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A new look at localic interpolation theorems. (English summary)
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The Katětov-Tong interpolation theorem states that a topological space X is normal if and only if for any pair of functions $f, g\colon X \to \mathbb{R}$, where f is upper and g is lower semicontinuous and $f \leq g$, there exists a continuous function $h\colon X \to \mathbb{R}$ such that $f \leq h \leq g$. The present paper provides a generalization and a new approach to this result based on frame homomorphisms.

A frame (also called a locale) is a complete lattice L satisfying the infinite distributive law:

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S\}$$

for all $x \in L$ and all $S \subseteq L$. A frame L is said to be normal if for any two elements $x, y \in L$ with $x \vee y = 1$ there exist $a, b \in L$ such that $x \vee a = 1 = y \vee b$ and $a \wedge b = 0$.

Let $\overline{\mathfrak{L}}(\mathbb{R})$, $\overline{\mathfrak{L}}_u(\mathbb{R})$ and $\overline{\mathfrak{L}}_l(\mathbb{R})$ be the frames of the real numbers with, respectively, the interval topology, the upper topology and the lower topology. They can be defined algebraically as follows. $\overline{\mathfrak{L}}_u(\mathbb{R})$ is generated by the set of generators $\{(-, \alpha) | \ \alpha \in \mathbb{Q}\}$ under the relations

(U_1) $\alpha \leq \beta \Rightarrow (-, \alpha) \leq (-, \beta)$,
(U_2) $\bigvee_{\beta < \alpha} (-, \beta) = (-, \alpha)$,
(U_3) $\bigvee_{\alpha \in \mathbb{Q}} (-, \alpha) = 1$.

The frames $\overline{\mathfrak{L}}_l(\mathbb{R})$ and $\overline{\mathfrak{L}}(\mathbb{R})$ are generated by sets $\{(\alpha, -) \mid \alpha \in \mathbb{Q}\}$ and $\{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{Q}\}$ under analogous relations.

Given any frame L there is a one-to-one correspondence between the continuous, the upper continuous and the lower continuous real functions on L and the frame homomorphisms $\overline{\mathfrak{L}}(\mathbb{R}) \to L$, $\overline{\mathfrak{L}}_u(\mathbb{R}) \to L$ and $\overline{\mathfrak{L}}(\mathbb{R}) \to L$ respectively. A relation of partial ordering between such homomorphisms can be introduced in a natural way.

The author starts with an extensive algebraic analysis of frames and then establishes his main theorem, which states that for a frame L the following conditions are equivalent:

(1) L is normal.

(2) Given any pair of frame homomorphisms $f\colon \overline{\mathfrak{L}}_u(\mathbb{R}) \to L$ and $g\colon \overline{\mathfrak{L}}_l(\mathbb{R}) \to L$ satisfying:

(a) $f \leq g$,
(b) for all $\alpha \in \mathbb{Q}$ there is a $\beta \in \mathbb{Q}$ such that $f(-, \alpha) \wedge g(\beta, -) = 0$,
(c) for all $\beta \in \mathbb{Q}$ there is an $\alpha \in \mathbb{Q}$ such that $f(-, \alpha) \wedge g(\beta, -) = 0$,

there exists a frame homomorphism $h\colon \overline{\mathfrak{L}}(\mathbb{R}) \to L$ such that $f \leq h \leq g$.

In a final section the author applies the above result to the frame $\mathcal{O}(X)$ consisting of all open subsets of a topological space X, and he shows how to deduce a localic version of the Katětov-Tong interpolation theorem as well as of the classical lemma of Urysohn.

Reviewed by *Hans Arwed Keller*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.