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The quantale of Galois connections. (English summary)
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The Galois connections dealt with in the paper are those sometimes called contravariant, or antitone. The notation $\mathrm{Gal}(A,B)$, where A and B are posets, stands for the set of all Galois maps of A into B.

Residuated pairs of mappings are known also as covariant Galois connections. It is well known that the composition of two residuated maps is again residuated. Due to this fact, the complete join semilattice of residuated self-maps of A is actually a quantale. In contrast, the set $\mathrm{Gal}(A,A)$ is not closed under composition. The author aims to show that this is not a serious disadvantage: he introduces a composition-like operation $\circ$ for Galois maps between complete lattices which then converts $\mathrm{Gal}(A,A)$ into a quantale. As an application, uniform structures are then described in terms of Galois connections.

Given two antitone maps $f\colon A\to B$ and $g\colon B\to C$ between complete lattices, their antitone composition $g\cdot f$ is defined by

$$(g\cdot f):=\bigvee\{c\in C\mid \exists b\in B\setminus\{0\}\colon f(a)\geq b\ \&\ g(b)\geq c\}.$$

It is easily shown that the set $\mathrm{Ant}(A,A)$ of antitone selfmaps is a quantale. The Galois composition $g\circ f$ turns out to be the least Galois map containing $g\cdot f$. Instead of a technically complicated demonstration that $\circ$ is associative and distributive over joins, it is shown that $\mathrm{Gal}(A,A)$ is a quotient of $\mathrm{Ant}(A,A)$.

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.