(P6) *if $A \ll B$ there exists a subset C of X such that $A \ll C \ll B$,*

(P7) *$A \ll B$ implies $X \setminus B \ll X \setminus A$,*

and whose morphisms are the functions $f : (X_1, \ll_1) \longrightarrow (X_2, \ll_2)$ for which $f^{-1}(A) \ll_1 f^{-1}(B)$ whenever $A \ll_2 B$. ■

The relations $\ll$ and the morphisms defined in 5.11 are usually called *proximities* and *proximal maps*, respectively.

A straightforward verification shows that, in case $(X, \mathcal{E}) \in \mathsf{FrWNear}$, $<_{\mathcal{E}}$ is a proximity on X.

We consider the converse problem of endowing a proximal space with a (functorial) framed Weil nearness structure.

Lemma 5.12. *Let $(X, \ll)$ be a proximal space.*

(a) *If $A \ll C \ll B$ then $(E_{A,C} \cap E_{C,B}) \circ (E_{A,C} \cap E_{C,B}) \subseteq E_{A,B}$.*

(b) *If $E = \bigcap_{i=1}^{n} E_{A_i,B_i}$, $E' = \bigcap_{i=1}^{n} E_{C_i,D_i}$ and, for every $i \in \{1, \ldots, n\}$, $A_i \ll C_i \ll D_i \ll B_i$, then, for every $x \in X$, $E'[x] \ll E[x]$.*

(c) *If $\bigcap_{i=1}^{n} E_{A_i,B_i} \subseteq E_{A,B}$ and, for every $i \in \{1, \ldots, n\}$, $A_i \ll B_i$, then $A \ll B$.*

Proof. (a) Let $(x, y), (y, z) \in E_{A,C} \cap E_{C,B}$ such that $(x, z) \notin X \setminus A \times X \setminus A$. In case $x \in A$, y is necessarily in C, which, in turn, implies that $z \in B$. Hence $(x, z) \in B \times B \subseteq E_{A,B}$.

The case $z \in A$ can be proved in a similar way.

(b) An easy computation shows that, for every $x \in X$, $E_{C,D}[x] \ll E_{A,B}[x]$ whenever $A \ll C \ll D \ll B$. Now a proof by induction on $n \geq 1$ is evident:

If $E = \bigcap_{i=1}^{n+1} E_{A_i,B_i}$ and $E' = \bigcap_{i=1}^{n+1} E_{C_i,D_i}$ with $A_i \ll C_i \ll D_i \ll B_i$ for every $i \in \{1, \ldots, n+1\}$, then, for every $x \in X$,

$$E'[x] = E_{C_1,D_1}[x] \cap \bigcap_{i=2}^{n+1} E_{C_i,D_i}[x].$$

By inductive hypothesis and by the case $n = 1$ already proved, we obtain

$$E'[x] \ll E_{A_1,B_1}[x] \cap \bigcap_{i=2}^{n+1} E_{A_i,B_i}[x] = E[x].$$

(c) For any $i \in \{1, \dots, n\}$ let C_i be such that $A_i \ll C_i \ll B_i$. An application of (a) yields

$$\bigcap_{i=1}^{n} (E_{A_i,C_i} \cap E_{C_i,B_i})^2 \subseteq \bigcap_{i=1}^{n} E_{A_i,B_i} \subseteq E_{A,B}.$$

Let

$$E = \bigcap_{i=1}^{n} (E_{A_i,C_i} \cap E_{C_i,B_i})$$

and define

$$X_1 = \{x \in X \mid E[x] \cap A = \emptyset\}$$

and

$$X_2 = \{x \in X \mid E[x] \cap A \neq \emptyset\}.$$

Note that $X_2 \neq \emptyset$ whenever $A \neq \emptyset$. Now we have $A \subseteq X \setminus \bigcup_{x \in X_1} E[x]$. For each $i \in \{1, \dots, n\}$ consider A'_i, C'_i, C''_i and B'_i such that

$$A_i \ll A'_i \ll C'_i \ll C_i \ll C''_i \ll B'_i \ll B_i.$$

From (b) we may conclude that, for every $x \in X$, $E'[x] \ll E[x]$, where E' denotes the entourage

$$\bigcap_{i=1}^{n} (E_{A'_i,C'_i} \cap E_{C''_i,B'_i}).$$

Then we have

$$A \subseteq X \setminus \bigcup_{x \in X_1} E[x] \subseteq X \setminus \bigcup_{x \in X_1} E'[x] \subseteq \bigcup_{x \in X_2} E'[x].$$

It is now easy to conclude that, due to the special form of E', there is a finite subset F_2 of X_2 such that

$$\bigcup_{x \in X_2} E'[x] = \bigcup_{x \in F_2} E'[x].$$

Indeed, since E' is of the form $\bigcap_{j=1}^{2n} E_{A''_j,B''_j}$ we have

$$\bigcup_{x \in X_2} E'[x] = \bigcup_{x \in X_2} \bigcap_{j=1}^{2n} (E_{A''_j,B''_j}[x]),$$

and it suffices now to form F_2 by choosing exactly one element from each non-empty set of the following 3^{2n} disjoint sets

$$X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap A_{2n-1}'' \cap A_{2n}''$$

$$X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap A_{2n-1}'' \cap (B_{2n}'' \setminus A_{2n}'')$$

$$X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap A_{2n-1}'' \cap (X \setminus B_{2n}'')$$

$$X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (B_{2n-1}'' \setminus A_{2n-1}'') \cap A_{2n}''$$

$$X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (B_{2n-1}'' \setminus A_{2n-1}'') \cap (B_{2n}'' \setminus A_{2n}'')$$

$$X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (B_{2n-1}'' \setminus A_{2n-1}'') \cap (X \setminus B_{2n}'')$$

$$X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (X \setminus B_{2n-1}'') \cap A_{2n}''$$

$$X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (X \setminus B_{2n-1}'') \cap (B_{2n}'' \setminus A_{2n}'')$$

$$X_2 \cap A_1'' \cap A_2'' \cap \ldots \cap (X \setminus B_{2n-1}'') \cap (X \setminus B_{2n}'')$$

$$\vdots$$

$$X_2 \cap (X \setminus B_1'') \cap (X \setminus B_2'') \cap \ldots \cap (X \setminus B_{2n-1}'') \cap (X \setminus B_{2n}''),$$

whose union is X_2.

Thus, by (b),

$$A \subseteq \bigcup_{x \in F_2} E'[x] \ll \bigcup_{x \in F_2} E[x].$$

Now, if $y \in E[x]$ for some $x \in F_2$, there is $a \in A$ with $(x, a) \in E$. Since E is symmetric,

$$(a, y) \in E^2 \subseteq \bigcap_{i=1}^{n} (E_{A_i, C_i} \cap E_{C_i, B_i})^2 \subseteq E_{A,B}$$

and, consequently, $y \in B$. Hence $\bigcup_{x \in F_2} E[x] \subseteq B$ and $A \ll B$. ∎

Proposition 5.13. *Suppose $(X, \ll)$ is a proximal space. Then*

$$\{E_{A,B} \mid A, B \subseteq X \text{ and } A \ll B\}$$

is a subbasis for a framed Weil uniformity $\mathcal{E}(\ll)$ on X. Furthermore, the proximity $<_{\mathcal{E}(\ll)}$ induced by $\mathcal{E}(\ll)$ coincides with $\ll$.

Proof. It is obvious that $\mathcal{E}(\ll)$ is a non-empty family of entourages of X. By Lemma 5.12 (a), $\mathcal{E}(\ll)$ is a Weil uniformity on X. The conclusion that $(X, \mathcal{E}(\ll))$ is framed follows immediately from assertion (c) of the same lemma:

$$\begin{aligned} x <_{\mathcal{E}(\ll)} A \quad &\Leftrightarrow \quad E_{x,A} \in \mathcal{E}(\ll) \\ &\Rightarrow \quad x \ll A \\ &\Rightarrow \quad \exists B \subseteq X : x \ll B \ll A \\ &\Rightarrow \quad x <_{\mathcal{E}(\ll)} B <_{\mathcal{E}(\ll)} A. \end{aligned}$$

The non-trivial part of the equivalence of the binary relations $<_{\mathcal{E}(\ll)}$ and $\ll$ is also an immediate corollary of Lemma 5.12 (c). ■

The fact that each $int(E_{A,B})$ belongs to $\mathcal{E}(\ll)$ whenever $A \ll B$ could be proved directly by observing that $E_{cl(A),int(B)} \subseteq int(E_{A,B})$ and by recalling the well-known result of proximities

$$A \ll B \Rightarrow cl(A) \ll int(B).$$

This is sufficient to conclude that $int(E) \in \mathcal{E}(\ll)$ whenever $E \in \mathcal{E}(\ll)$.

For any $(X, \mathcal{E}) \in$ FrWNear satisfying

$$\text{(NW6)} \quad \forall E \in \mathcal{E}\ \exists A_1, B_1, \ldots, A_n, B_n \subseteq X : \left(\bigcap_{i=1}^{n} E_{A_i,B_i} \subseteq E \text{ and } \bigcap_{i=1}^{n} E_{A_i,B_i} \in \mathcal{E} \right),$$

the Weil nearness $\mathcal{E}(<_{\mathcal{E}})$ induced by $<_{\mathcal{E}}$ coincides with $\mathcal{E}$. Thus, the proximal spaces may be identified as the framed Weil nearness spaces satisfying (NW6). The same happens for morphisms:

Proposition 5.14. *Let $(X_1, \ll_1)$ and $(X_2, \ll_2)$ be proximal spaces. A map $f : X_1 \longrightarrow X_2$ is a proximal map from $(X_1, \ll_1)$ to $(X_2, \ll_2)$ if and only if it is a Weil nearness map from $(X_1, \mathcal{E}(\ll_1))$ to $(X_2, \mathcal{E}(\ll_2))$.*

Proof. Suppose $E \in \mathcal{E}(\ll_2)$ and let $A_1, B_1, \ldots, A_n, B_n \subseteq X_2$ such that $\bigcap_{i=1}^n E^{X_2}_{A_i,B_i} \subseteq E$ and $A_i \ll_2 B_i$ for every $i \in \{1, \ldots, n\}$. Then, for each i, $f^{-1}(A_i) \ll_1 f^{-1}(B_i)$ and, therefore,

$$\bigcap_{i=1}^n E^{X_1}_{f^{-1}(A_i),f^{-1}(B_i)} \in \mathcal{E}(\ll_1).$$

To prove that $(f \times f)^{-1}(E) \in \mathcal{E}(\ll_1)$ it suffices now to check that it contains

$$\bigcap_{i=1}^n E^{X_1}_{f^{-1}(A_i),f^{-1}(B_i)},$$

which is straightforward since each $E^{X_1}_{f^{-1}(A_i),f^{-1}(B_i)}$ is equal to $(f \times f)^{-1}(E^{X_2}_{A_i,B_i})$.

Conversely, suppose that $A \ll_2 B$. Then $E^{X_2}_{A,B} \in \mathcal{E}(\ll_2)$ and, consequently,

$$E^{X_1}_{f^{-1}(A),f^{-1}(B)} = (f \times f)^{-1}(E^{X_2}_{A,B}) \in \mathcal{E}(\ll_1).$$

By Lemma 5.12, $f^{-1}(A) \ll_1 f^{-1}(B)$. ■

Corollary 5.15. *The categories* Prox *and* $\mathsf{FrWNear}_{(\mathsf{NW6})}$ *are isomorphic.* ■

Note that the category $\mathsf{FrWNear}_{(\mathsf{NW6})}$ is a bireflective subcategory of $\mathsf{FrWNear}$:

Given $(X, \mathcal{E})$ in $\mathsf{FrWNear}$ we already know that $(X, \mathcal{E}(<_\mathcal{E}))$ belongs to $\mathsf{FrWNear}_{(\mathsf{NW6})}$. Since $\mathcal{E}(<_\mathcal{E}) \subseteq \mathcal{E}$,

$$1_X : (X, \mathcal{E}) \longrightarrow (X, \mathcal{E}(<_\mathcal{E}))$$

is in $\mathsf{FrWNear}$. This is the bireflective map of $(X, \mathcal{E})$ in $\mathsf{FrWNear}_{(\mathsf{NW6})}$; indeed, if $f : (X, \mathcal{E}) \longrightarrow (X', \mathcal{E}')$ belongs to $\mathsf{FrWNear}$, with $(X', \mathcal{E}') \in \mathsf{FrWNear}_{(\mathsf{NW6})}$, $f : (X, \mathcal{E}(<_\mathcal{E})) \longrightarrow (X', \mathcal{E}')$ is also in $\mathsf{FrWNear}$: for any $E \in \mathcal{E}'$ we may write $\bigcap_{i=1}^n E^{X'}_{A_i,B_i} \subseteq E$ where each $E^{X'}_{A_i,B_i} \in \mathcal{E}'$. Therefore

$$E^X_{f^{-1}(A_i),f^{-1}(B_i)} = (f \times f)^{-1}(E^{X'}_{A_i,B_i})$$

belongs to $\mathcal{E}$ and, since

$$\bigcap_{i=1}^n E^X_{f^{-1}(A_i),f^{-1}(B_i)} \subseteq (f \times f)^{-1}(E),$$

$(f \times f)^{-1}(E) \in \mathcal{E}(<_\mathcal{E})$.

6. Proximal frames

The notion of spatial proximity in terms of the relation $\ll$ is immediately manageable from a lattice-theoretical point of view:

Definitions 6.1. (Frith [29])

(1) Let L be a frame and let $\ll$ be a binary relation on L. The pair $(L, \ll)$ is a *proximal frame* provided that:

(P1) $1 \ll 1$ and $0 \ll 0$;

(P2) $x \ll y$ implies $x \prec y$;

(P3) $x \leq y \ll z \leq w$ implies $x \ll w$;

(P4) $x_1 \ll y$ and $x_2 \ll y$ imply $x_1 \vee x_2 \ll y$;

(P5) $x \ll y_1$ and $x \ll y_2$ imply $x \ll y_1 \wedge y_2$;

(P6) if $x \ll y$ there is a $z \in L$ such that $x \ll z \ll y$;

(P7) $x \ll y$ implies $y^* \ll x^*$;

(P8) $x = \bigvee\{y \in L \mid y \ll x\}$ for every $x \in L$.

(2) Let $(L_1, \ll_1)$ and $(L_2, \ll_2)$ be proximal frames. A *proximal frame homomorphism* $f : (L_1, \ll_1) \longrightarrow (L_2, \ll_2)$ is a frame map $f : L_1 \longrightarrow L_2$ satisfying

$$\forall\, x, y \in L_1 \Big(x \ll_1 y \Rightarrow f(x) \ll_2 f(y)\Big).$$

(3) We denote by PFrm the category of proximal frames and proximal frame homomorphisms.

It will be instructive to see how an analogue of the isomorphism between Prox and $\mathsf{FrWNear}_{(\mathsf{NW6})}$ may be established for frames. With this we obtain a new characterization of proximal frames.

Proposition 6.2. *If $(L, \mathcal{E})$ is a Weil uniform frame then $(L, \overset{\mathcal{E}}{\lhd})$ is a proximal frame.*

Proof. Condition (P7) is a consequence of the implication

$$E \circ (x \oplus x) \subseteq y \oplus y \Rightarrow E^{-1} \circ (y^* \oplus y^*) \subseteq x^* \oplus x^*.$$

The other conditions were proved in Proposition 4.8 of Chapter I. ∎

Remark 6.3. When $x \vee y = 1$, the C-ideal $(x \oplus x) \vee (y \oplus y)$ is a Weil entourage of L. The converse is also true since it is a corollary of the following property of the C-ideal generated by a down-set of $L \times L$, which can be proved in a similar way to Lemma I.4.2, by considering

$$\mathbf{E} = \{E \in \mathcal{D}(L \times L) \mid A \subseteq E \subseteq k(A) \text{ and } x \vee y \leq c \text{ for every } (x, y) \in E \setminus \mathbb{O}\} :$$

let $c \in L$; if $A \in \mathcal{D}(L \times L)$ is such that $x \vee y \leq c$ whenever $(x, y) \in A \setminus \mathbb{O}$, then also $x \vee y \leq c$ for every $(x, y) \in k(A) \setminus \mathbb{O}$.

Thus, in any proximal frame $(L, \ll)$, since $\ll$ is stronger than $\prec$,

$$E_{x,y} := (x^* \oplus x^*) \vee (y \oplus y)$$

is a Weil entourage whenever $x \ll y$. These entourages are very important as we shall see. For instance, they characterize, for any Weil nearness $\mathcal{E}$ on L, the relation $\overset{\mathcal{E}}{\lhd}$:

Proposition 6.4. *Let $\mathcal{E}$ be a Weil nearness on L. Then $x \overset{\mathcal{E}}{\lhd} y$ if and only if $E_{x,y} \in \mathcal{E}$.*

Proof. Assume that there is $E \in \mathcal{E}$ satisfying $E \circ (x \oplus x) \subseteq y \oplus y$. It follows that, whenever $(a, a) \in E$, $a \leq x^* \vee y$ and, consequently, that $x^* \vee y = 1$. Then $E \cap E^{-1} \subseteq E_{x,y}$. Indeed, if $(a, b) \in E \cap E^{-1}$ and $b \leq x^*$ we have:

- in case $a \leq x^*$, $(a, b) \in x^* \oplus x^* \subseteq E_{x,y}$;
- otherwise, if $a \wedge x \neq 0$ then $(b, a) \in E \circ (x \oplus x)$ thus $b \leq y$. Hence $b \leq x^* \wedge y$. But $(1, x^* \wedge y) = (x^* \vee y, x^* \wedge y) \in E_{x,y}$ so $(a, b) \in E_{x,y}$.

On the other hand, if $b \wedge x \neq 0$ then $a \leq y$. In case $a \leq x^*$ we have $(a, b) \leq (x^* \wedge y, 1) \in E_{x,y}$. If $a \not\leq x^*$ then $b \leq y$ and, again, $(a, b) \in E_{x,y}$.

The reverse implication follows from the inclusion $E_{x,y} \circ (x \oplus x) \subseteq y \oplus y$. ■

From now on, for any down-set E of $L \times L$ and any $a \in L$, we denote by $E[a]$ the element

$$\bigvee \{b \in L \mid (a, b) \in E\}.$$

In particular, we have:

Proposition 6.5. *Let $x, y \in L$ with $x \prec y$. Then*

$$E_{x,y}[a] = \begin{cases} 1 & \text{if } a \in [0, x^* \wedge y] \\ x^* & \text{if } a \in [0, x^*] \cap (L \setminus [0, y]) \\ y & \text{if } a \in (L \setminus [0, x^*]) \cap [0, y] \\ x^* \wedge y & \text{if } a \in L \setminus ([0, x^*] \cup [0, y]). \end{cases}$$

Proof. First of all, let us point out that, since $E_{x,y} \circ (x \oplus x) \subseteq y \oplus y$,

$$\text{if } (a, b) \in E_{x,y} \text{ and } b \wedge x \neq 0 \text{ then } a \leq y. \tag{6.5.1}$$

Now, assume that $(a, b) \in E_{x,y}$ with $a \leq x^* \wedge y$. The pair $(x^* \wedge y, 1)$ is in $E_{x,y}$ so $(a, 1) \in E_{x,y}$ and, therefore, $E_{x,y}[a] = 1$.

The case $a \in [0, x^*] \cap (L \setminus [0, y])$ is also clear: by 6.5.1, necessarily $b \leq x^*$; since $(a, x^*) \in E_{x,y}$, $E_{x,y}[a] = x^*$.

If $a \in (L \setminus [0, x^*]) \cap [0, y]$ we have $(a, y) \in E_{x,y}$. Furthermore, for any $(a, b) \in E_{x,y}$, $(b, a) \in E_{x,y}$ and, since $a \wedge x \neq 0$, $b \leq y$. Hence $E_{x,y}[a] = y$ whenever $a \not\leq x^*$ and $a \leq y$.

Finally, in case $a \wedge x \neq 0$ and $a \not\leq y$ we have, for any $(a, b) \in E_{x,y}$, by 6.5.1, $b \leq y$ and $b \wedge x = 0$, i.e., $b \leq x^* \wedge y$. Since $(a, x^* \wedge y) \leq (1, x^* \wedge y) \in E_{x,y}$ we may conclude that, in this case, $E_{x,y}[a] = x^* \wedge y$. ■

The following obvious property will also be useful later on.

Lemma 6.6. *Let $E_1, E_2, \ldots, E_n \in \mathcal{D}(L \times L)$. Then*

$$\Big(\bigcap_{i=1}^{n} E_i\Big)[x] = \bigwedge_{i=1}^{n} (E_i[x]).$$ ■

The proof of next lemma is modelled after the proof of Lemma 5.12 except that, where the latter uses points, we think out means of avoiding them.

Lemma 6.7. *Let $(L, \ll)$ be a proximal frame.*

(a) *If $x \ll z \ll y$ then $(E_{x,z} \cap E_{z,y}) \circ (E_{x,z} \cap E_{z,y}) \subseteq E_{x,y}$.*

(b) *If $E = \bigcap_{i=1}^{n} E_{x_i,y_i}$, $E' = \bigcap_{i=1}^{n} E_{z_i,w_i}$ and, for every $i \in \{1, \ldots, n\}$, $x_i \ll z_i \ll w_i \ll y_i$, then, for every $a \in L$, $E'[a] \ll E[a]$.*

(c) *If $\bigcap_{i=1}^{n} E_{x_i,y_i} \subseteq E_{x,y}$ and, for every $i \in \{1, \ldots, n\}$, $x_i \ll y_i$, then $x \ll y$.*

Proof. (a) Since $(L \oplus L, \cap, \bigvee)$ is a frame, we have

$$\begin{aligned} E_{x,z} \cap E_{z,y} &= (x^* \wedge z^* \oplus x^* \wedge z^*) \vee (x^* \wedge y \oplus x^* \wedge y) \vee (z \wedge y \oplus z \wedge y) \\ &= ((x \vee z)^* \oplus (x \vee z)^*) \vee (x^* \wedge y \oplus x^* \wedge y) \vee (z \wedge y \oplus z \wedge y). \end{aligned}$$

On the other hand, an easy computation shows that, for any (a, b) and (b, c) in

$$((x \vee z)^* \oplus (x \vee z)^*) \cup (x^* \wedge y \oplus x^* \wedge y) \cup (z \wedge y \oplus z \wedge y),$$

with $a, b, c \neq 0$, the pair (a, c) belongs to $E_{x,y}$. This suffices to conclude that

$$(E_{x,z} \cap E_{z,y}) \circ (E_{x,z} \cap E_{z,y}) \subseteq E_{x,y}.$$

(b) Let us prove this assertion by induction over $n \geq 1$. When $x \ll z \ll w \ll y$ we have

$$E_{x,y}[a] = \begin{cases} 1 & \text{if } a \in [0, x^* \wedge y] \\ x^* & \text{if } a \in [0, x^*] \cap (L \setminus [0, y]) \\ y & \text{if } a \in (L \setminus [0, x^*]) \cap [0, y] \\ x^* \wedge y & \text{if } a \in L \setminus ([0, x^*] \cup [0, y]). \end{cases}$$

and

$$E_{z,w}[a] = \begin{cases} 1 & \text{if } a \in [0, z^* \wedge w] \\ z^* & \text{if } a \in [0, z^*] \cap (L \setminus [0, w]) \\ w & \text{if } a \in (L \setminus [0, z^*]) \cap [0, w] \\ z^* \wedge w & \text{if } a \in L \setminus ([0, z^*] \cup [0, w])). \end{cases}$$

The properties of $\ll$ ensure us that, for every $a \in L$, $E_{z,w}[a] \ll E_{x,y}[a]$ and the case $n = 1$ is proved.

We now assume that the formula is true for a positive integer n and prove that it is also true for $n+1$. If $E = \bigcap_{i=1}^{n+1} E_{x_i,y_i}$ and $E' = \bigcap_{i=1}^{n+1} E_{z_i,w_i}$ with $x_i \ll z_i \ll w_i \ll y_i$ for every $i \in \{1, \ldots, n+1\}$, we have by Lemma 6.6 that, for every $a \in L$,

$$E'[a] = \Big(\big(\bigcap_{i=1}^{n} E_{z_i,w_i}\big)[a]\Big) \wedge E_{z_{n+1},w_{n+1}}[a].$$

Therefore

$$E'[a] \ll \Big(\big(\bigcap_{i=1}^{n} E_{x_i,y_i}\big)[a]\Big) \wedge E_{x_{n+1},y_{n+1}}[a] = E[a].$$

(c) For any $i \in \{1, \ldots, n\}$ let z_i be such that $x_i \ll z_i \ll y_i$. By (a) we have

$$\bigcap_{i=1}^{n} (E_{x_i,z_i} \cap E_{z_i,y_i})^2 \subseteq \bigcap_{i=1}^{n} E_{x_i,y_i} \subseteq E_{x,y}.$$

Let $E = \bigcap_{i=1}^{n} (E_{x_i,z_i} \cap E_{z_i,y_i})$ and define

$$L_1 = \{a \in L \setminus \{0\} \mid E[a] \wedge x = 0\}$$

and

$$L_2 = \{a \in L \setminus \{0\} \mid E[a] \wedge x \neq 0\}.$$

Since E is a Weil entourage, L_2 is empty only when $x = 0$. Now we have $x \leq \bigwedge_{a \in L_1} (E[a]^*)$. For each $i \in \{1, \ldots, n\}$ consider $x_i', z_i', z_i'', y_i' \in L$ such that

$$x_i \ll x_i' \ll z_i' \ll z_i \ll z_i'' \ll y_i' \ll y_i.$$

By (b) we may conclude that, for every $a \in L$, $E'[a] \ll E[a]$, where E' denotes the Weil entourage $\bigcap_{i=1}^{n} (E_{x_i',z_i'} \cap E_{z_i'',y_i'})$. Then we have

$$x \leq \bigwedge_{a \in L_1} (E[a]^*) \leq \bigwedge_{a \in L_1} (E'[a]^*) = \Big(\bigvee_{a \in L_1} E'[a]\Big)^*.$$

Since E' is a Weil entourage we have

$$\Big(\bigvee_{a \in L_1} E'[a]\Big) \vee \Big(\bigvee_{a \in L_2} E'[a]\Big) = 1,$$

which implies

$$\Big(\bigvee_{a\in L_1} E'[a]\Big)^* \leq \bigvee_{a\in L_2} E'[a].$$

Therefore

$$x \leq \bigvee_{a\in L_2}\Big(\Big(\bigcap_{i=1}^{n}(E_{x'_i,z'_i}\cap E_{z''_i,y'_i})\Big)[a]\Big).$$

Let us write the latter element in the form $\bigvee_{a\in L_2}\Big((\bigcap_{i=1}^{2n} E_{u_j,v_j})[a]\Big)$ where, for every $j\in\{1,\ldots,2n\}$, $u_j \ll v_j$. Then,

$$x \leq \bigvee_{a\in L_2}\bigwedge_{j=1}^{2n}(E_{u_j,v_j}[a]).$$

Choose exactly one element of each non-empty set of the following 4^{2n} disjoint sets (whose union is L_2)

$$L_2\cap[0,u_1^*\wedge v_1]\cap[0,u_2^*\wedge v_2]\cap\ldots\cap[0,u_{2n-1}^*\wedge v_{2n-1}]\cap[0,u_{2n}^*\wedge v_{2n}]$$

$$L_2\cap[0,u_1^*\wedge v_1]\cap[0,u_2^*\wedge v_2]\cap\ldots\cap[0,u_{2n-1}^*\wedge v_{2n-1}]\cap[0,u_{2n}^*]\cap(L\setminus[0,v_{2n}])$$

$$L_2\cap[0,u_1^*\wedge v_1]\cap[0,u_2^*\wedge v_2]\cap\ldots\cap[0,u_{2n-1}^*\wedge v_{2n-1}]\cap(L\setminus[0,u_{2n}^*])\cap[0,v_{2n}]$$

$$L_2\cap[0,u_1^*\wedge v_1]\cap[0,u_2^*\wedge v_2]\cap\ldots\cap[0,u_{2n-1}^*\wedge v_{2n-1}]\cap(L\setminus([0,u_{2n}^*]\cup[0,v_{2n}]))$$

$$L_2\cap[0,u_1^*\wedge v_1]\cap[0,u_2^*\wedge v_2]\cap\ldots\cap[0,u_{2n-1}^*]\cap(L\setminus[0,v_{2n-1}])\cap[0,u_{2n}^*\wedge v_{2n}]$$

$$L_2\cap[0,u_1^*\wedge v_1]\cap[0,u_2^*\wedge v_2]\cap\ldots\cap[0,u_{2n-1}^*]\cap(L\setminus[0,v_{2n-1}])\cap[0,u_{2n}^*]\cap(L\setminus[0,v_{2n}])$$

$$L_2\cap[0,u_1^*\wedge v_1]\cap[0,u_2^*\wedge v_2]\cap\ldots\cap[0,u_{2n-1}^*]\cap(L\setminus[0,v_{2n-1}])\cap(L\setminus[0,u_{2n}^*])\cap[0,v_{2n}]$$

$$L_2\cap[0,u_1^*\wedge v_1]\cap[0,u_2^*\wedge v_2]\cap\ldots\cap[0,u_{2n-1}^*]\cap(L\setminus[0,v_{2n-1}])\cap(L\setminus([0,u_{2n}^*]\cup[0,v_{2n}]))$$

$$L_2 \cap [0, u_1^* \wedge v_1] \cap [0, u_2^* \wedge v_2] \cap \ldots \cap (L \setminus [0, u_{2n-1}^*]) \cap [0, v_{2n-1}] \cap [0, u_{2n}^* \wedge v_{2n}]$$

$$\vdots$$

$$L_2 \cap (L \setminus ([0, u_1^*] \cup [0, v_1])) \cap (L \setminus ([0, u_2^*] \cup [0, v_2])) \cap \ldots \cap (L \setminus ([0, u_{2n}^*] \cup [0, v_{2n}])),$$

and denote by F_2 the set constituted by them. Clearly

$$\bigvee_{a \in L_2} \bigwedge_{j=1}^{2n} (E_{u_j, v_j}[a]) = \bigvee_{a \in F_2} \bigwedge_{j=1}^{2n} (E_{u_j, v_j}[a]).$$

It is now already possible to apply (b) and conclude that

$$x \leq \bigvee_{a \in F_2} E'[a] \ll \bigvee_{a \in F_2} E[a].$$

Finally, we only have to verify that $\bigvee_{a \in F_2} E[a] \leq y$:

If $(a, b) \in E$ with $a \in F_2$, then there is $c \in L$ such that $(a, c) \in E$ and $c \wedge x \neq 0$. But $a \neq 0$ and E is symmetric, thus $(c, b) \in E^2 \subseteq E_{x,y}$. Hence $b \leq y$, because $c \wedge x \neq 0$, which shows that $\bigvee_{a \in F_2} E[a] \leq y$ holds. ■

This lemma implies immediately the following proposition.

Proposition 6.8. *Let $(L, \ll)$ be a proximal frame. Then*

$$\{E_{x,y} \mid x, y \in L, x \ll y\}$$

is a subbasis for a Weil uniformity $\mathcal{E}(\ll)$ on L. Moreover, the proximity $\overset{\mathcal{E}(\ll)}{\lhd}$ induced by $\mathcal{E}(\ll)$ coincides with $\ll$. ■

The following proposition is also obvious.

Proposition 6.9. *If $(L, \mathcal{E})$ is a Weil uniform frame satisfying*

$$\text{(UW5)}\ \forall E \in \mathcal{E}\ \ \exists x_1, y_1, \ldots, x_n, y_n \in L : \left(\bigcap_{i=1}^{n} E_{x_i, y_i} \subseteq E \ \text{ and } \bigcap_{i=1}^{n} E_{x_i, y_i} \in \mathcal{E} \right),$$

the Weil uniformity $\mathcal{E}(\overset{\mathcal{E}}{\lhd})$ induced by $\overset{\mathcal{E}}{\lhd}$ coincides with $\mathcal{E}$. ■

Let us denote by $\mathsf{WUFrm}_{(\mathsf{UW5})}$ the full subcategory of WUFrm of all Weil uniform frames satisfying (UW5). This is a bicoreflective subcategory of WUFrm; if $(L,\mathcal{E})$ is a Weil uniform frame and $\mathcal{E}_P$ is the set of all entourages E of L for which there exist $x_1,y_1,\ldots,x_n,y_n \in L$ such that $\bigcap_{i=1}^n E_{x_i,y_i} \subseteq E$ and $\bigcap_{i=1}^n E_{x_i,y_i} \in \mathcal{E}$, then $1_L : (L,\mathcal{E}_P) \longrightarrow (L,\mathcal{E})$ is the bicoreflection of $(L,\mathcal{E})$ with respect to $\mathsf{WUFrm}_{(\mathsf{UW5})}$.

The preceding propositions allows us to present a new characterization of proximal frames:

Theorem 6.10. *The categories* PFrm *and* $\mathsf{WUFrm}_{(\mathsf{UW5})}$ *are isomorphic.*

In order to complete the proof of this theorem we start by recalling a simple and well-known lemma.

Lemma 6.11. *Suppose that $f : L_1 \longrightarrow L_2$ is a frame homomorphism and let $x,y \in L_1$ with $x \prec y$. Then $f(y)^* \leq f(x^*)$.*

Proof. We have

$$\begin{aligned} x \prec y &\Leftrightarrow x^* \vee y = 1 \\ &\Rightarrow f(x^*) \vee f(y) = 1 \\ &\Rightarrow f(x^*) \vee f(y)^{**} = 1. \end{aligned}$$

Hence $f(y)^* \prec f(x^*)$ and, therefore, $f(y)^* \leq f(x^*)$. ■

Proof of the theorem. According to Propositions 6.2, 6.8 and 6.9 it remains to show that a frame map $f : L_1 \longrightarrow L_2$ between two proximal frames $(L_1,\ll_1)$ and $(L_2,\ll_2)$ is a proximal frame homomorphism if and only if it is a Weil uniform homomorphism from $(L_1,\mathcal{E}(\ll_1))$ to $(L_2,\mathcal{E}(\ll_2))$. So, let f be a proximal frame homomorphism. For any $E \in \mathcal{E}(\ll_1)$ we may write $\bigcap_{i=1}^n E_{x_i,y_i} \subseteq E$, where $x_i \ll_1 y_i$ for every $i \in \{1,\ldots,n\}$. Then

$$(f\oplus f)(E) \supseteq \bigcap_{i=1}^n (f\oplus f)(E_{x_i,y_i}) = \bigcap_{i=1}^n \Big((f(x_i^*)\oplus f(x_i^*)) \vee (f(y_i)\oplus f(y_i)) \Big).$$

Now consider, for each $i \in \{1,\ldots,n\}$, $z_i, w_i \in L$ such that $x_i \ll_1 z_i \ll_1 w_i \ll_1 y_i$. By the lemma, we have

$$(f(x_i^*)\oplus f(x_i^*)) \vee (f(y_i)\oplus f(y_i)) \supseteq (f(z_i)^*\oplus f(z_i)^*) \vee (f(w_i)\oplus f(w_i)) = E_{f(z_i),f(w_i)}.$$

So $(f \oplus f)(E) \supseteq \bigcap_{i=1}^{n} E_{f(z_i),f(w_i)}$. Since, by hypothesis, $f(z_i) \ll_2 f(w_i)$, then $(f \oplus f)(E) \in \mathcal{E}(\ll_2)$.

Conversely, if $x \ll_1 y$ then $(f \oplus f)(E_{x,y}) \in \mathcal{E}(\ll_2)$. As $(f \oplus f)(E_{x,y}) \subseteq E_{f(x),f(y)}$, the entourage $E_{f(x),f(y)}$ also belongs to $\mathcal{E}(\ll_2)$ and we may write $\bigcap_{i=1}^{n} E_{x_i,y_i} \subseteq E_{f(x),f(y)}$ for $x_1, y_1, \ldots, x_n, y_n \in L_2$ with $x_i \ll_2 y_i$ for every $i \in \{1, \ldots, n\}$. It suffices now to apply Lemma 6.7 (c) to conclude that $f(x) \ll_2 f(y)$. ■

In [29], Frith proved that the two concepts of proximity and totally bounded uniformity are still equivalent for frames. Since a uniform frame is totally bounded when it has a basis of finite covers, this leads us to the following definition:

Definition 6.12. A Weil entourage E is *finite* provided that there exist elements $x_1, y_1, \ldots, x_n, y_n$ in L such that $x_i \prec y_i$, for every $i \in \{1, \ldots, n\}$, and $\bigcap_{i=1}^{n} E_{x_i,y_i} = E$.

This is the "right" notion of finite Weil entourage if one wants to have Theorem 6.10 rephrased as follows:

> "*The category* PFrm *is isomorphic to the full subcategory of* WUFrm *of Weil uniform frames with a basis of finite entourages*".

By Remark 6.3, the conditions $x_i \prec y_i$ in Definition 6.12 are redundant since each E_{x_i,y_i} by containing E is also a Weil entourage.

For purposes of reference we shall denote by $FWEnt(L)$ the filter of $(WEnt(L), \subseteq)$ generated by all finite Weil entourages.

When L is normal, this notion of finiteness may be stated in a more natural way:

Proposition 6.13. *Assume that the frame L is normal. Then $FWEnt(L)$ is precisely the filter of $WEnt(L)$ generated by*

$$\Big\{\bigvee_{i=1}^{n}(x_i \oplus x_i) \mid n \in \mathbb{N}, x_1, \ldots, x_n \in L, \bigvee_{i=1}^{n} x_i = 1\Big\}.$$

Proof. Consider $E \in FWEnt(L)$. Then there are $x_1, y_1, \ldots, x_n, y_n \in L$ such that $x_i \prec y_i$ for every $i \in \{1, \ldots, n\}$ and $\bigcap_{i=1}^{n} E_{x_i,y_i} \subseteq E$. Since each E_{x_i,y_i} belongs to the set

$$\Big\{\bigvee_{i=1}^{n}(x_i \oplus x_i) \mid n \in \mathbb{N}, x_1, \ldots, x_n \in L, \bigvee_{i=1}^{n} x_i = 1\Big\},$$

E belongs to the filter generated by it.

Conversely, consider $\bigvee_{i=1}^{n}(x_i \oplus x_i)$ with $\bigvee_{i=1}^{n} x_i = 1$. From the normality of L we may ensure the existence of $y_1, \ldots, y_n \in L$ such that $\bigvee_{i=1}^{n} y_i = 1$ and $y_i \prec x_i$ for any $i \in \{1, \ldots, n\}$. Since $(L \oplus L, \cap, \bigvee)$ is a frame, one can easily prove, by induction on $n \geq 1$, that

$$\bigcap_{i=1}^{n} E_{y_i,x_i} = \bigvee_{z_1 \in \{y_1^*, x_1\}} \cdots \bigvee_{z_n \in \{y_n^*, x_n\}} (z_1 \wedge \ldots \wedge z_n \oplus z_1 \wedge \ldots \wedge z_n).$$

But $(y_1^* \wedge \ldots \wedge y_n^*) \oplus (y_1^* \wedge \ldots \wedge y_n^*) = \mathbb{O}$. Hence $\bigcap_{i=1}^{n} E_{y_i,x_i} \subseteq \bigvee_{i=1}^{n}(x_i \oplus x_i)$ and therefore $\bigvee_{i=1}^{n}(x_i \oplus x_i) \in FWEnt(L)$. ■

Now, as for nearnesses [12], the following basic results are also valid for this notion of finiteness, as one would expect:

Proposition 6.14. *For a compact regular frame L, the filters $FWEnt(L)$ and $WEnt(L)$ coincide and form the unique Weil nearness on L. Furthermore, this nearness is a uniformity.*

Proof. We already observed in Remark 3.2 that $WEnt(L)$ is a Weil nearness on L whenever L is regular.

By compactness, every Weil entourage E contains $\bigvee_{i=1}^{n}(x_i \oplus x_i)$ for some finite cover $\{x_1, \ldots, x_n\}$ of L. Since any compact regular frame is normal, we may conclude from Proposition 6.13 that $WEnt(L) = FWEnt(L)$. Now, to prove the uniqueness, it suffices to show that every Weil nearness $\mathcal{E}$ on L contains all entourages of the type $\bigvee_{i=1}^{n}(x_i \oplus x_i)$ for $\bigvee_{i=1}^{n} x_i = 1$. So, let $\mathcal{E}$ be a Weil nearness on L and consider a finite cover $\{x_1, \ldots, x_n\}$ of L. By compactness, for every $i \in \{1, \ldots, n\}$, there exist $y_1^i, \ldots, y_{j_i}^i \in L$ such that

$$y_k^i \overset{\mathcal{E}}{\triangleleft} x_i \text{ for } k \in \{1, \ldots, j_i\}$$

and

$$\bigvee_{i=1}^{n} \bigvee_{k=1}^{j_i} y_k^i = 1.$$

Let $E_k^i \in \mathcal{E}$ be such that $E_k^i \circ (y_k^i \oplus y_k^i) \subseteq x_i \oplus x_i$ and consider a symmetric $E \in \mathcal{E}$ such that $E \subseteq \bigcap_{i=1}^{n} \bigcap_{k=1}^{j_i} E_k^i$. An easy computation shows that $E \subseteq \bigvee_{i=1}^{n}(x_i \oplus x_i)$: let

$(a,b) \in E$ with $a, b \neq 0$. Since $b \neq 0$, there are $i \in \{1,\ldots,n\}$ and $k \in \{1,\ldots,j_i\}$ such that $b \wedge y^i_k \neq 0$. Then, $(a, b \wedge y^i_k) \in E$ and $(b \wedge y^i_k, y^i_k) \in y^i_k \oplus y^i_k$ imply that $a \leq x_i$. By symmetry, $b \leq x_i$ also. In conclusion, $\bigvee_{i=1}^n (x_i \oplus x_i) \in \mathcal{E}$, as we required.

Finally, let us prove that $FWEnt(L)$ is a uniformity. In the proof of Proposition 6.13 above we showed that, whenever $\bigvee_{i=1}^n x_i = 1$, there exist $y_1, \ldots, y_n$ such that $\bigvee_{i=1}^n y_i = 1$, $y_i \prec x_i$ for every $i \in \{1,\ldots,n\}$ and $\bigcap_{i=1}^n E_{y_i,x_i} \subseteq \bigvee_{i=1}^n (x_i \oplus x_i)$. This inclusion may be improved in order to imply that $FWEnt(L)$ is a uniformity; in fact,

$$\Big(\bigcap_{i=1}^n E_{y_i,x_i}\Big)^2 \subseteq \bigvee_{i=1}^n (x_i \oplus x_i)$$

as we now prove.

Using Lemma I.4.2, we may write $\left(\bigcap_{i=1}^n E_{y_i,x_i}\right)^2$ in the form

$$\bigcap_{z_1 \in \{y_1^*, x_1\}} \cdots \bigcap_{z_n \in \{y_n^*, x_n\}} (z_1 \wedge \ldots \wedge z_n \oplus z_1 \wedge \ldots \wedge z_n) \circ$$
$$\circ \bigcap_{z_1 \in \{y_1^*, x_1\}} \cdots \bigcap_{z_n \in \{y_n^*, x_n\}} (z_1 \wedge \ldots \wedge z_n \oplus z_1 \wedge \ldots \wedge z_n).$$

Now consider $(a,b) \in (z_1 \wedge \ldots \wedge z_n) \oplus (z_1 \wedge \ldots \wedge z_n)$ and $(b,c) \in (z_1' \wedge \ldots \wedge z_n') \oplus (z_1' \wedge \ldots \wedge z_n')$, for some $z_1, z_1' \in \{y_1^*, x_1\} \ldots, z_n, z_n' \in \{y_n^*, x_n\}$ with $a, b, c \neq 0$. Since $y_1^* \wedge \ldots \wedge y_n^* = 0$, there must exist $j \in \{1,\ldots,n\}$ such that $z_j = z_j' = x_j$, which implies $(a,c) \in x_j \oplus x_j$. ∎

Proposition 6.15. *$FWEnt(L)$ is a Weil uniformity in case L is regular and normal.*

Proof. It is contained in the proof of the previous result. ∎

Remark 6.16. We have reserved for the Weil entourages defined in 6.12 the adjective *finite*, guided by the theorem that states an isomorphism between the categories PFrm and $\mathsf{WUFrm}_{(\mathsf{UW5})}$.

Note that we have an equivalence in Proposition 6.15 if we replace the filter of $WEnt(L)$ generated by

$$\{\bigvee_{i=1}^n (x_i \oplus x_i) \mid n \in \mathbb{N}, x_1, \ldots, x_n \in L, \bigvee_{i=1}^n x_i = 1\}$$

for $FWEnt(L)$.

Recall the axiomatization of proximal spaces in terms of the infinitesimal relation δ (Definition 5.9). We conclude this chapter with a brief discussion of the analogous problem for frames.

Proposition 6.17. *If $(L, \ll)$ is a proximal frame, the binary relation on L given by*

$$x\delta y \equiv x \not\ll y^*$$

satisfies the following properties:

(I1) *$x\delta y$ implies $y\delta x$;*

(I2) *$x\delta(y \vee z)$ if and only if $x\delta y$ or $x\delta z$;*

(I3) *$x\delta y$ implies $x \neq 0$ and $y \neq 0$;*

(I4) *if $x\not\delta y$ then there is $z \in L$ such that $x\not\delta z$ and $y\not\delta z^*$;*

(I5) *$x^* \vee y^* \neq 1$ implies $x\delta y$;*

(I6) *for every $x \in L$, $x = \bigvee\{y \in L \mid y \leq x \text{ and } y\not\delta x^*\}$.*

Proof. (I1) We have

$$\begin{aligned} x\not\delta y &\Leftrightarrow x \ll y^* \\ &\Rightarrow y^{**} \ll x^* \quad \text{by property (P7)} \\ &\Rightarrow y \ll x^* \quad \text{by property (P3)} \\ &\Leftrightarrow y\not\delta x. \end{aligned}$$

(I2) If $x\delta(y \vee z)$ then $x \not\ll (y \vee z)^* = y^* \wedge z^*$. This implies, by (P5), that $x \not\ll y^*$ or $x \not\ll z^*$. Conversely, if $x\delta y$, $x \not\ll (y \vee z)^*$ by (P3). Thus $x\delta(y \wedge z)$. In case $x\delta z$, we have also $x\delta(y \vee z)$, similarly.

(I3) $x \not\ll y^*$ implies, by (P1) and (P3), $x \neq 0$ and $y^* \neq 1$, that is, $x \neq 0$ and $y \neq 0$.

(I4) If $x \ll y^*$ there exists, by property (P5), $w \in L$ such that $x \ll w \ll y^*$. Let $z := w^*$; then $x \ll w \leq w^{**} = z^*$, whence $x\not\delta z$. Moreover $y \leq y^{**} \ll w^* = z \leq z^{**}$ and, consequently, $y\not\delta z^*$.

(I5) Since

$$x^* \vee y^* \neq 1 \Leftrightarrow x \not\prec y^*$$

we may conclude by (P2) that $x\delta y$.

(I6) By (P8) we have $x = \bigvee\{y \in L \mid y \ll x\} \leq \bigvee\{y \in L \mid y \leq x, y \not\delta x^*\} \leq x$. ■

Note that, since $x \wedge y \neq 0$ implies $x^* \vee y^* \neq 1$, (I5) says, in particular, that $x\delta y$ whenever $x \wedge y \neq 0$. When L is Boolean the converse is also true and condition (I5) is equivalent to the condition

$$x \wedge y \neq 0 \text{ implies } x\delta y.$$

We say that a binary relation δ satisfying properties (I1)-(I6) is an *infinitesimal relation* and that, in this case, (L, δ) is an *infinitesimal frame*.

The correspondence of 6.17 is invertible for Boolean frames:

Proposition 6.18. *If (L, δ) is an infinitesimal frame, the binary relation $\ll$ given by*

$$x \ll y \equiv x \not\delta y^*$$

is a proximity on L if and only if L is Boolean.

Proof. If $\ll$ is a proximity then, for any $x \in L$, $x^{**} = \bigvee\{y \in L \mid y \ll x^{**}\}$. But

$$y \ll x^{**} \Leftrightarrow y \not\delta x^{***} \Leftrightarrow y \ll x.$$

Consequently, $x^{**} = \bigvee\{y \in L \mid y \ll x\} = x$ and L is Boolean.

Since in any Boolean frame the DeMorgan law $(x_1 \wedge x_2)^* = x_1^* \vee x_2^*$ also holds, the proof that $\ll$ is a proximity when L is Boolean follows immediately from the properties of δ. ■

These two results show that on Boolean frames there is a one-to-one correspondence between proximities and infinitesimal relations.

Proposition 6.19. *Let $f : (L_1, \ll_1) \longrightarrow (L_2, \ll_2)$ be a frame map between proximal Boolean frames. The following assertions are equivalent:*

(i) *f is a proximal frame homomorphism;*

(ii) *for every $x, y \in L_1$, $x \not\delta_{\ll_1} y$ implies $f(x) \not\delta_{\ll_2} f(y)$.*

Proof. It is an immediate consequence of the fact that, for any frame map $f : L_1 \longrightarrow L_2$, in case L_1 is Boolean, $f(x^*) = f(x)^*$ for every $x \in L_1$. ■

We define an *infinitesimal homomorphism* $f : (L_1, \delta_1) \longrightarrow (L_2, \delta_2)$ between infinitesimal frames as a frame map $f : L_1 \longrightarrow L_2$ for which $f(x) \not\delta_2 f(y)$ whenever $x \not\delta_1 y$.

Corollary 6.20. *The full subcategory of* PFrm *of proximal Boolean frames is isomorphic to the category of infinitesimal Boolean frames and infinitesimal homomorphisms.*

■

In conclusion, it is possible to generalize for Boolean frames the description of proximal spaces in terms of infinitesimal relations.

Lemma 6.21. *Let $\mathcal{E}$ be a Weil nearness on L. If $(x \oplus y) \cap E = \mathbb{O}$ for some $E \in \mathcal{E}$ then $x \leq y^*$ and $y \leq x^*$.*

Proof. Since $(x \oplus y) \cap E = \mathbb{O}$ is equivalent to $(y \oplus x) \cap E^{-1} = \mathbb{O}$, it suffices to show one of the inequalities. Let us prove that $x \leq y^*$, i.e., that $x \wedge y = 0$. We have $x \wedge y = \bigvee\{x \wedge y \wedge a \mid (a, a) \in E\}$. But, for any $(a, a) \in E$, $(x \wedge y \wedge a, x \wedge y \wedge a) \in (x \oplus y) \cap E = \mathbb{O}$, so $x \wedge y = 0$. ■

Proposition 6.22. *Let $\mathcal{E}$ be a Weil nearness on L. The following assertions are equivalent:*

(i) *$x \overset{\mathcal{E}}{\lhd} y^*$;*

(ii) *$(x \oplus y) \cap E = \mathbb{O}$ for some $E \in \mathcal{E}$;*

(iii) *there is $E \in \mathcal{E}$ such that, for every non-zero $x' \in \downarrow\{x\}$, $E[x'] \wedge y = 0$;*

(iv) *there is $E \in \mathcal{E}$ such that, for every non-zero $y' \in \downarrow\{y\}$, $x \wedge E[y'] = 0$.*

Proof. The equivalence between (i) and (ii) is a consequence of the equivalence

$$(x \oplus y) \cap E^{-1} = \mathbb{O} \Leftrightarrow E \circ (x \oplus x) \subseteq y^* \oplus y^*$$

that we prove next. So, let $(x \oplus y) \cap E^{-1} = \mathbb{O}$ and consider $(a, b) \in E$ and $(b, c) \in x \oplus x$ with $a, b, c \neq 0$. Then $(b, a \wedge y) \in E^{-1} \cap (x \oplus y) = \mathbb{O}$ and $a \wedge y = 0$, that is, $a \leq y^*$. On the other hand, by Lemma 6.21, $c \leq x \leq y^*$. Hence $(a, c) \in y^* \oplus y^*$. The reverse implication is also clear.

The equivalence (ii)$\Leftrightarrow$ (iii) is obvious.

By symmetry, (ii) is then also equivalent to (iv). ■

As it is well-known, given a uniform space $(X, \mathcal{E})$, there is a way of defining a proximity on X by $A\delta B$ if and only if one of the three following equivalent conditions holds:

(i) for every $E \in \mathcal{E}$, $(A \times B) \cap E \neq \emptyset$;

(ii) for every $E \in \mathcal{E}$ there is $a \in A$ such that $E[a] \cap B \neq \emptyset$;

(iii) for every $E \in \mathcal{E}$ there is $b \in B$ such that $A \cap E[b] \neq \emptyset$.

Proposition 6.22 together with Propositions 6.2 and 6.17 say that any Weil uniformity $\mathcal{E}$ on L induces an infinitesimal relation δ on L by $x\delta y$ if and only if one of the three following equivalent conditions is satisfied:

(i) for every $E \in \mathcal{E}$, $(x \oplus y) \cap E \neq \mathbb{O}$;

(ii) for every $E \in \mathcal{E}$ there exists $x' \in \downarrow\{x\} \setminus \{0\}$ such that $E[x'] \wedge y \neq 0$;

(iii) for every $E \in \mathcal{E}$ there exists $y' \in \downarrow\{y\} \setminus \{0\}$ such that $x \wedge E[y'] \neq 0$.

We may avoid the use of $\downarrow\{x\}$ and $\downarrow\{y\}$ in (ii) and (iii) above, by replacing the following equivalent conditions

(ii') for every $E \in \mathcal{E}$, $st(x, E) \wedge y \neq 0$

and

(iii') for every $E \in \mathcal{E}$, $x \wedge st(y, E) \neq 0$

for, respectively, (ii) and (iii), as we show in the last result of this dissertation:

Proposition 6.23. *Let $\mathcal{E}$ be a Weil uniformity on L. Then the following assertions are equivalent:*

(i) *for every $E \in \mathcal{E}$ there exists $x' \in \downarrow\{x\} \setminus \{0\}$ such that $E[x'] \wedge y \neq 0$;*

(ii) *for every $E \in \mathcal{E}$, $st(x, E) \wedge y \neq 0$.*

Proof. (i)⇒(ii): Let $E \in \mathcal{E}$ and consider a symmetric Weil entourage $F \in \mathcal{E}$ such that $F^2 \subseteq E$. By hypothesis, there is $x' \in \downarrow\{x\} \setminus \{0\}$ such that $F[x'] \wedge y \neq 0$. Therefore, there is $z \in L$ such that $(x', z) \in F$ and $z \wedge y \neq 0$. Since z and x' are non-zero we may conclude that $(x' \vee z, x' \vee z) \in F^2 \subseteq E$. Also, $(x' \vee z) \wedge x \neq 0$ and $(x' \vee z) \wedge y \neq 0$. Hence $st(x, E) \wedge y \neq 0$.

(ii)⇒(i): For any $E \in \mathcal{E}$, $st(x, E) \wedge y \neq 0$ means that there is $(z, z) \in E$ such that $z \wedge x \neq 0$ and $z \wedge y \neq 0$. It suffices now to take $z \wedge x$ for x'. ■

Notes on Chapter IV:

(1) In Section 2 we observed that the open and spectrum functors, conveniently adapted to the categories FrNear and NFrm, establish a dual adjunction between them. With respect to the correlative Weil structures, the "spectrum" of a Weil nearness frame is a framed Weil nearness space and the frame of a framed Weil nearness space is a Weil nearness frame and these correspondences are functorial. However, as we observed in Section 4, these functors do not define a dual adjunction. This is a surprising circumstance which is further evidence for the thesis that, in many situations, covers are better than entourages.

(2) One of the advantages of entourage-like theories is that the symmetry is visible and so the corresponding non-symmetric versions are evident and pleasantly manageable. For example, we saw that by dropping in Definition I.4.5 the symmetry condition (UW3) and replacing $\overline{\mathcal{E}} = \mathcal{E} \cup \{E^{-1} \mid E \in \mathcal{E}\}$ for $\mathcal{E}$ in (UW4), we obtain a category of Weil structured frames which is isomorphic to the category of quasi-uniform frames of Frith. In the same

way, from Weil nearness frames, a theory of "*Weil quasi-nearness frames*", as sought for by Frith at the end of [29], may be nicely established. The corresponding theory of "*Weil quasi-nearness spaces*", similar to the theory of "quasi-nearness spaces" presented by Frith in the last chapter of [29], can be also developed just by dropping axiom (NW0) in Definition 4.3.

The "*quasi-proximal frames*" and "*quasi-proximal spaces*" arise similarly, by dropping the symmetry of $\mathcal{E}$ in the definition of the objects of $\mathsf{WUFrm}_{(\mathsf{UW5})}$. The isomorphism of Chapter III between QWUFrm and QUFrm yields an isomorphism between this category of "quasi-proximal frames" and the one of Frith ([29], p. 68).

(3) We concluded in Section 5 that WNear contains the categories of

- symmetric topological spaces and continuous maps,
- uniform spaces and uniformly continuous maps
- proximal spaces and proximal maps

in a nice way: they are either bireflective or bicoreflective full subcategories of WNear. However the category Top of topological spaces is not a subcategory of WNear. Other useful topological structures, namely the non-symmetric ones of quasi-uniform spaces and quasi-proximal spaces are also not embeddable in WNear.

Nevertheless, in the realm of those "Weil quasi-nearness spaces" mentioned above, it is possible to consider all those spaces of topological and uniform nature. It turns out that the category of Weil quasi-nearness spaces contains nicely all the non-symmetric categories referred to above as well as the category Top, i.e., it is a unified theory of (non-symmetric) topology and uniformity.

(4) The isomorphism between PFrm and $\mathsf{WUFrm}_{(\mathsf{UW5})}$ suggested us the notion of finite Weil entourage. It also justifies that we name the Weil nearness frames satisfying (UW5) as "*Weil contigual frames*". This is the analogous notion to the contigual frames of Dube [18] in the setting of nearness frames. One natural question is this: are these two categories equivalent? Another interesting problem is whether the corresponding category of "*Weil contigual spaces*" (as a full subcategory of WNear) is equivalent to the full subcategory of Near of contigual nearness spaces of Herrlich [34] and, consequently, equivalent to the classical category of contigual spaces and contigual maps

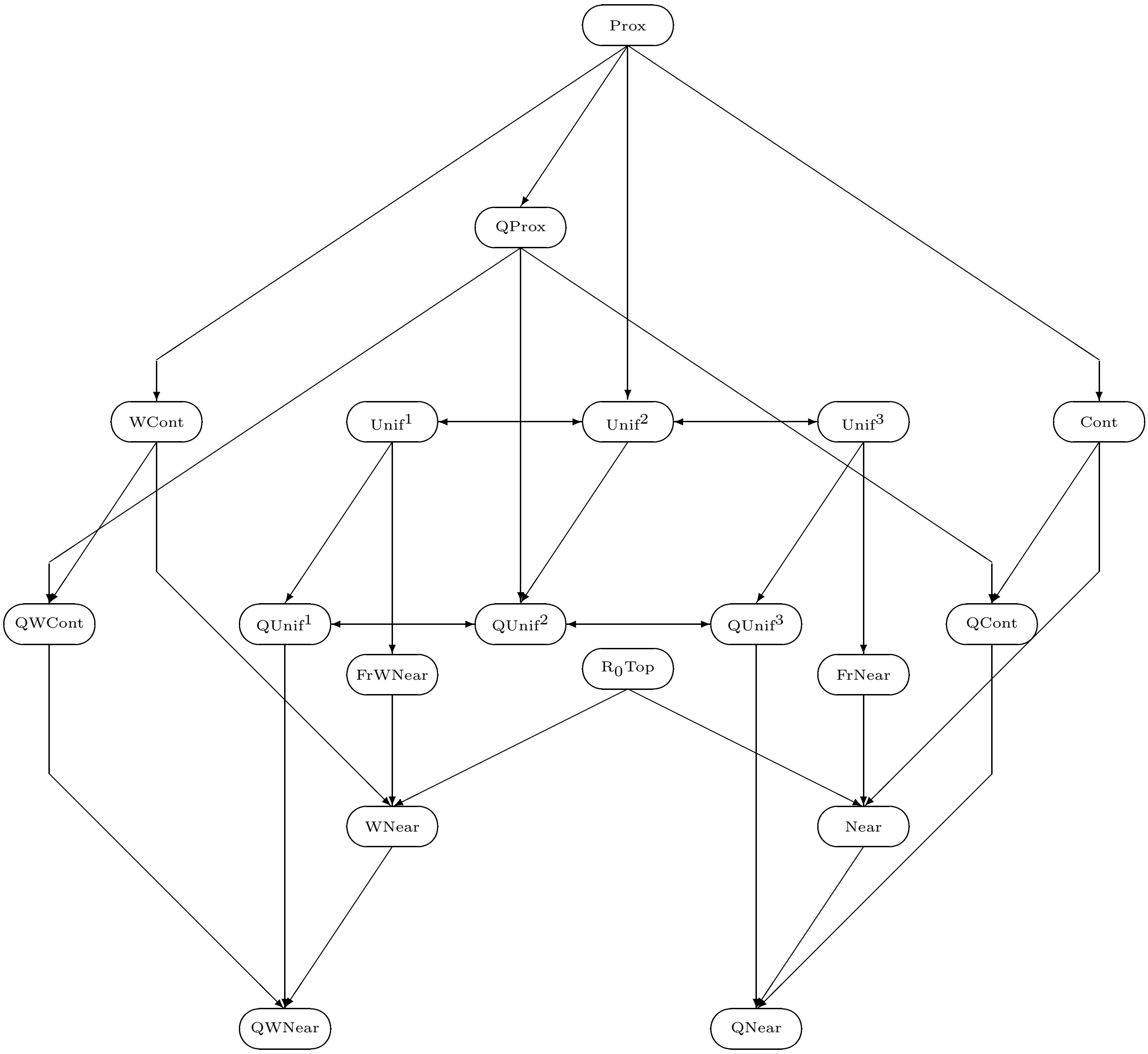
in the sense of Ivanova and Ivanov [41].

APPENDIX

HIERARCHY OF NEARNESS STRUCTURES ON SETS AND FRAMES

This appendix consists of two diagrams. The first one summarizes the hierarchy of nearness structures on sets in the senses of Tukey and Weil. The second diagram is the corresponding diagram for frames.

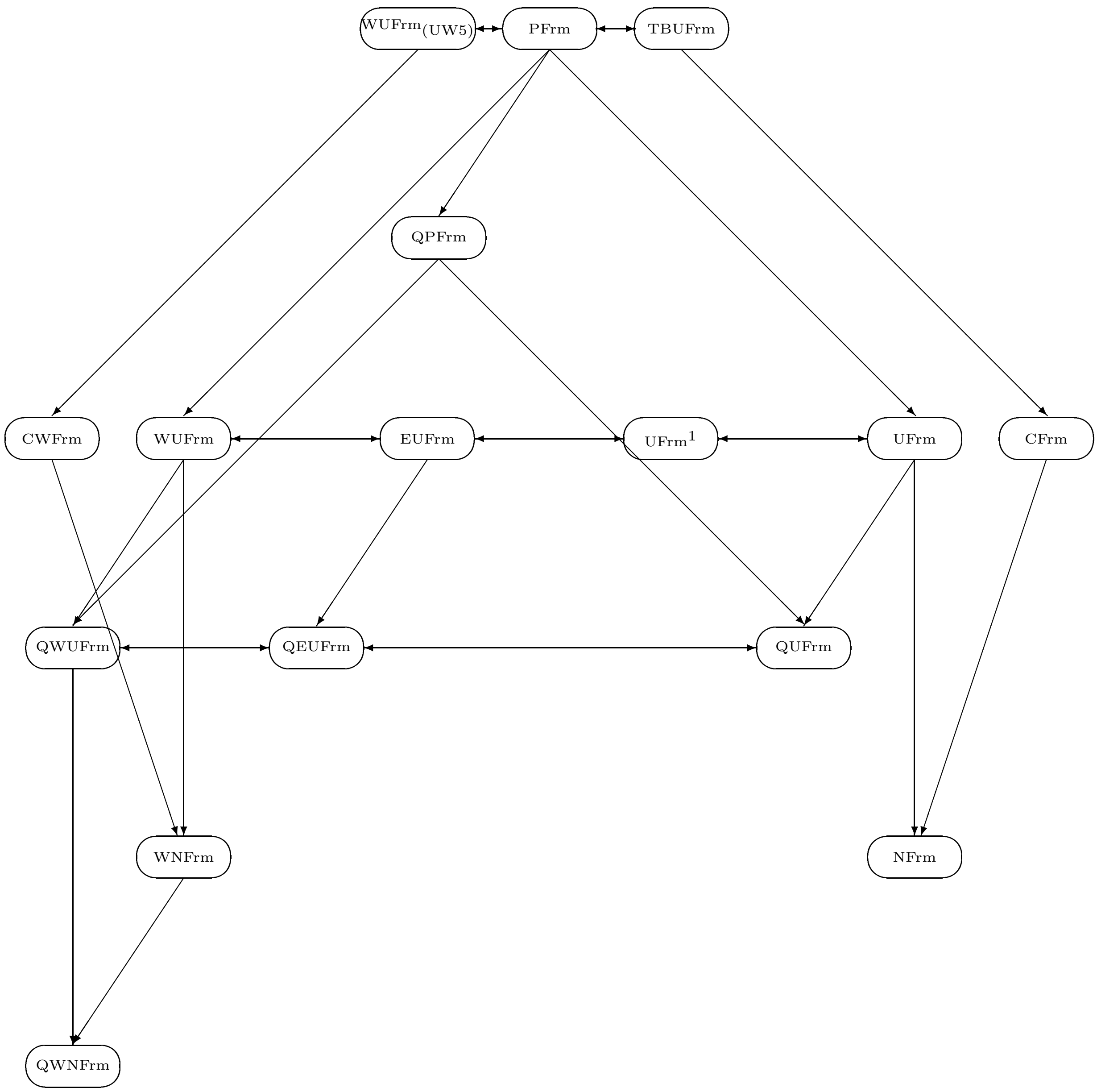
In these diagrams, $\mathcal{A} \longrightarrow \mathcal{B}$ and $\mathcal{A} \longleftrightarrow \mathcal{B}$ mean that category $\mathcal{A}$ is fully embeddable in category $\mathcal{B}$ and that categories $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, respectively.



[1]in the sense of Weil

[2]in the sense of Bourbaki

[3]in the sense of Tukey



[1] gauge frames

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