TORSION THEORIES

(english version)

by

Jorge Manuel Senos da Fonseca Picado

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Supervisor: *Prof. Manuela Sobral*

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SUMMARY

The concept of torsion had its origin in the theory of abelian groups. There a group is torsion if and only if all its elements have finite order. This notion could be directly extended to modules over an integral domain. In the theory of rings and of modules of quotients, to each ring of quotients of a ring R there is an associated notion of torsion for modules over R ([43], II.3).

Several authors, as Dlab in [20] and Levy in [35], extended this notion to modules over more general rings. However these definitions are too closely linked with some particular classes of modules and, apparently, not connected.

In 1966 Dickson [17] gave a definition of torsion theory in abelian categories which generalizes and unifies the existing ones. As Herrlich and Strecker refer in [27], quoting Bass, " Virtually all algebraic notions in Category Theory are parodies of their parents in the most classical of categories... the category of left A-modules. "

The extension of some fundamental ideas and constructions of one area of Mathematics to other is one of the aims of the theory of Categories. This was Barr's purpose in [4], where he gave a definition of torsion theory for non-abelian categories and a great variety of examples encompassed by these theories. Also Cassidy, Hébert and Kelly, in 1985, following a different path from the one of Barr, introduced another definition of torsion theory which arose from a close analysis of the relations between reflective subcategories and factorization systems of a category.

This dissertation, which is mainly based on [11], summarizes some of the results in this area.

In the first chapter, where we use essentially [17] and [15], we give an approach to torsion theories in abelian categories. In the first section we present Dickson's definition and stress the first analogy between the theory of torsion/torsion free subcategories in this context and the one of connectednesses/disconectednesses in the category of topological spaces as it appears in [3]. The concept of hereditary torsion theories is a very important one. Indeed, they are equivalent to, for example, "idempotent filters of ideals" in (Bourbaki, *Eléments de*

Mathématique, fasc. 27, Chap. 1,2 (1961)) and [24] nowadays called Gabriel topologies, "left exact torsion radicals" in (Maranda, *Injective Structures*, Trans. AMS, 110 (1964)), "idempotent kernel functors" in (Goldman, *Rings and modules of quotients*, J. Algebra, 13 (1969)) and to "left exact reflectors" in [24]. This is the topic of the second section where we give a characterization theorem extracted from [11] and [43]. Finally in section 3, where we use mainly [15], we present, with some detail, a classification in the category of abelian groups of all torsion subcategories contained in the subcategory of torsion groups (in the classical sense) and of all hereditary torsion subcategories.

The second chapter starts with the definitions of prefactorization and factorization systems as well as the statements and proof of the results that will enable us to establish the one-to-one correspondence between reflective subcategories and prefactorization systems of a category. All these results are from [11], being 1.8 a generalization of (2.4) in [11]. We point out that this bijection develops an idea referred by Bousfield in ([6], 2.5). Observation 2.5 of [11] led us to the study of the links between the notions introduced in the preceding section and the categories of fractions. Under very general conditions on a category, namely that it is a finitely well-complete category, it is proved in [11] that the bijection we referred to above is a bijection between reflective subcategories and factorization systems. This is the subject we deal with in 3.2, 3.3 and 3.4. In 3.5 we present an easy proof given by Cassidy, Hébert and Kelly in [11] of a result of Day [14]. Section 4 as well as the first one consist of a short account of the essential steps in [11] towards the formulation of the notion of torsion theory for categories with initial and terminal objects. There we present a detailed description of the reflective and of the coreflective factorization systems associated with a factorization system and exhibit some examples. At this point it is possible to study torsion theories in categories with initial and terminal objects. This is the topic of section 5. The definition 5.2, which can be formulated in any category, was suggested by the study of [11] and aims to stress the differences and the similarities with the characterization of torsion in abelian categories. In 5.4 we prove that it is a generalization of the definition of Cassidy, Hébert and Kelly [11]. The remaining results are from [11] with exception of 5.7 and of example (c) where we study the torsion theories of the category of graphs and in preorder categories defined by complete lattices, respectively. Section 6 contains a brief survey of torsion theories in categories without zero object as well as sufficient conditions for a category without zero object to have only as torsion theories the trivial ones. As a direct consequence we conclude that the notion of torsion becomes trivial in topological categories giving us a justification for the fact of being the analogue theory of connectednesses/disconnectednesses that plays an important role in such categories. Except for 7.10 (ii), which was inspired by the similar result for abelian categories, all the results in section 7 are from [11] and deal with torsion theories in categories with a zero object. In 8.5,

8.6, 8.7, 8.8 and 8.10 we collect results of an approach given by Tholen in [45] for the study of connections in the context of factorizations of sources and apply them to the special case we are interested in: the torsion theories in categories with zero object. With these results, and still inspired in [45], we obtain theorem 8.11 and its corollaries 8.12 and 8.13, which give us a characterization of torsion theories in terms of $\mathcal{E}$-connections and $\mathcal{E}$-reflections. Finally, we are able to relate in proposition 8.14 the definition given by Cassidy, Hébert and Kelly [11] with Barr's definition [4].

The third chapter deals with torsion theories and localizations. In section 1 we first recall the definition of certain types of reflectors given in [11] and its characterizations, giving particular attention to the reflectors with stable units (1.6) whose characterization is based on results of [34] and on the observation that condition 1.9 (iv) in this paper coincides with the definition of reflector with stable units. The example 1.8 is a detailed proof of theorem 1 of [10]. The relation between localizing subcategories and hereditary torsion theories is studied in section 2. Its main goal is to present a result of [11] where it is shown that in abelian categories with enough injectives there exists a one-to-one correspondence between localizing subcategories and hereditary torsion theories. It also contains some examples of non-abelian categories where this correspondence is not bijective. The third and last section is concerned with the idea of setting up a link with the classical results on localizations suggested in the introduction of [11]. For that our main tools are two results of [5] which we recall in 3.2 and 3.4. With these results we describe localizing subcategories of some special categories in terms of a topology adequate for each case. We also make a short reference to the case of toposes.

INTRODUCTION

Throughout this dissertation we shall denote categories by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, ... , etc. The class of objects of a category $\mathcal{A}$ will be denoted by $\mathrm{Obj}\mathcal{A}$, whereas, $\mathrm{Mor}\mathcal{A}$ will stand for the class of morphisms of $\mathcal{A}$. Capital and small Latin letters will be used for objects and morphisms of a category, respectively, and so we will abbreviate $A \in \mathrm{Obj}\mathcal{A}$ and $f \in \mathrm{Mor}\mathcal{A}$ to $A \in \mathcal{A}$ and $f \in \mathcal{A}$. For two objects A and B in $\mathcal{A}$, $\mathrm{Hom}_{\mathcal{A}}(A,B)$ will denote the set of morphisms from A to B. We shall simply write $\mathrm{Hom}_{\mathcal{A}}(A,B)=0$ when $\mathrm{Hom}_{\mathcal{A}}(A,B)=\{0\}$.

By a subcategory we always mean an isomorphism closed full subcategory, and so identified by its objects.

Our general references for Category Theory are [27] and [36].

We shall denote by $\mathrm{Iso}\mathcal{A}$, $\mathrm{Mono}\mathcal{A}$, $\mathrm{ExtMono}\mathcal{A}$, $\mathrm{RegMono}\mathcal{A}$, $\mathrm{StrMono}\mathcal{A}$, $\mathrm{Epi}\mathcal{A}$, $\mathrm{RegEpi}\mathcal{A}$ and $\mathrm{StrEpi}\mathcal{A}$ the classes of isomorphisms, monomorphisms, extremal monomorphisms, regular monomorphisms, strong monomorphisms, epimorphisms, regular epimorphisms and strong epimorphisms of the category $\mathcal{A}$, respectively.

CHAPTER I

TORSION THEORIES IN ABELIAN CATEGORIES

1. TORSION THEORIES: GENESIS AND EVOLUTION

An element g of an abelian group is called a torsion element if it has finite order, i.e. if the ideal $\{n \in \mathbb{Z} \mid ng = 0\}$ of $\mathbb{Z}$ is non-zero. An abelian group G is torsion if all of its elements are torsion elements, which is equivalent to say that $\mathrm{Hom}(G,\mathbb{Q}) = 0$, $\mathbb{Q}$ being the group of rationals. It is important to point out that every group G has a maximal torsion subgroup G_t for which $(G/G_t)_t = 0$, i.e. G/G_t is a torsion-free group.

This definition of torsion can easily be extended to modules over integral domains. However, for other rings this concept may have no relevance. Using this definition, the categories of modules have non-zero torsion-free objects only if the ring has no zero divisors. Indeed, if d and d' are non-zero elements of the ring R such that $dd' = 0$ then, for every element m of a left R-module, either $d'm = 0$ or $d(d'm) = 0$.

Skirting this and other difficulties, several authors generalized the concept of torsion to any categories of modules, e. g. Dlab in [20] and Levy in [35].

The notion of torsion theory that we shall present in this chapter is due to Dickson [17], who gave an axiomatic definition of torsion in abelian categories. This definition avoids certain pathologies that are in general associated to definitions too closely related with a special abelian category, and unifies important classical torsion theories like for example those of Goldie and of Lambek [42]. Furthermore, it allows the formulation of the corresponding primary decomposition theorems. For additional information we refer to Dickson [16], [17] and [18].

Throughout this chapter we shall denote by $\mathcal{A}$ an abelian wellpowered category in which there exist the product $\prod_{i \in I} (A/A_i)$ and the coproduct $\coprod_{i \in I} A_i$, for every family $\{A_i : i \in I\}$ of subobjects of an object A in $\mathcal{A}$.

In an abelian category, every morphism $f: X \longrightarrow Y$ has a factorization

$$X \xrightarrow{e} \mathrm{Im}\, f \xrightarrow{m} Y$$

where m is a monomorphism and e is an epimorphism ([36], VIII.3.1). We shall use $\ker f$ and $\mathrm{Ker}\, f$ to denote the kernel morphism of f and the corresponding object, i.e.

$$\mathrm{Ker}\, f \xrightarrow{m} X \quad \text{where} \quad m = \ker f.$$

The assumptions on the category $\mathcal{A}$ imply that the set $\mathcal{S}(A)$ of subobjects of each $\mathcal{A}$-object A is a complete lattice for the usual order relation: for each family $\{A_i : i \in I\}$ of elements of $\mathcal{S}(A)$,

$$\bigvee_{i \in I} A_i = \mathrm{Im}\left(\coprod_{i \in I} A_i \longrightarrow A \right)$$

and

$$\bigwedge_{i \in I} A_i = \mathrm{Ker}\left(A \longrightarrow \prod_{i \in I} (A/A_i) \right) \qquad \text{([43], IV, §2 and §4).}$$

<u>DEFINITION 1.1</u>. A pair $(\mathcal{C}, \mathcal{B})$ of subcategories of $\mathcal{A}$ is a <u>torsion theory of $\mathcal{A}$</u> provided that:

(T1) $\mathrm{Hom}_{\mathcal{A}}(C,B) = 0$, for each $C \in \mathcal{C}$ and $B \in \mathcal{B}$.

(T2) For every object A of $\mathcal{A}$ there exists an exact sequence

$$0 \longrightarrow C \longrightarrow A \longrightarrow B \longrightarrow 0$$

with $C \in \mathcal{C}$ and $B \in \mathcal{B}$.

Then we shall say that $\mathcal{C}$ is a <u>torsion subcategory</u> and its objects are called the <u>torsion objects</u> while $\mathcal{B}$ is a <u>torsion-free subcategory</u> and its objects are called <u>torsion-free objects</u>.

It is clear that if $(\mathcal{C}, \mathcal{B})$ is a torsion theory then the exact sequence in (T2) is unique up to isomorphism.

<u>EXAMPLES 1.2</u>. (a) Let $\mathcal{A}b$ be the category of abelian groups, $\mathcal{T}ors$ and $\mathcal{T}ors\mathcal{F}$ be the subcategories of $\mathcal{A}b$ whose objects are the torsion abelian groups and the torsion-free abelian groups, respectively. Then $(\mathcal{T}ors, \mathcal{T}ors\mathcal{F})$ is a torsion theory of $\mathcal{A}b$.

(b) Consider the category R-Mod of left modules over a ring R with identity. A subset S of R is called a <u>left denominator set</u> if it is multiplicatively closed and satisfies the following conditions:

(S1) $\forall a \in S, \forall r \in R, \exists b \in S, \exists t \in R : ta = br$.

(S2) If $ra = 0$ with $a \in S$ then $br = 0$ for some $b \in S$.

If we consider a left denominator set S and define for each $M \in R\text{-Mod}$ the submodule

$$t(M) = \{ m \in M \mid am = 0 \text{ for some } a \in S \}$$

then the pair $(\mathcal{C}, \mathcal{B})$ where

$$\mathcal{C} = \{ M \in R\text{-Mod} \mid t(M) = M \} \text{ and } \mathcal{B} = \{ M \in R\text{-Mod} \mid t(M) = 0 \}$$

is a torsion theory of the category R-Mod. The modules of $\mathcal{C}$ are usually called S-torsion modules and they are precisely the modules for which the respective quotient module $M[S^{-1}]$ is the zero module ([43], II.3).

When S is the set of regular elements of the ring, we obtain the definition of torsion for modules over rings with zero divisors, originally given by Levy in [35].

(c) The pair $(\mathcal{D}, \mathcal{R})$, where $\mathcal{D}$ is the subcategory of divisible abelian groups and $\mathcal{R}$ is the subcategory of reduced abelian groups, is also a torsion theory of $\mathcal{A}b$. This is an important example because every abelian group G has a direct decomposition $G = D \oplus R$, where D is the maximal divisible subgroup of G and $R \in \mathcal{R}$ ([23], 21.2).

<u>DEFINITION 1.3</u>. A subcategory $\mathcal{B}$ of the category $\mathcal{A}$ is called <u>reflective</u> if the inclusion functor, $E : \mathcal{B} \longrightarrow \mathcal{A}$, has a left adjoint, $R : \mathcal{A} \longrightarrow \mathcal{B}$, i.e., for each object A of $\mathcal{A}$, there exists a universal morphism, $r_A : A \longrightarrow R(A)$, from A to E. This morphism is called the <u>reflection</u> of A in $\mathcal{B}$, and the functor R is the <u>reflector</u>. If all reflection morphisms belong to a given class $\mathcal{E}$ of morphisms in $\mathcal{A}$, $\mathcal{B}$ is called an <u>$\mathcal{E}$-reflective subcategory</u> of $\mathcal{A}$.

<u>REMARKS 1.4</u>. (i) By duality we obtain the notions of coreflective subcategory, coreflection and of coreflector.

(ii) It is easy to check that, given a reflector $R : \mathcal{A} \longrightarrow \mathcal{B}$, there exists a reflector $R' : \mathcal{A} \longrightarrow \mathcal{B}$, naturally isomorphic to R, such that r'_B is the identity for each object B of $\mathcal{B}$ (cf., for example, [30], II.7.6). Since there is no loss of generality, in the following, we shall always consider reflectors that satisfy this condition.

(iii) From (T1) and (T2) it follows that $\mathcal{B}$ is an Epi-reflective subcategory of $\mathcal{A}$. Indeed, it is easy to see that the morphism $A \longrightarrow B$ in (T2) is the reflection of A in $\mathcal{B}$. Then, by duality, $\mathcal{C}$ is a Mono-coreflective subcategory of $\mathcal{A}$.

The family $\mathcal{S}_{\mathcal{A}}$ of subcategories of $\mathcal{A}$ partially ordered by inclusion is a (possibly large) complete lattice. Let us consider the operators

$$\bullet^{\rightarrow}: \mathcal{S}_{\mathcal{A}} \longrightarrow \mathcal{S}_{\mathcal{A}}$$
$$\mathcal{B} \longmapsto \mathcal{B}^{\rightarrow} := \left\{ A \in \mathrm{Obj}\,\mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(B,A) = 0 \text{ for all } B \in \mathrm{Obj}\,\mathcal{B} \right\}$$

and

$$\bullet^{\leftarrow}: \mathcal{S}_{\mathcal{A}} \longrightarrow \mathcal{S}_{\mathcal{A}}$$
$$\mathcal{B} \longmapsto \mathcal{B}^{\leftarrow} := \left\{ A \in \mathrm{Obj}\,\mathcal{A} \mid \mathrm{Hom}_{\mathcal{A}}(A,B) = 0 \text{ for all } B \in \mathrm{Obj}\,\mathcal{B} \right\}.$$

The following proposition, whose proof is straightforward, brings together some of their properties.

PROPOSITION 1.5. *Let* $\mathcal{B}$, $\mathcal{B}_1$ *and* $\mathcal{B}_2$ *be subcategories of* $\mathcal{A}$. *Then:*

(i) $\mathcal{B} \subset \mathcal{B}^{\leftarrow\rightarrow}$.

(ii) $\mathcal{B} \subset \mathcal{B}^{\rightarrow\leftarrow}$.

(iii) $\mathcal{B}_1 \subset \mathcal{B}_2 \Rightarrow (\mathcal{B}_2{}^{\rightarrow} \subset \mathcal{B}_1{}^{\rightarrow}$ *and* $\mathcal{B}_2{}^{\leftarrow} \subset \mathcal{B}_1{}^{\leftarrow})$.

(iv) $\mathcal{B}^{\leftarrow\rightarrow\leftarrow} = \mathcal{B}^{\leftarrow}$ *and* $\mathcal{B}^{\rightarrow\leftarrow\rightarrow} = \mathcal{B}^{\rightarrow}$. □

Hence, $\bullet^{\rightarrow}: \mathcal{S}_{\mathcal{A}}{}^{op} \longrightarrow \mathcal{S}_{\mathcal{A}}$ $\qquad$ $\bullet^{\leftarrow}: \mathcal{S}_{\mathcal{A}} \longrightarrow \mathcal{S}_{\mathcal{A}}{}^{op}$

are functors for which the pair $(\bullet^{\rightarrow}, \bullet^{\leftarrow})$ is a Galois connection ([13], p. 44), which is equivalent to say that

$$\mathcal{B}_1 \subset \mathcal{B}_2{}^{\rightarrow} \Leftrightarrow \mathcal{B}_2 \subset \mathcal{B}_1{}^{\leftarrow}.$$

REMARK 1.6. The functors $T = \bullet^{\rightarrow\leftarrow}$ and $F = \bullet^{\leftarrow\rightarrow}$ are closure operators ([13], p. 42) in the complete lattice $\mathcal{S}_{\mathcal{A}}$, to which correspond the following closure systems

$$\mathcal{T} = \left\{ \mathcal{B} \in \mathcal{S}_{\mathcal{A}} \mid \mathcal{B}^{\rightarrow\leftarrow} = \mathcal{B} \right\} \quad \text{and} \quad \mathcal{F} = \left\{ \mathcal{B} \in \mathcal{S}_{\mathcal{A}} \mid \mathcal{B}^{\leftarrow\rightarrow} = \mathcal{B} \right\}$$

in the sense of Cohn ([13], p. 41).

LEMMA 1.7. *If* $(\mathcal{C},\mathcal{B})$ *is a torsion theory of a category* $\mathcal{A}$ *then* $\mathcal{C}^{\rightarrow} = \mathcal{B}$ *and* $\mathcal{B}^{\leftarrow} = \mathcal{C}$.

Proof: By (T1) $\mathcal{B} \subset \mathcal{C}^{\rightarrow}$ and $\mathcal{C} \subset \mathcal{B}^{\leftarrow}$. Since, for every object A, there exists an exact sequence

$$0 \longrightarrow C \xrightarrow{f} A \xrightarrow{g} B \longrightarrow 0,$$

then, if $A \in \mathcal{C}^{\rightarrow}$, we have that $f = 0$, and so g is an isomorphism. Hence $A \in \mathcal{B}$ and therefore $\mathcal{C}^{\rightarrow} = \mathcal{B}$. The other inclusion follows by duality. □

PROPOSITION 1.8. *A pair* $(\mathcal{C},\mathcal{B})$ *of subcategories of* $\mathcal{A}$ *is a torsion theory of* $\mathcal{A}$ *if and only if the following conditions are satisfied:*

(i) $\mathcal{C}\cap\mathcal{B}=\{0\}$.

(ii) $\mathcal{C}$ *is closed for quotient objects, i.e. if the sequence* $C\longrightarrow A\longrightarrow 0$ *is exact and* $C\in\mathcal{C}$ then $A\in\mathcal{C}$.

(iii) $\mathcal{B}$ *is closed for subobjects, i.e. if the sequence* $0\longrightarrow A\longrightarrow B$ *is exact and* $B\in\mathcal{B}$ *then* $A\in\mathcal{B}$.

(iv) *For every object* A *of* $\mathcal{A}$ *there exists an exact sequence* $0\longrightarrow C\longrightarrow A\longrightarrow B\longrightarrow 0$, with $C\in\mathcal{C}$ *and* $B\in\mathcal{B}$.

Proof: Let $(\mathcal{C},\mathcal{B})$ be a torsion theory of $\mathcal{A}$. Condition (i) is obviously true and (iv) is the condition (T2) of 1.1.

If
$$C\xrightarrow{g}A\longrightarrow 0$$
is exact, i.e. g is an epimorphism, then $A\in\mathcal{B}^{\leftarrow}$ because, for every $B\in\mathrm{Obj}\,\mathcal{B}$, if $f\in\mathrm{Hom}_{\mathcal{A}}(A,B)$ we have $f.g=0$ and so $f=0$. Thus $A\in\mathcal{C}$, by 1.7, and we have proved (ii).

Condition (iii) follows by duality.

Conversely, if $(\mathcal{C},\mathcal{B})$ satisfies (i), (ii), (iii) and (iv) then (T2) holds. It remains to prove (T1). Let $C\in\mathcal{C}$, $B\in\mathcal{B}$ and $f\in\mathrm{Hom}_{\mathcal{A}}(C,B)$. Consider the (Epi,Mono)-factorization, m.e, of f. By (ii) and (iii), $\mathrm{Im}\,f\in\mathcal{C}\cap\mathcal{B}$ and so $\mathrm{Im}\,f=0$, i.e. $f=0$. □

DEFINITION 1.9. A subcategory $\mathcal{D}$ of $\mathcal{A}$ is called closed under extensions if, for every exact sequence $0\longrightarrow D_1\longrightarrow A\longrightarrow D_2\longrightarrow 0$, $A\in\mathcal{D}$ whenever $D_1,D_2\in\mathcal{D}$.

LEMMA 1.10. *If the subcategory* $\mathcal{C}$ *is of the form* $\mathcal{B}^{\leftarrow}$, *for some subcategory* $\mathcal{B}$, *then* $\mathcal{C}$ *is closed under extensions and the formation of all coproducts that exist in* $\mathcal{A}$.

Proof: If we consider an exact sequence
$$0\longrightarrow C_1\longrightarrow A\longrightarrow C_2\longrightarrow 0$$
with $C_1,C_2\in\mathcal{C}$, then, since for every object $B\in\mathcal{B}$ the contravariant functor $\mathrm{Hom}(-,B)$ is left exact, we have the exact sequence
$$0\longrightarrow\mathrm{Hom}(C_2,B)\longrightarrow\mathrm{Hom}(A,B)\longrightarrow\mathrm{Hom}(C_1,B).$$
But $\mathrm{Hom}(C_2,B)=\mathrm{Hom}(C_1,B)=0$ so that $\mathrm{Hom}(A,B)=0$, i.e. $A\in\mathcal{B}^{\leftarrow}=\mathcal{C}$.

Finally, let $\{C_i : i\in I\}$ be a family of objects of $\mathcal{C}$ whose coproduct exists in $\mathcal{A}$. Then $\coprod_{i\in I}C_i\in\mathcal{B}^{\leftarrow}=\mathcal{C}$. Indeed, for every object B of $\mathcal{B}$, we have
$$\mathrm{Hom}(\coprod_{i\in I}C_i,B)\cong\prod_{i\in I}\mathrm{Hom}(C_i,B)=0.$$
□

<u>THEOREM 1.11</u>. $(\mathcal{C},\mathcal{B})$ *is a torsion theory of* $\mathcal{A}$ *if and only if* $\mathcal{C}^{\rightarrow}=\mathcal{B}$ *and* $\mathcal{B}^{\leftarrow}=\mathcal{C}$.

<u>Proof</u>: The necessity was proved in 1.7. To prove its sufficiency it is enough to show (T2), since (T1) trivially holds. Let A be an object of $\mathcal{A}$ and $\mathcal{S}(A)\cap\mathcal{C}=\{A_i : i\in I\}$, where $\mathcal{S}(A)$ denotes the set of its subobjects. Consider the following diagram

$$\begin{tikzcd}
 & \coprod_{i\in I} A_i \arrow[dl, "e"'] \arrow[dd, dashed, "f"] & A_i \arrow[l, "u_i"'] \arrow[ddl, "m_i"] \\
A_t \arrow[dr, "m"'] & & \\
 & A &
\end{tikzcd}$$

where f is the unique morphism such that $f.u_i=m_i$, for every $i\in I$, and $m.e$ is the corresponding (Epi,Mono)-factorization. By 1.10 we have $\coprod_{i\in I} A_i\in\mathcal{C}$ and, in a similar way to that one used in the proof of 1.8(ii), we can conclude that $A_t\in\mathcal{C}$, since e is an epimorphism. To finish the proof it remains to show that $A/A_t\in\mathcal{B}=\mathcal{C}^{\rightarrow}$. If $C\in\mathcal{C}$ and $g\in\mathrm{Hom}_{\mathcal{A}}(C,A/A_t)$ then $\mathrm{Im}\,g\in\mathcal{C}$ and it is a subobject of A/A_t. Thus $\mathrm{Im}\,g\cong B/A_t$, for some subobject B of A greater than or equal to A_t, with respect to the usual order relation in the class of subobjects of A. Since A_t and $B/A_t\in\mathcal{C}$, and, by 1.10, $\mathcal{C}$ is closed under extensions, then $B\in\mathcal{C}$ so that $B\in\{A_i : i\in I\}$. We have already observed that in the lattice $\mathcal{S}(A)$, $\bigvee_{i\in I} A_i=A_t$. Hence we can conclude that $B=A_t$, which implies that $g=0$. □

With this theorem we can obtain very many examples of torsion theories of $\mathcal{A}$: every subcategory $\mathcal{B}$ of $\mathcal{A}$ generates the torsion theory $(\mathcal{B}^{\leftarrow},\mathcal{B}^{\leftarrow\rightarrow})$ and cogenerates the torsion theory $(\mathcal{B}^{\rightarrow\leftarrow},\mathcal{B}^{\rightarrow})$. This unifies the classical torsion theories. For example, in a category of modules over a ring R, the <u>Goldie torsion theory</u> is the pair $(\mathcal{B}^{\rightarrow\leftarrow},\mathcal{B}^{\rightarrow})$, where $\mathcal{B}$ is the subcategory of modules M/L with L an essential submodule of M, and the <u>Lambek torsion theory</u> is the pair $(\mathcal{B}^{\leftarrow},\mathcal{B}^{\leftarrow\rightarrow})$, where $\mathcal{B}$ is the subcategory defined by the injective envelope of R (see definition 2.2). In $\mathcal{A}b$, the torsion theory generated by the group $\mathbb{Q}$ of rational numbers is the torsion theory we referred to in 1.2(a) and 1.2(c) is the torsion theory it cogenerates.

COROLLARY 1.12. *Let $\mathcal{B}$ and $\mathcal{C}$ be subcategories of the category $\mathcal{A}$. Then:*

(i) *$\mathcal{C}$ is a torsion subcategory if and only if it is closed under quotient objects, extensions, and all the coproducts that exist in $\mathcal{A}$.*

(ii) *$\mathcal{B}$ is a torsion-free subcategory if and only if it is closed under subobjects, extensions, and all the products that exist in $\mathcal{A}$.*

Proof: (i) If $\mathcal{C}$ is a torsion subcategory we have already observed (1.8(ii) and 1.10) that $\mathcal{C}$ is closed under quotient objects, extensions, and all the coproducts that exist in $\mathcal{A}$.

Conversely, it is obvious that $(\mathcal{C},\mathcal{C}^{\rightarrow})$ verifies (T1). It also satisfies (T2) (see proof of 1.11), and so it is a torsion theory.

(ii) follows from (i), by duality. □

EXAMPLES 1.13. Let p be a prime number.

(a) An abelian group is called a p-group if the order of its elements is a power of p. The subcategory p-$\mathcal{A}$b of p-groups is a torsion subcategory of $\mathcal{A}$b because it satisfies 1.12(i). This subcategory is equal to $\{\mathbb{Z}_p\}^{\rightarrow\leftarrow}$, where $\mathbb{Z}_p$ denotes the cyclic group of order p. Indeed $\{\mathbb{Z}_p\}^{\rightarrow\leftarrow} \subset$ p-$\mathcal{A}$b because $\mathbb{Z}_p$ is a p-group and $\{\mathbb{Z}_p\}^{\rightarrow\leftarrow}$ is the smallest torsion subcategory that contains $\mathbb{Z}_p$. Conversely, given a p-group G, denote by $<g>$ the subgroup of G generated by $\{g\}$. The morphism

$$\varphi: \coprod_{g\in G} <g> \longrightarrow G \text{ defined by } \varphi\big((n_g\, g)_{g\in G}\big) = \sum_{g\in G} n_g\, g$$

is an epimorphism. But, for each $g \in G$, there exists a $n \in \mathbb{N}$ such that $<g> \cong \mathbb{Z}_{p^n}$. On the other hand, the sequence

$$0 \longrightarrow \mathbb{Z}_{p^n} \longrightarrow \mathbb{Z}_{p^{n+1}} \longrightarrow \mathbb{Z}_p \longrightarrow 0$$

is exact, for every $n \in \mathbb{N}$, and so, since $\{\mathbb{Z}_p\}^{\rightarrow\leftarrow}$ is closed under extensions, it can be proved by induction that $\mathbb{Z}_{p^n} \in \{\mathbb{Z}_p\}^{\rightarrow\leftarrow}$, for every $n \in \mathbb{N}$. It follows that $G \in \{\mathbb{Z}_p\}^{\rightarrow\leftarrow}$ because $\{\mathbb{Z}_p\}^{\rightarrow\leftarrow}$ is closed under epimorphic images and φ is an epimorphism.

(b) The subcategory of divisible p-groups also satisfies 1.12(i) ([23], 20). This torsion subcategory coincides with $\{\mathbb{Z}(p^\infty)\}^{\rightarrow\leftarrow}$, where $\mathbb{Z}(p^\infty)$ is the multiplicative group of the complex roots of the equations $x^{p^n} = 1$, $n \in \mathbb{N}$, because every divisible p-group is a direct sum of copies of $\mathbb{Z}(p^\infty)$ ([23], 23.1).

REMARK 1.14. The condition "$\mathcal{B}$ *is closed under extensions*" is equivalent to the following:

For each epimorphism $e: A \longrightarrow B_2$, with $B_2 \in \mathcal{B}$, if the diagram

$$\begin{tikzcd} B_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ A \arrow[r, "e"'] & B_2 \end{tikzcd}$$

is a pullback and $B_1 \in \mathfrak{B}$ then $A \in \mathfrak{B}$. Therefore, 1.12(ii) allows us to observe a first close relation between the theory of connectednesses / disconnectednesses in $\mathcal{T}op$ [3] and the torsion / torsion-free theory on abelian categories. A disconnectedness $\mathfrak{D}$ in $\mathcal{T}op$ is a subcategory closed under products and subobjects, and such that for each surjective morphism $f: X \longrightarrow Y$, with $Y \in \mathfrak{D}$, if in the following pullback where $\theta y(*) = y$,

$$\begin{tikzcd} f^{-1}(y) \arrow[r] \arrow[d] & \{*\} \arrow[d, "\theta y"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

$f^{-1}(y) \in \mathfrak{D}$, for every $y \in Y$, then $X \in \mathfrak{D}$ ([3], 2.12).

2. HEREDITARY TORSION THEORIES

<u>DEFINITION 2.1</u>. A subcategory $\mathcal{C}$ of $\mathcal{A}$ is called <u>hereditary</u> if it is closed under subobjects.

If the subcategory $\mathcal{C}$ of a torsion theory $(\mathcal{C}, \mathfrak{B})$ is hereditary we say that $(\mathcal{C}, \mathfrak{B})$ is an <u>hereditary torsion theory</u>.

<u>EXAMPLES</u>. The torsion theories of the examples 1.2 (a) and (b) are clearly hereditary.

The Lambek torsion theory is also hereditary since M is a Lambek torsion module if and only if $\mathrm{Hom}(N,A) = 0$, for every submodule N of M.

1.2(c) is an example of a non-hereditary torsion theory.

DEFINITION 2.2. We shall say that a monomorphism $i: B \longrightarrow A$ is an essential extension of B if $B \cap A' \neq 0$, for every non-zero subobject A' of A. An injective envelope of an object B is an essential extension of B, $\alpha: B \longrightarrow E(B)$, such that $E(B)$ is an injective object, i.e. for every monomorphism $f: X \longrightarrow Y$, the map

$$\begin{aligned} \mathrm{Hom}(f,E(B)): \mathrm{Hom}_{\mathcal{A}}(Y,E(B)) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(X,E(B)) \\ g \quad &\longmapsto \quad g.f \end{aligned}$$

is surjective.

The category $\mathcal{A}$ is said to have injective envelopes if every object of $\mathcal{A}$ has an injective envelope.

It can be easily proved that if an injective envelope exists then it is unique up to isomorphism.

THEOREM 2.3. *Let* $(\mathcal{C},\mathcal{B})$ *be a torsion theory of the abelian category* $\mathcal{A}$ *and consider the following conditions*:

(i) $(\mathcal{C},\mathcal{B})$ *is hereditary.*

(ii) $\mathcal{B}$ *is closed under essential extensions.*

(iii) $\mathcal{B}$ *is closed under injective envelopes.*

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) *and, if* $\mathcal{A}$ *has injective envelopes, the conditions are equivalent.*

Proof:

(i) $\Rightarrow$ (ii): Assume $\mathcal{C}$ is hereditary. Consider $B \in \mathcal{B}$ and let $i: B \longrightarrow A$ be an essential extension of B. We prove now that $A \in \mathcal{B} = \mathcal{C}^{\rightarrow}$. For every morphism $f: C \longrightarrow A$, with $C \in \mathcal{C}$, we consider the following pullback

$$\begin{tikzcd} D \arrow[r, "\bar{f}"] \arrow[d, "\bar{i}"'] & B \arrow[d, "i"] \\ C \arrow[r, "f"'] & A \end{tikzcd}$$

Since $\bar{i}$ is a monomorphism, then $D \in \mathcal{C}$ and so $\bar{f} = 0$. In the following diagram,

$$\begin{tikzcd}
D \arrow[rr, "\bar{f}=0"] \arrow[dr, dashed, "t"] \arrow[ddd, "\bar{i}"'] & & B \arrow[ddd, "i"] \\
& \mathrm{Im}\, f \cap B \arrow[ur, "m'"] \arrow[d, "i'"] \arrow[dl, phantom, "(2)"] \arrow[dr, phantom, "(1)"] & \\
& \mathrm{Im}\, f \arrow[dr, "m"] & \\
C \arrow[ur, "e"] \arrow[rr, "f"'] & & A
\end{tikzcd}$$

since (1) is a pullback t is the unique morphism for which it commutes. Furthermore, it can be easily observed that (2) is also a pullback. Thus t is an epimorphism and so $\mathrm{Im}\, f \cap B = 0$. Since i is essential then $\mathrm{Im}\, f = 0$, i.e. $f = 0$.

<u>(ii)$\Rightarrow$(iii)</u>: It is obvious.

Suppose now that $\mathcal{A}$ has injective envelopes.

<u>(iii)$\Rightarrow$(i)</u>: Let $j: C' \longrightarrow C$ be a monomorphism where $C \in \mathcal{C}$. To prove that $C' \in \mathcal{C} = \mathcal{B}^{\leftarrow}$ we consider $g: C' \longrightarrow B$, with $B \in \mathcal{B}$, and let $\alpha: B \longrightarrow E(B)$ be the injective envelope of B in $\mathcal{A}$. Since $E(B)$ is injective, there exists a morphism $f: C \longrightarrow E(B)$ such that the diagram

$$\begin{tikzcd}
C' \arrow[r, "j"] \arrow[d, "g"'] & C \arrow[ddl, "f"] \\
B \arrow[d, "\alpha"'] & \\
E(B) &
\end{tikzcd}$$

commutes. Hence, since $E(B) \in \mathcal{B}$, $f = 0$ and so $g = 0$. $\square$

REMARK 2.4. A family $\{A_i : i \in I\}$ of subobjects of an object A is called directed if for every pair $(i,j) \in I \times I$ there exists a $k \in I$ such that $A_i \leq A_k$ and $A_j \leq A_k$, for the usual order relation in the class of subobjects of A.

Every cocomplete abelian category, with a generator, for which the equality

$$\Big(\sum_{i \in I} A_i\Big) \cap B = \sum_{i \in I} (A_i \cap B)$$

holds, for every directed family $\{A_i : i \in I\}$ of subobjects of an object A and every subobject B of A, has injective envelopes ([39], III.3.2). Therefore, any Grothendieck category [43] has injective envelopes. These categories are a natural extension of categories of modules and are precisely the categories which arise as categories of modules of quotients ([43], chapter X).

PROPOSITION 2.5. *If $\mathcal{A}$ has injective envelopes and $\mathcal{C}$ is a subcategory of $\mathcal{A}$ closed under subobjects and quotient objects then the torsion theory $(\mathcal{C}^{\rightarrow\leftarrow}, \mathcal{C}^{\rightarrow})$, cogenerated by $\mathcal{C}$, is hereditary.*

Proof: By 2.3 we only need to show that $\mathcal{C}^{\rightarrow}$ is closed under injective envelopes. If $B \in \mathcal{C}^{\rightarrow}$, let $\alpha : B \longrightarrow E(B)$ be an injective envelope of B, and let $f : C \longrightarrow E(B)$, with $C \in \mathcal{C}$. Then $\operatorname{Im} f \in \mathcal{C}$ and $\operatorname{Im} f \cap B \in \mathcal{C}$, by hypothesis. Hence, $\operatorname{Im} f \cap B$ is a subobject of B that belongs to $\mathcal{C}$ and so $\operatorname{Im} f \cap B = 0$. By essentiality of α, $\operatorname{Im} f = 0$, i.e. $f = 0$. □

EXAMPLE. By proposition 2.5 we can immediately conclude that the Goldie torsion theory is hereditary.

3. TORSION THEORIES IN THE CATEGORY OF ABELIAN GROUPS

We finish this first chapter by presenting the complete classification in $\mathcal{A}b$ of the torsion subcategories contained in the subcategory $\mathcal{T}ors$ of usual torsion groups, and of the hereditary torsion subcategories.

Let us denote by P the set of prime numbers. We call any formal product

$$\prod_{p \in P} p^{e(p)}, \text{ where } e(p) \text{ is } 0, 1 \text{ or } \infty$$

a Steinitz number and we denote the set of Steinitz numbers by N. Let U be the class of torsion subcategories of $\mathcal{A}b$ contained in $\mathcal{T}ors$. Consider $\mathcal{C} \in U$ and $p \in P$. If there is a

non-zero p-group G in $\mathcal{C}$, consider his decomposition $G=D\oplus R$, where D is the maximal divisible subgroup of G (1.2(c)). In case $R=0$, G is a divisible p-group and then a copy of $\mathbb{Z}(p^\infty)$ belongs to $\mathcal{C}$ since every divisible p-group is a direct sum of copies of $\mathbb{Z}(p^\infty)$, as we referred to in 1.13(b). Otherwise $pR\neq R$. Indeed, for any prime number q, distinct of p, we have $qG=G$ since G is a p-group, so $G=qD\oplus qR=D\oplus qR$, which implies $qR=R$. Hence if pR is equal to R, R is divisible, i.e. $R=0$, a contradiction. Therefore $pR\neq R$, and the projection $R\longrightarrow R/pR$ is a non-zero map. Since every non-zero element of R/pR has order p, R/pR may be written as a direct sum in which all summands are isomorphic to $\mathbb{Z}_p$. Thus, in this case, $\mathcal{C}$ contains a copy of $\mathbb{Z}_p$.

These facts enable us to define the following map:

$$\begin{aligned}\Delta : U &\longrightarrow N\\ \mathcal{C} &\longmapsto \prod_{p\in P} p^{e(p)}\end{aligned}\qquad \text{where } e(p)=\begin{cases}0 & \text{if } \mathcal{C} \text{ does not contain a non-zero p-group.}\\ 1 & \text{if } \mathcal{C} \text{ contains a copy of } \mathbb{Z}_p.\\ \infty & \text{if } \mathcal{C} \text{ contains a copy of } \mathbb{Z}(p^\infty) \text{ and}\\ & \quad\text{does not contain a copy of } \mathbb{Z}p.\end{cases}$$

<u>THEOREM 3.1</u>. *The map Δ is a bijection.*

<u>Proof</u>: Consider the map

$$\begin{aligned}\Omega :\ N &\longrightarrow U\\ \coprod_{p\in P} p^{e(p)} &\longmapsto \mathcal{G}^{\rightarrow\leftarrow}\end{aligned}$$

with $\mathcal{G}=\{G_p \mid p\in P\}$ and where, for every $p\in P$, G_p is equal to 0, $\mathbb{Z}_p$ or $\mathbb{Z}(p^\infty)$, whenever $e(p)$ is 0, 1 or ∞, respectively. For every $\mathcal{C}\in U$,

$$\Omega\Delta(\mathcal{C})=\Omega\Big(\prod_{p\in P} p^{e(p)}\Big)=\mathcal{G}^{\rightarrow\leftarrow}.$$

First we show that $\Omega\Delta=\mathrm{Id}_U$, i.e. $\mathcal{C}=\mathcal{G}^{\rightarrow\leftarrow}$.

The inclusion $\mathcal{G}^{\rightarrow\leftarrow}\subset\mathcal{C}$ holds trivially because $\mathcal{G}\subset\mathcal{C}$, $\mathcal{C}$ is a torsion subcategory and $\mathcal{G}^{\rightarrow\leftarrow}$ is the smallest torsion subcategory that contains $\mathcal{G}$. We now prove the reverse inclusion. Let $C\in\mathcal{C}$. The primary decomposition theorem can be applied to C because $\mathcal{C}\subset\mathcal{T}ors$, and so we have $C=\coprod_{p\in P} C_p$, where each C_p is a p-group. It is obvious that each C_p belongs to $\mathcal{C}$. In order to prove that $C\in\mathcal{G}^{\rightarrow\leftarrow}$ it suffices to show that each C_p belongs to $\mathcal{G}^{\rightarrow\leftarrow}$ because this subcategory is closed under coproducts. Since C_p is a p-group then C_p belongs to $\{\mathbb{Z}(p^\infty)\}^{\rightarrow\leftarrow}$ if it is divisible or belongs to $\{\mathbb{Z}_p\}^{\rightarrow\leftarrow}$ otherwise (recall 1.13(a) and (b)). Thus $C_p\in\mathcal{G}^{\rightarrow\leftarrow}$, as we claimed and so $\Omega\Delta=\mathrm{Id}_U$.

Furthermore, $\Delta\Omega = \mathrm{Id}_N$. Indeed, for every Steinitz number $\prod_{p \in P} p^{e(p)}$ we have

$$\Delta\Omega\Big(\prod_{p \in P} p^{e(p)}\Big) = \Delta(\mathcal{G}^{\rightarrow\leftarrow}).$$

Since $\{\mathbb{Z}_p\}^{\rightarrow\leftarrow} = \{\text{p-groups}\}$ and $\{\mathbb{Z}(p^\infty)\}^{\rightarrow\leftarrow} = \{\text{divisible p-groups}\}$ (1.13 (a) (b)), it is obvious that

$$\mathcal{G}^{\rightarrow\leftarrow} = \Big(\bigcup_{p \in P} \mathcal{C}_p\Big)^{\rightarrow\leftarrow}$$

where $\mathcal{C}_p$, for every $p \in P$, is the subcategory $\{0\}$, the subcategory of p-groups or the subcategory of divisible p-group, whenever $e(p)$ is equal to 0, 1 or ∞, respectively. But by ([15], 2.5), $G \in \mathcal{G}^{\rightarrow\leftarrow}$ if and only if $G = \coprod_{p \in P} G_{\mathcal{C}_p}$, where $G_{\mathcal{C}_p}$ denotes the maximal subgroup of G that belongs to $\mathcal{C}p$.

Then, it is easy to check that

$$\Delta(\mathcal{G}^{\rightarrow\leftarrow}) = \prod_{p \in P} p^{e(p)},$$

and so $\Delta\Omega = \mathrm{Id}_N$. □

Finally, we present the classification of the hereditary torsion subcategories of $\mathcal{A}b$. We say that a torsion subcategory of $\mathcal{A}b$ is <u>proper</u> if it does not contain all abelian groups.

<u>THEOREM 3.2</u>. *The map* Δ *defines a one-to-one correspondence between the set* U' *of the proper hereditary torsion subcategories of* $\mathcal{A}b$ *and the set* N' *of the Steinitz numbers for which* $e(p)$ *is equal to* 0 *or* 1, *for every* $p \in P$.

<u>Proof</u>: Let $\mathcal{C} \in U'$. If $\mathcal{C}$ contains a group G with an infinite order element then $\mathbb{Z}$ belongs to $\mathcal{C}$, because G has a subgroup isomorphic to $\mathbb{Z}$. Then $\mathcal{C} = \mathcal{A}b$, since any group is an homomorphic image of a direct sum of copies of $\mathbb{Z}$. Hence $\mathcal{C} \subset \mathcal{T}ors$ and so $U' \subset U$. Therefore, by 3.1 it suffices to prove that $\Delta(U') = N'$.

<u>$\Delta(U') \subset N'$</u>: Consider

$$\prod_{p \in P} p^{e(p)} = \Delta(\mathcal{C}), \text{ with } \mathcal{C} \in U'.$$

If for some $p \in P$, $e(p) = \infty$, then $\mathbb{Z}(p^\infty) \in \mathcal{C}$. Since $\mathbb{Z}(p^\infty)$ has a subgroup isomorphic to $\mathbb{Z}_p$ and this subgroup belongs to $\mathcal{C}$, we have that $e(p) = 1$, which is contradictory. Thus $e(p)$ is 0 or 1, for every $p \in P$.

<u>$N' \subset \Delta(U')$</u>: Let $\prod_{p \in P} p^{e(p)} \in N'$. By 3.1, there is a $\mathcal{C} \in U$ such that

$$\prod_{p \in P} p^{e(p)} = \Delta(\mathcal{C}).$$

We show that $\mathcal{C} \in U'$. As we observed in 3.1, $\mathcal{C} = \mathcal{G}^{\rightarrow\leftarrow}$, with $\mathcal{G} = \{G_p \mid p \in P\}$ where, for every $p \in P$, G_p is equal to 0 or $\mathbb{Z}_p$, if $e(p)$ is equal to 0 or 1, respectively. Just, as in 3.1, it is obvious that

$$\mathcal{G}^{\rightarrow\leftarrow} = \Big(\bigcup_{p \in P} \mathcal{C}_p\Big)^{\rightarrow\leftarrow}$$

where $\mathcal{C}_p$, for every $p \in P$, is the subcategory $\{0\}$ or the subcategory of p-groups, as $e(p)$ is equal to 0 or 1, respectively. But, since each $\mathcal{C}_p$ is closed under subobjects and quotient objects,

$$\bigcup_{p \in P} \mathcal{C}_p$$

also verifies these conditions and so $\mathcal{G}^{\rightarrow\leftarrow} = \mathcal{C}$ is hereditary, by 2.5. □