CHAPTER III

LOCALIZATIONS AND TORSION THEORIES

1. CHARACTERIZATION OF SOME CLASSES OF REFLECTORS

We have already observed in II.3.4 that if $\mathcal{A}$ is (FWC) and $\mathfrak{B}$ is a reflective subcategory of $\mathcal{A}$ then $\Phi\mathfrak{B}$ is a factorization system. However if we drop the condition relative to the existence of intersections of strong monomorphisms then $\Phi\mathfrak{B}$ will be a factorization system only for some special type of reflective subcategories. In this section we will do a brief digression through some of these reflectors, giving particular attention to the localizations, whose torsion theory has special interesting properties if it exists, and the reflectors with stable units defined in [11], which we develop here a little more.

Let $\mathcal{A}$ be a finitely complete category, and $(r_A : A \longrightarrow R(A))_{A \in \mathrm{Obj}\mathcal{A}}$ the class of reflection morphisms from $\mathcal{A}$ to a subcategory $\mathfrak{B}$ We denote by $\mathfrak{F} = (\mathcal{E}, \mathcal{M})$ the prefactorization system $\Phi\mathfrak{B}$. Throughout this section, for each $A \in \mathrm{Obj}\mathcal{A}$ and each $g : B \longrightarrow R(A)$ with $B \in \mathrm{Obj}\mathfrak{B}$, the diagram

$$\begin{tikzcd} P(g) \arrow[r, "g_1"] \arrow[d, "g_0"'] & A \arrow[d, "r_A"] \\ B \arrow[r, "g"'] & R(A) \end{tikzcd}$$

represents the pullback of g along r_A.

It is clear that given a morphism $f : A \longrightarrow A'$ there exists a unique morphism $\bar{f}$ such that the following diagram

$$\begin{tikzcd}
A \arrow[dr, dashed, "\bar{f}"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "r_A"'] & & \\
& P(R(f)) \arrow[r, "R(f)_1"] \arrow[d, "R(f)_0"'] & A' \arrow[d, "r_{A'}"] \\
& R(A) \arrow[r, "R(f)"'] & R(A')
\end{tikzcd}$$

commutes.

DEFINITION 1.1. The reflector R is called simple if $\bar{f} \in \mathcal{E}$ for each $\mathcal{A}$-morphism f.

Since $R(f) \in \mathrm{Mor}\,\mathcal{B} \subset \mathcal{M}$ then $R(f)_1 \in \mathcal{M}$ so, if the reflector is simple, $\mathcal{F}$ is a factorization system.

PROPOSITION 1.2. *The following assertions are equivalent:*

(i) *The reflector* R *is simple.*

(ii) $f: A \longrightarrow A' \in \mathcal{M}$ *if and only if* (A, r_A, f) *is the pullback of* $(R(f), r_{A'})$.

(iii) *For each* $g: B \longrightarrow R(A)$ *in* $\mathcal{B}$, $(P(g), r_{P(g)}, 1)$ *is the pullback of* $(g_0, R(g_0))$.

(iv) *For each* $g: B \longrightarrow R(A)$ *in* $\mathcal{B}$, *if* $R(g_0)$ *is a split epimorphism then it is an isomorphism.*

Proof: See ([11], 4.1). □

REMARK 1.3. When $\mathcal{A}$ is preadittive the conditions (i)-(iv) are equivalent to

(v) For each $g: B \longrightarrow R(A)$ in $\mathcal{B}$, the morphism $R(g_0)$ is a monomorphism.

In fact, it is obvious that the implication ((v) $\Rightarrow$ (iv)) is always true. In preadittive categories ((iii) $\Rightarrow$ (v)) holds, because m is a monomorphism if and only if $\ker m = 0$.

PROPOSITION 1.4. *The following conditions are equivalent and imply that* R *is simple*:

(i) *For each* $g: B \longrightarrow R(A)$ *in* $\mathfrak{B}$, $R(g_0)$ *is an isomorphism.*

(ii) *The reflector* R *preserves the pullback of any pair of morphisms provided that one of them belongs to* $\mathcal{M}$.

(iii) *Any pullback of a morphism of* $\mathcal{E}$ *along a morphism of* $\mathcal{M}$ *lies in* $\mathcal{E}$.

Proof: See ([11], 4.3). □

DEFINITIONS 1.5. (1) A reflector is said to be semi-left-exact provided that it satisfies one and then all the equivalent conditions of 1.4.

(2) We say that a reflector has stable units if the pullback of any reflection morphism belongs to $\mathcal{E}$.

Denote by $S(\mathfrak{B})$ the full subcategory of $\mathcal{A}$ defined by the objects X whose reflection r_X is a monomorphism. By a pullback over an object of $S(\mathfrak{B})$ we mean a pullback of two morphisms whose common codomain is an object of $S(\mathfrak{B})$.

PROPOSITION 1.6. *The following assertions are equivalent and imply that the reflector* R *is semi-left-exact.*

(i) R *has stable units.*

(ii) $\mathcal{E}$ *is stable under pullbacks over objects of* $S(\mathfrak{B})$.

(iii) R *preserves pullbacks over objects of* $S(\mathfrak{B})$.

(iv) R *preserves finite products and equalizers of pairs of morphisms with codomain in* $S(\mathfrak{B})$.

Proof: (i) implies 1.4(i) and, therefore, that R is semi-left-exact.

(i) ⇒ (ii): Consider a pullback

$$\begin{tikzcd} D \arrow[r, "\bar{e}"] \arrow[d, "\bar{f}"'] & C \arrow[d, "f"] \\ A \arrow[r, "e"'] & X \end{tikzcd}$$

where $e \in \mathcal{E}$ and $X \in S(\mathcal{B})$. It is obvious that $(D,\bar{e},\bar{f})$ is also the pullback of $(r_X.e, r_X.f)$, since r_X is a monomorphism. But Re is an isomorphism and $r_X.e = Re.r_A$ hence, by hypothesis, $\bar{e} \in \mathcal{E}$.

<u>(ii) ⇒ (iii)</u>: Let

$$\begin{tikzcd} D \arrow[r, "g'"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\ A \arrow[r, "g"'] & X \end{tikzcd} \qquad (1)$$

be a pullback with $X \in S(\mathcal{B})$.

Consider the pullback $(P,\bar{g},\bar{f})$ of Rg and Rf. Then there exists a unique morphism t such that $\bar{g}.t = Rg'$ and $\bar{f}.t = Rf'$.

$$\begin{tikzcd} R(D) \arrow[dr, dashed, "t"] \arrow[drr, "Rg'"] \arrow[ddr, "Rf'"'] & & \\ & P \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & R(C) \arrow[d, "Rf"] \\ & R(A) \arrow[r, "Rg"'] & R(X) \end{tikzcd} \qquad (2)$$

To prove (iii) it suffices to check that t is an isomorphism.

By definition of reflection morphism, there exist unique morphisms g_1 and f_1 such that $g_1.r_P = \bar{g}$ and $f_1.r_P = \bar{f}$. According to this fact and since (2) is a pullback, it is obvious that r_P is a monomorphism, i.e. $P \in S(\mathcal{B})$.

Now we prove that in the following bipullback (see [34]),

$$\begin{tikzcd} A \arrow[d, "r_A"'] & K \arrow[l, "a"'] \arrow[r, "c"] \arrow[d, "u"] & C \arrow[d, "r_C"] \\ R(A) & P \arrow[l, "\bar{f}"] \arrow[r, "\bar{g}"'] & R(C) \end{tikzcd} \qquad (3)$$

$u \in \mathcal{E}$.

Note that a bipullback is the limit of a diagram of the following form

$$\begin{tikzcd}
\cdot \arrow[d] & & \cdot \arrow[d] \\
\cdot & \cdot \arrow[l] \arrow[r] & \cdot
\end{tikzcd}$$

But, as can be easily observed, every bipullback can be constructed from three pullbacks. In particular, the bipullback above can be obtained from the three following pullbacks

$$\begin{tikzcd}
 & & K \arrow[dl, "r'_C"'] \arrow[dr, "r'_A"] & & \\
A \arrow[d, "r_A"'] & \cdot \arrow[l, "p"'] \arrow[dr, "\bar{r}_A"] & & \cdot \arrow[r, "q"] \arrow[dl, "\bar{r}_C"'] & C \arrow[d, "r_C"] \\
R(A) & & P \arrow[ll, "\bar{f}"] \arrow[rr, "\bar{g}"'] & & R(C)
\end{tikzcd}$$

taking $u = \bar{r}_A . r'_C = \bar{r}_C . r'_A$, $a = p . r'_C$ and $c = q . r'_A$.

By hypothesis, $\bar{r}_A, \bar{r}_C \in \mathcal{E}$, and we already know that $P \in S(\mathfrak{B})$, hence $r'_A, r'_C \in \mathcal{E}$. Since $\mathcal{E}$ is closed under composition we can conclude that $u \in \mathcal{E}$, as required.

Since $r_X . g . a = Rg . r_A . a = Rg . \bar{f} . u = Rf . \bar{g} . u = Rf . r_C . c = r_X . f . c$ and r_X is a monomorphism we conclude that $g . a = f . c$. Therefore, in (1) there exists a unique morphism d such that $f' . d = a$ and $g' . d = c$:

$$\begin{tikzcd}
K \arrow[dr, dashed, "d"] \arrow[drr, "c"] \arrow[ddr, "a"'] & & \\
 & D \arrow[r, "g'"] \arrow[d, "f'"'] & C \arrow[d, "f"] \\
 & A \arrow[r, "g"'] & X
\end{tikzcd}$$

By lemma II.1.14, $\mathcal{E}^{\perp}=\mathcal{B}$, so there exists a unique morphism t' such that the following diagram is commutative:

$$\begin{tikzcd}
K \arrow[r, "u"] \arrow[d, "d"'] & P \arrow[d, dashed, "t'"] \\
D \arrow[r, "r_D"'] & R(D)
\end{tikzcd}$$

We assert that $t.t'=1$ and $t'.t=1$. In order to prove it we begin by observing that d is an isomorphism.

Since (2) is a pullback and $Rf.r_C.g'=r_X.f.g'=r_X.g.f'=Rg.r_A.f'$, there exists a unique morphism $s:D\longrightarrow P$ such that $\bar{f}.s=r_A.f'$ e $\bar{g}.s=r_C.g'$.

But $r_A.f'=Rf'.r_D=\bar{f}.t.r_D$ and $r_C.g'=Rg'.r_D=\bar{g}.t.r_D$ so $s=t.r_D$.

Consequently, in the bipullback (3) there exists a unique morphism m such that $u.m=s$, $a.m=f'$ and $c.m=g'$.

$$\begin{tikzcd}
 & D \arrow[dl, "f'"'] \arrow[d, dashed, "m"] \arrow[dr, "g'"] \arrow[dd, bend right, "s"'] & \\
A \arrow[d, "r_A"'] & K \arrow[l, "a"'] \arrow[r, "c"] \arrow[d, "u"] & C \arrow[d, "r_C"] \\
R(A) & P \arrow[l, "\bar{f}"] \arrow[r, "\bar{g}"'] & R(C)
\end{tikzcd}$$

From $f'.d.m=a.m=f'$ and $g'.d.m=c.m=g'$ we obtain $d.m=1$. On the other hand,

$$\bar{f}.u.m.d=\bar{f}.s.d=\bar{f}.t.r_D.d=Rf'.r_D.d=r_A.f'.d=r_A.a=\bar{f}.u \quad \text{and}$$
$$\bar{g}.u.m.d=\bar{g}.s.d=\bar{g}.t.r_D.d=Rg'.r_D.d=r_C.g'.d=r_C.c=\bar{g}.u$$

implies that $u.m.d=u$, since $\{\bar{f},\bar{g}\}$ is a monosource.

From the equalities

- $c.m.d=g'.d=c$
- $a.m.d=f'.d=a$
- $u.m.d=u$

we conclude, by definition of bipullback, that $m.d=1$ and, therefore, d is an isomorphism. Then $t'.t.r_D=t'.u.m=r_D.d.m=r_D$ which implies that $t'.t=1$. But $Rf'.t'.u=Rf'.r_D.d=\bar{f}.t.r_D.d=\bar{f}.u.m.d=\bar{f}.u$. Moreover $Rf'.t'.u=Rf'.r_D.d=r_A.f'.d$ which implies that $Rf'.t'=\bar{f}$, because $u\in\mathcal{E}$ and $R(A)\in\mathcal{E}^{\perp}$. In an analogous way we infer that $Rg'.t'=\bar{g}$. But then

- $\bar{f}.t.t'=Rf'.t'=\bar{f}$
- $\bar{g}.t.t'=Rg'.t'=\bar{g}$

whence $t.t'=1$ since (2) is a pullback.

<u>(iii) ⇒ (iv)</u>: To prove that R preserves finite products it suffices to observe that if $(A\times C,p_1,p_2)$ is the product of two objects A and C then

$$\begin{tikzcd} A\times C \arrow[r, "p_2"] \arrow[d, "p_1"'] & C \arrow[d] \\ A \arrow[r] & T \end{tikzcd}$$

is a pullback and that $T\in\mathcal{B}\subset S(\mathcal{B})$.

Finally, if $A\xrightarrow{h}B\underset{g}{\overset{f}{\rightrightarrows}}X$ is an equalizer with $X\in S(\mathcal{B})$ then

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "f.h=g.h"'] & B \arrow[d, "(f,g)"] \\ X \arrow[r, "\Delta"'] & X\times X \end{tikzcd}$$

is a pullback with $\Delta=(1_X,1_X)$, using the notation of ([36], p. 69).

Consider the products $(X\times X,\pi_1,\pi_2)$ and $(R(X)\times R(X),p_1,p_2)$, and the morphism $(r_X)^2:=(r_X.\pi_1,r_X.\pi_2)$. The fact that r_X is a monomorphism implies, immediately, that $(r_X)^2$ is also a monomorphism. But it is clear that there exists a morphism t such that $t.r_{X\times X}=(r_X)^2$. Hence $r_{X\times X}$ is a monomorphism, i.e. $X\times X\in S(\mathcal{B})$, and then the pullback fulfils the conditions of hypothesis. Therefore Rh is the equalizer of Rf and Rg.

<u>(iv) ⇒ (i)</u>: Clearly ((iv) ⇒ (iii)) since a pullback

$$\begin{tikzcd} P \arrow[r, "\bar{g}"] \arrow[d, "\bar{f}"'] & C \arrow[d, "f"] \\ A \arrow[r, "g"'] & X \end{tikzcd}$$

is the equalizer of $A \times C \underset{g.p_1}{\overset{f.p_2}{\rightrightarrows}} X$, where p_1 and p_2 are the projections of $A \times C$ in A and C, respectively.

On the other hand, it is obvious that ((ii) $\Rightarrow$ (i)). Therefore, it remains to prove the implication ((iii) $\Rightarrow$ (ii)). So, consider a pullback

$$\begin{tikzcd} D \arrow[r, "\bar{e}"] \arrow[d, "\bar{f}"'] & C \arrow[d, "f"] \\ A \arrow[r, "e"'] & X \end{tikzcd}$$

with $e \in \mathcal{E}$ and $X \in S(\mathcal{B})$. Let us prove that $\bar{e} \in \mathcal{E}$.

By hypothesis $(RD, R\bar{e}, R\bar{f})$ is the pullback of Re and Rf. Hence there exists a unique morphism t such that the following diagram is commutative:

$$\begin{tikzcd} C \arrow[dr, dashed, "t"] \arrow[drr, "r_C"] \arrow[ddr, "(Re)^{-1}.Rf.r_C"'] & & \\ & R(D) \arrow[r, "R\bar{e}"] \arrow[d, "R\bar{f}"'] & R(C) \arrow[d, "Rf"] \\ & R(A) \arrow[r, "Re"'] & R(X) \end{tikzcd}$$

By definition of reflection morphism there exists a unique morphism $\bar{t}$ such that $\bar{t}.r_C = t$. Then $R\bar{e}.\bar{t}.r_C = R\bar{e}.t = r_C$ and so $R\bar{e}.\bar{t} = 1$.

We have also that

- $R\bar{e}.t.\bar{e} = r_C.\bar{e} = R\bar{e}.r_D$

and

- $R\bar{f}.t.\bar{e} = (Re)^{-1}.Rf.r_C.\bar{e} = (Re)^{-1}.Rf.R\bar{e}.r_D = R\bar{f}.r_D$.

Thus $t.\bar{e} = r_D$ which implies that $r_D = \bar{t}.r_C.\bar{e} = \bar{t}.R\bar{e}.r_D$, so $\bar{t}.R\bar{e} = 1$. In conclusion, $R\bar{e}$ is an isomorphism, i.e. $\bar{e} \in \mathcal{E}$. □

DEFINITION 1.7. We say that a reflector is a localization if it is left exact, i.e. if it preserves finite limits.

In the sequel a reflective subcategory of $\mathcal{A}$ such that the reflector is a localization will be called a localizing subcategory of $\mathcal{A}$.

EXAMPLE 1.8. The only localizing subcategories of $\mathcal{T}op$ are $\mathcal{T}op$ itself, the subcategory $\mathcal{I}nd$ of indiscrete topological spaces and $\mathcal{T}er$. It is obvious that $\mathcal{T}op$ and $\mathcal{T}er$ are localizing. Let $\mathcal{B}$ be a localizing subcategory of $\mathcal{T}op$ different from $\mathcal{T}op$ and $\mathcal{T}er$. The corresponding reflector is a monofunctor, because being left exact it preserves monomorphisms. The subcategory $\mathcal{B}$ contains a space B with at least two points b_1 and b_2, since $\mathcal{B} \neq \mathcal{T}er$. Let us show that $\mathcal{B}$ is monoreflective. If D is a discrete space, $r_D : D \longrightarrow R(D)$ is a monomorphism: if $d_1, d_2 \in D$ and $f : D \longrightarrow B$ is such that $f(d_1) \neq f(d_2)$, then there exists a morphism $\bar{f}$ such that $\bar{f}.r_D = f$ and consequently d_1 and d_2 have different images by r_D. For any space X, consider the corresponding discrete space D with the same underlying set and the identity map $i : D \longrightarrow X$. Then $r_X.i = R(i).r_D$ and, since R is a monofunctor, $R(i)$ is a monomorphism and the same occurs with $R(i).r_D$. Thus r_X is a monomorphism because i is bijective.

By the dual of ([26], Prop. 1) $\mathcal{B}$ is bireflective and so, according to ([38], Theorem 1) it contains $\mathcal{I}nd$. But the Sierpiński space S does not belong to $\mathcal{B}$ because $\mathcal{T}op$ is the bireflective hull of S. Then it is obvious that the indiscrete space with two points, I_2, is the reflection of S in $\mathcal{B}$. Since the reflector preserves products, the reflection of $S \times S$ in $\mathcal{B}$ is $I_2 \times I_2 = I_4$. Now we shall see that this fact allows us to conclude that the reflection of the discrete space D_2 with two points is equal to I_2. Let us denote by m the map $D_2 \longrightarrow S \times S$ defined by $m(d_1) = (0,1)$ and $m(d_2) = (1,0)$, assuming that $\{1\}$ is the non-trivial open set of S. This map is injective, continuous and $m^{-1}|_{m(D_2)}$ is continuous. Therefore, m is a regular monomorphism since it is an embedding. Then $R(m)$ is a regular monomorphism, because R preserves equalizers. In the following diagram

$$\begin{tikzcd} D_2 \arrow[r, "m"] \arrow[d, "r_{D_2}"'] & SxS \arrow[d, "r_{SxS}"] \\ R(D_2) \arrow[r, "R(m)"'] & R(SxS)=I_4 \end{tikzcd}$$

$R(D_2)$ has two elements since the reflection morphism r_{D_2} is a bijection, and we obtain a homeomorphism by corestricting it to a subspace of I_4, since $R(m)$ is an embedding. Hence, $R(D_2)=I_2$.

Finally, we prove that $\mathcal{B} \subset \mathcal{I}nd$. Assume that for some $X \in \mathcal{B}$ there exists an open set U of X such that $U \neq \emptyset$ and $U \neq X$. If we consider the map $f: D_2 \longrightarrow X$ such that $f^{-1}(U)=\{d_1\}$ we conclude that there is a morphism $\bar{f}: I_2 \longrightarrow X$ such that $\bar{f}.r_{D_2}=f$, which is impossible, as can be immediately observed.

PROPOSITION 1.9. *The following conditions are equivalent and imply that* R *has stable units*:

(i) R *is a localization.*

(ii) $\mathcal{E}$ *is stable under pullbacks.*

Proof: (i) $\Rightarrow$ (ii): It is trivial, since $\mathcal{E}=\Sigma_R$.

(ii) $\Rightarrow$ (i): It suffices to prove that R preserves pullbacks because the terminal object, since it belongs to $\mathcal{B}$, is preserved by R.

It is clear that R has stable units so $\Phi\mathcal{B}=(\mathcal{E},\mathcal{M})$ is a factorization system. Given $f: A \longrightarrow B$ and $h: C \longrightarrow B$ let $(P,\bar{f},\bar{h})$ be their pullback. Consider the $(\mathcal{E},\mathcal{M})$-factorizations, $f=m_1.e_1$ and $h=m_2.e_2$. It is obvious that the following diagram is commutative:

$$\begin{tikzcd} & P=P_4 \arrow[r, "e_1''"] \arrow[d, "e_2''"'] \arrow[dd, bend right=60, "\bar{h}"'] \arrow[rr, bend left=60, "\bar{f}"] \arrow[dr, phantom, "(4)"] & P_2 \arrow[r, "m_1''"] \arrow[d, "e_2'"] \arrow[dr, phantom, "(2)"] & C \arrow[d, "e_2"] \\ & P_3 \arrow[r, "e_1'"'] \arrow[d, "m_2''"'] \arrow[dr, phantom, "(3)"] & P_1 \arrow[r, "m_1'"'] \arrow[d, "m_2'"] \arrow[dr, phantom, "(1)"] & \bar{C} \arrow[d, "m_2"] \\ & A \arrow[r, "e_1"'] & \bar{A} \arrow[r, "m_1"'] & B \end{tikzcd}$$

where each square (1), (2), (3) and (4) is a pullback.

By 1.4(ii) R preserves (1), (2) and (3) and, by hypothesis, $e_1', e_2', e_1'', e_2'' \in \mathcal{E} = \Sigma_R$ thus R also preserves (4). Therefore R preserves the outer pullback. □

REMARK 1.10. In Theorem II.3.5, if F is left exact then it is obvious that Q is left exact too. Using the characterization 1.9 it is possible to prove that, in these conditions, R is also left exact. Indeed, if $(\mathcal{E},\mathcal{M})$ is the factorization system $(G(\mathrm{Mor}\mathcal{C})^{\uparrow}, G(\mathrm{Mor}\mathcal{C})^{\uparrow\downarrow})$ associated with the adjunction $F \dashv G$, we know that $\mathcal{E} = \Sigma_F$. Since $\mathcal{B} = \mathcal{M}/T$, the factorization system $\Phi\mathcal{B}$ is equal to $\Phi\Psi(\mathcal{E},\mathcal{M}) = (\overset{\circ}{\mathcal{E}},\overset{\circ}{\mathcal{M}})$ and $\overset{\circ}{\mathcal{E}} = \Sigma_R$. But we saw in the proof of II.3.5 that $(\mathcal{E},\mathcal{M})$ is reflective, thus $\overset{\circ}{\mathcal{E}} = \mathcal{E}$ and since F preserves pullbacks it is obvious that R satisfies 1.9 (ii).

These facts imply that Theorem II.3.5 of B. Day generalizes a result of Lawvere-Tierney (cf. [32], Theorem 4.14) about factorizations of geometric morphisms between toposes.

DEFINITION 1.11. A factorization system $(\mathcal{E},\mathcal{M})$ is called local if it is reflective and the class $\mathcal{E}$ is stable under pullbacks.

As a corollary of 1.9 and of some results in II.1 we have the following:

COROLLARY 1.12. *In a finitely complete category there exists a one-to-one correspondence between the localizations and the local factorization systems.* □

We can derive from the preceding results the following chain of implications:

R is a localization $\Rightarrow$ R has stable units $\Rightarrow$ R is semi-left-exact $\Rightarrow$ R is simple $\Rightarrow$ $\Phi\mathcal{B}$ is a factorization system.

The following examples show that the converse of each one of these implications is false:

(a) The reflector from $\mathcal{G}rp$ to $\mathcal{A}b$ which sends each group G to the quotient of G by its commutator subgroup is not simple because 1.2(iv) is not true if A is the symmetric group of order three and $B = 0$. However, $\Phi\mathcal{B}$ is a factorization system since $\mathcal{G}rp$ is a (FWC)-category.

(b) The reflector from category $\mathcal{A}_3$ of II.4.3 (d) to subcategory $\{a,b,1\}$ is simple but

is not semi-left-exact:

Indeed, the pullback of $c \longrightarrow 1 \in \mathcal{E}$ along $b \longrightarrow 1 \in \mathcal{M}$ is equal to $0 \longrightarrow b$ which does not belong to $\mathcal{E}$.

(c) Let G be a group with a cardinal greater than one and consider the category $\mathcal{A}$ of G-sets whose objects are the pairs $(X,\cdot)$, where X is a set and $\cdot : G \times X \longrightarrow X$ is an action of G in X verifying $1 \cdot x = x$ and $(gg') \cdot x = g \cdot (g' \cdot x)$ for every $g, g' \in G$ and every $x \in X$, and whose morphisms are the maps which preserve the actions. A G-set $(X,\cdot)$ is called <u>discrete</u> if $g \cdot x = x$, for every pair $(g,x) \in G \times X$. In each G-set, $(X,\cdot)$, let us define the following equivalence relation:

$$x \sim x' \text{ if and only if there exists a } g \in G \text{ such that } x = g \cdot x'.$$

Of course, $X/\sim$ is a G-set for the action $g \cdot [x] = [g \cdot x]$. If $\mathcal{B}$ denotes the subcategory of discrete G-sets, the functor

$$\begin{aligned} R : \mathcal{A} &\longrightarrow \mathcal{B} \\ X &\longmapsto X/\sim \\ (f : X \longrightarrow Y) &\longmapsto (X/\sim \longrightarrow Y/\sim) \\ & \qquad\quad [x] \longmapsto [f(x)] \end{aligned}$$

is a reflector from $\mathcal{A}$ to $\mathcal{B}$. This reflector is semi-left-exact since it satisfies 1.4(i). However, it does not have stable units because, for the G-set G whose action is defined by the group operation, the pullback of $r_G : G \longrightarrow \{e_G\}$ along r_G does not belong to $\mathcal{E}$.

(d) Consider the category $\mathcal{T}op$ and the subcategory $\mathcal{T}op_0$ of T_0-spaces. The reflector R from $\mathcal{T}op$ to $\mathcal{T}op_0$ which sends each topological space (X,τ_X) to $X/\sim$, with quotient topology, where $\sim$ is the equivalence relation defined by

$$x \sim y \text{ if and only if } x \in \overline{\{y\}} \text{ and } y \in \overline{\{x\}},$$

which is also equivalent to

$$\forall\, U \in \tau_X,\ x \in U \Leftrightarrow y \in U,$$

is not a localization as we remarked in 1.8. The same conclusion follows from the fact that this reflector does not preserve monomorphisms. Indeed, it suffices to consider a monomorphism from a T_0-space X with cardinal greater than one into a topological space Y with the indiscrete topology, because $R(X) = X$ and $R(Y) \cong \{*\}$ where $\{*\}$ is the terminal object of $\mathcal{T}op$.

On the other hand, R has stable units. Indeed, consider the pullback

$$\begin{tikzcd} P \arrow[r, "\bar{r}"] \arrow[d, "\bar{f}"'] & Y \arrow[d, "f"] \\ X \arrow[r, "r_X"'] & X/\sim \end{tikzcd}$$

i.e. P is the set $\{(x,y) \in X \times Y \mid r_X(x) = f(y)\}$, with the subspace topology with respect to the product, and $\bar{r}$ and $\bar{f}$ are the projections to Y and X, respectively. Let us check that

$$\begin{aligned} R\bar{r} : P/\sim &\longrightarrow Y/\sim \\ [(x,y)] &\longmapsto [y] \end{aligned}$$

is an isomorphism.

First we note that $\bar{r}$ is surjective. Indeed, for each $y \in Y$, if we consider $x \in X$ such that $r_X(x) = f(y)$ then $\bar{r}((x,y)) = y$. Now let us prove that $\bar{r}$ is open. Denoting the projections from $X \times Y$ to X and Y by p_X and p_Y, respectively, the collection of all sets $p_X(V_i) \cap p_Y(W_j)$, where V_i ranges over all members of τ_X and W_j over all members of τ_Y, is a basis for the topology of $X \times Y$. So, the collection of all sets $P \cap p_X^{-1}(V_i) \cap p_Y^{-1}(W_j)$, for each $V_i \in \tau_X$ and $W_j \in \tau_Y$, is a basis for the topology of P. Let $U = P \cap p_X^{-1}(V) \cap p_Y^{-1}(W)$, for some open set V of X and some open set W of Y. To prove that $\bar{r}(U)$ is an open set of Y it is enough to observe that $\bar{r}(U) = f^{-1}(r_X(V)) \cap W$, which is an open set of Y because r_X is open, being $r_X^{-1}(r_X(V)) = V$ for every $V \in \tau_X$.

Hence, the morphism $\bar{r}$ is open and so it is an extremal epimorphism. But then $R\bar{r}$ is an extremal epimorphism since $R\bar{r}.r_P = r_Y.\bar{r}$ is an extremal epimorphism. Thus, to conclude that $R\bar{r}$ is an isomorphism, it suffices to prove that it is a monomorphism.

Let $[(x_0,y_0)],[(x_1,y_1)] \in P/\sim$ such that $R\bar{r}([(x_0,y_0)]) = R\bar{r}([(x_1,y_1)])$. But $R\bar{r}.r_P = r_Y.\bar{r}$. Then r_Y equalizes $\bar{r}((x_0,y_0))$ and $\bar{r}((x_1,y_1))$, i.e. $r_Y(y_0) = r_Y(y_1)$ and so $g.r_Y(y_0) = g.r_Y(y_1)$, where g denotes the unique morphism such that $g.r_Y = f$, whose existence is guaranteed by definition of reflection morphism. Thus $r_Y(y_0) = r_Y(y_1)$ and $f(y_0) = f(y_1)$ and, consequently, $r_X(x_0) = r_X(x_1)$, since $(x_0,y_0),(x_1,y_1) \in P$. Let us prove that $[(x_0,y_0)] = [(x_1,y_1)]$, i.e. $(x_1,y_1) \in \overline{\{(x_0,y_0)\}}$ and $(x_0,y_0) \in \overline{\{(x_1,y_1)\}}$. If (x_1,y_1) would not belong to $\overline{\{(x_0,y_0)\}}$, then, for some open set U of P, $(x_1,y_1) \in U$ and $(x_0,y_0) \notin U$. The open set U is a union of open sets of type $\bar{f}^{-1}(V) \cap \bar{r}^{-1}(W)$ where V is an open set of X and W an open set of Y. Without loss of generality, we can assume that $U = \bar{f}^{-1}(V) \cap \bar{r}^{-1}(W)$. Then $x_1 \in V$ and $y_1 \in W$ since $(x_1,y_1) \in \bar{f}^{-1}(V) \cap \bar{r}^{-1}(W)$, and

$x_0 \in V$ *and* $y_0 \in W$ *since* $r_X(x_0)=r_X(x_1)$ and $r_Y(y_0)=r_Y(y_1)$. Then $(x_0,y_0) \in U$, which is absurd. Thus $(x_1,y_1) \in \overline{\{(x_0,y_0)\}}$. Analogously we conclude that $(x_0,y_0) \in \overline{\{(x_1,y_1)\}}$.

Therefore, R has stable units, as we claimed.

2. LOCALIZATIONS AND HEREDITARY TORSION THEORIES

In this section we assume that the categories $\mathcal{A}$ and $\mathcal{A}^{op}$ are finitely complete.

The torsion theories are not necessarily generated by localizing categories. However, when this happens, we have a torsion theory with interesting properties.

Next definition generalizes I.2.1 since in an abelian category any monomorphism is regular.

DEFINITION 2.1. A subcategory $\mathcal{C}$ of $\mathcal{A}$ is hereditary provided that it is closed under regular subobjects and, in this case, if $(\mathcal{C},\mathcal{B})$ is a torsion theory then it is called a hereditary torsion theory.

Given a localizing subcategory $\mathcal{B}$, consider the corresponding factorization system $\mathcal{F}=(\mathcal{E},\mathcal{M})=\Phi\mathcal{B}$, the subcategory $\mathcal{C}=I/\mathcal{E}$ and the subcategory $\mathcal{B}'$ defined in II.4. Recall that $\mathcal{B}^{\leftarrow}=\mathcal{C}$ and $\mathcal{C}^{\rightarrow}=\mathcal{B}'$.

PROPOSITION 2.2. *The subcategory* $\mathcal{C}$ *is hereditary and, therefore, the torsion theory* $(\mathcal{C},\mathcal{B}')$ *generated by a localizing subcategory* $\mathcal{B}$ *is hereditary whenever it exists.*

Proof: Consider a regular subobject C' of $C \in \mathcal{C}$, being $i: C' \longrightarrow C$ the equalizer of $f,g: C \longrightarrow A$. Since the reflector R from $\mathcal{A}$ to $\mathcal{B}$ is a localization then $R(i)$ is the equalizer of $R(f)$ and $R(g)$. But $R(C) \cong R(I)$, by II.5.1 (ii), and $R(I)$ is the initial object of $\mathcal{B}$ hence $R(f)=R(g)$ and $R(C') \cong R(I)$. Thus, $C' \in \mathcal{C}$. □

DEFINITION 2.3. Given a subcategory $\mathcal{B}$ of $\mathcal{A}$, we denote by $\mathcal{B}^{\#}$ the closure of $\mathcal{B}$ under subobjects, i.e. the subcategory of $\mathcal{A}$ whose objects are the $\mathcal{A}$-objects for which there exist a monomorphism with codomain in $\mathcal{B}$.

If we assume that $\mathcal{A}$ has zero object and that all strong epimorphisms are cokernels then the classes $\mathrm{SEpi}\mathcal{A}$ and $\mathrm{RegEpi}\mathcal{A}$ coincide and so $\mathrm{RegEpi}\mathcal{A}$ is closed under composition. According to the dual of ([12], I.3.5 (iii)), we have the following:

LEMMA 2.4. *If $\mathcal{A}$ has zero object and all strong epimorphisms are cokernels then* $(\mathrm{StrEpi}\mathcal{A},\mathrm{Mono}\mathcal{A})$ *is a factorization system of* $\mathcal{A}$. □

PROPOSITION 2.5. *Assume that $\mathcal{A}$ has zero object and all strong epimorphisms are cokernels. Let $\mathcal{B}$ be a localizing subcategory of $\mathcal{A}$. If the hereditary torsion theory* $(\mathcal{C},\mathcal{B}')$ *generated by $\mathcal{B}$ exists then it is normal and* $\mathcal{B}'=\mathcal{B}^{\#}$.

Proof: The reflector R from $\mathcal{A}$ to $\mathcal{B}$ is normal, since it is left exact. Hence $\mathcal{C}=\mathcal{B}^{\leftarrow}$ is coreflective and each coreflection morphism $s_A:S(A)\longrightarrow A$ is the kernel of the respective reflection morphism r_A (cf. II.7.7).

For every object A of $\mathcal{B}'$, r_A is a monomorphism since, by the dual of the proposition II.5.1 (ii), $\mathcal{C}^{\rightarrow}=\mathcal{B}'$ is the subcategory whose class of objects is equal to $\{A\in \mathrm{Obj}\mathcal{A} \mid S(A)\cong 0\}$. Therefore $\mathcal{B}'\subset\mathcal{B}^{\#}$. The converse inclusion is obvious because $\mathcal{B}'$ contains $\mathcal{B}$ and it is closed under subobjects (see II.7.2).

It remains to prove that if the torsion theory $(\mathcal{C},\mathcal{B}')$ exists (which happens, for instance, when $\overline{\mathcal{F}}$ and $\overset{\circ}{\overline{\mathcal{F}}}$ are factorization systems), then it is normal.

The previous lemma allows us to consider the $(\mathrm{StrEpi}\mathcal{A},\mathrm{Mono}\mathcal{A})$-factorization $m_A.r_A^{\#}$ of r_A, for each object A of $\mathcal{A}$. The morphism $r_A^{\#}$ is a reflection morphism from A to $\mathcal{B}^{\#}$. Indeed, considering $f:A\longrightarrow C$ and $C\in\mathcal{B}^{\#}$, if $i:C\longrightarrow B$ is a monomorphism with $B\in\mathcal{B}$ then there exists a unique morphism g such that $g.r_A=i.f$. Consequently, since $r_A^{\#}\in\mathrm{StrEpi}\mathcal{A}=(\mathrm{Mono}\mathcal{A})^{\uparrow}\cap\mathrm{Epi}\mathcal{A}$, there exists a unique morphism h such that the diagram

$$\begin{tikzcd}
A \arrow[r, "r_A^{\#}"] \arrow[dd, "f"'] & R^{\#}(A) \arrow[d, "m_A"] \arrow[ddl, dashed, "h"] \\
 & R(A) \arrow[d, "g"] \\
C \arrow[r, "i"'] & B
\end{tikzcd}$$

commutes. Therefore, $f = h.r_A^{\#}$ factors through $r_A^{\#}$. The unicity follows because $r_A^{\#}$ is epimorphic.

Since $s_A = \ker(m_A.r_A^{\#})$ and m_A is a monomorphism then $s_A = \ker r_A^{\#}$. But, by hypothesis, $r_A^{\#}$ is a cokernel and so $r_A^{\#} = \operatorname{coker} s_A$. Therefore, $(\mathcal{C}, \mathcal{B}^{\#}) = (\mathcal{C}, \mathcal{B}')$ is normal. □

The correspondence $\mathcal{B} \longmapsto (\mathcal{C}, \mathcal{B}')$ defined in the class of localizing subcategories of $\mathcal{A}$ is not, in general, injective. For instance, in $\mathcal{T}op$ we proved that the only localizing subcategories are $\mathcal{T}op$ itself, $\mathcal{T}er$ and $\mathcal{I}nd$ (1.8) to which correspond, respectively, the hereditary torsion theories $(\mathcal{T}op^{\leftarrow}, \mathcal{T}op^{\leftarrow\rightarrow}) = (\mathcal{I}ni, \mathcal{T}op)$, $(\mathcal{T}er^{\leftarrow}, \mathcal{T}er^{\leftarrow\rightarrow}) = (\mathcal{T}op, \mathcal{T}er)$ and $(\mathcal{I}nd^{\leftarrow}, \mathcal{I}nd^{\leftarrow\rightarrow}) = (\mathcal{I}ni, \mathcal{T}op)$.

The same situation occurs in the topological categories $\mathcal{CR}eg$ and $\mathcal{P}rox$ which have only the two trivial torsion theories and the localizing subcategories $\mathcal{CR}eg$, $\mathcal{T}er$, $\mathcal{I}nd$ and $\mathcal{P}rox$, $\mathcal{T}er$, $\mathcal{I}nd$, respectively (cf. [10], 3). However, when $\mathcal{A}$ is abelian that correspondence defines a bijection between the class of the localizing subcategories and the class of the hereditary torsion theories provided that $\mathcal{A}$ has sufficient injectives. To prove this assertion we shall make use of the following lemmas:

<u>LEMMA 2.6</u>. *If pushouts in $\mathcal{A}$ preserve monomorphisms then* $(\mathrm{Epi}\mathcal{A}, \mathrm{StrMono}\mathcal{A})$ *is a factorization system of $\mathcal{A}$.*

<u>Proof</u>: It suffices to prove that $\mathcal{A}$ has $(\mathrm{Epi}\mathcal{A}, \mathrm{RegMono}\mathcal{A})$-factorizations since then the classes $\mathrm{StrMono}\mathcal{A}$ and $\mathrm{RegMono}\mathcal{A}$ are equal and therefore, as we did in lemma 2.4, applying ([12], I.3.5 (iii)) we conclude what is required.

Consider a morphism $f: A \longrightarrow B$ of $\mathcal{A}$ and let $(x, y: B \longrightarrow C)$ be its cokernel pair and $p: E \longrightarrow B$ be the equalizer of x and y. Then there exists a morphism q such that $f = p.q$. Since p is a regular monomorphism it suffices to show that q is an epimorphism. So, let us assume that $(u, v: E \longrightarrow F)$ is the cokernel pair of q and consider the pushouts

$$\begin{tikzcd} E \arrow[r, "u"] \arrow[d, "p"'] & F \arrow[d, "p'"] \\ B \arrow[r, "u'"'] & X \end{tikzcd} \qquad \begin{tikzcd} E \arrow[r, "p"] \arrow[d, "v"'] & B \arrow[d, "v'"] \\ F \arrow[r, "q'"'] & Y \end{tikzcd} \qquad \begin{tikzcd} F \arrow[r, "q'"] \arrow[d, "p'"'] & Y \arrow[d, "p''"] \\ X \arrow[r, "q''"'] & Z \end{tikzcd}$$

Then

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[d, "q"'] & E \arrow[r, "p"] & B \arrow[d, "v'"] \\
E \arrow[d, "p"'] & & Y \arrow[d, "p''"] \\
B \arrow[r, "u'"'] & X \arrow[r, "q''"'] & Z
\end{tikzcd}$$

is a pushout, so there exists an isomorphism i such that $i.x = q''.u'$ and $i.y = p''.v'$ which implies that $q''.p'.u = q''.u'.p = i.x.p = i.y.p = p''.v'.p = p''.q'.v = q''.p'.v$. By hypothesis, the pushouts preserve monomorphisms thus q'' and p' are monomorphisms. Therefore $u = v$ and so q is an epimorphism. □

<u>REMARKS 2.7</u>. (i) The dual of 2.6 enables us to replace in 2.5 the condition "*$\mathcal{A}$ has zero object and all strong epimorphisms are cokernels*" by the one "*pullbacks in $\mathcal{A}$ preserve epimorphisms*".

(ii) If a subcategory $\mathcal{B}$ of $\mathcal{A}$ is reflective then a morphism of $\mathcal{B}$ belongs to Mono$\mathcal{A}$ if and only if it belongs to Mono$\mathcal{B}$. In fact, it is obvious that this condition is necessary. Conversely, if $m: B \longrightarrow B'$ belongs to Mono$\mathcal{B}$ and if $f,g: A \longrightarrow B$ are parallel morphisms of $\mathcal{A}$ such that $m.f = m.g$ then $m.Rf.r_A = m.Rg.r_A$, i.e. $m.Rf = m.Rg$ and so, by hypothesis, $Rf = Rg$ which implies that $f = g$. Thus, $m \in$ Mono$\mathcal{A}$.

Let us assume now that $\mathcal{A}$ is an abelian category.

If $\mathcal{B}$ is localizing, we observed in 2.5 that in the pair $(\mathcal{C}, \mathcal{B}')$, $\mathcal{C}$ is coreflective, $\mathcal{B}'$ is reflective, each coreflection morphism $s_A: S(A) \longrightarrow A$ is the kernel of the reflection morphism $r'_A = r_A^{\#}: A \longrightarrow R'(A)$ and $r'_A = \operatorname{coker} s_A$. Considering $r'_A: A \longrightarrow R'(A)$, $f: A' \longrightarrow R'(A)$ and the diagram

$$\begin{tikzcd}
 & P \arrow[r, "x"] \arrow[d, "y"] & A' \arrow[d, "f"] \\
S(A) \arrow[ur, "j"] \arrow[r, "s_A"'] & A \arrow[r, "r'_A"'] & R'(A)
\end{tikzcd}$$

where the square is a pullback and j the unique morphism such that $y.j = s_A$ and $x.j = 0$, it is easy to check that $j = \ker x$. But x is the pullback of a cokernel and so, since $\mathcal{A}$ is abelian, x is a cokernel and then $x = \operatorname{coker} j$. The fact that R' is a left adjoint implies that $R'(x)$ is the cokernel of $R'(j): R'S(A) \longrightarrow R'(P)$ in $\mathfrak{B}'$. Obviously $R'S(A) = 0$ since $\mathfrak{B}' = \mathcal{C}^{\rightarrow}$, hence $R'(x)$ is an isomorphism. We proved that R' has stable units and therefore $\overset{\circ}{\overline{\mathcal{F}}}$ is a factorization system.

Now, by duality, we can conclude that the coreflector has also stable units and that $\overline{\mathcal{F}}$ is a factorization system.

In conclusion, if $\mathcal{A}$ is abelian then any localizing subcategory $\mathfrak{B}$ generates an hereditary torsion theory $(\mathcal{C}, \mathfrak{B}')$. So, we can define a map Z from the class of localizing subcategories of $\mathcal{A}$ to the class of hereditary torsion theories of $\mathcal{A}$ by $Z(\mathfrak{B}) = (\mathcal{C}, \mathfrak{B}')$.

<u>LEMMA 2.8</u>. *If* $(\mathcal{C}, \mathfrak{B}')$ *is an hereditary torsion theory of the abelian category* $\mathcal{A}$ *then* $\mathrm{Mono}\mathfrak{B}'$ *is closed under pushouts in* $\mathfrak{B}'$, $\mathrm{StrMono}\mathfrak{B}' = \mathrm{RegMono}\mathfrak{B}'$ *and* $(\mathrm{Epi}\mathfrak{B}', \mathrm{StrMono}\mathfrak{B}')$ *is a factorization system of* $\mathfrak{B}'$.

<u>Proof</u>: Let us assume that

$$\begin{tikzcd} B \arrow[r, "i"] \arrow[d, "f"'] & B' \arrow[d, "g"] \\ B'' \arrow[r, "j"'] & A \end{tikzcd}$$

is the pushout in $\mathcal{A}$ of the morphisms i and f of $\mathfrak{B}'$, such that $i \in \mathrm{Mono}\mathfrak{B}'$. Then $(R'(A), R'(j), R'(g))$ is the pushout in $\mathfrak{B}'$ of i and f. But, according to 2.7(ii), $i \in \mathrm{Mono}\mathcal{A}$. Thus $j \in \mathrm{Mono}\mathcal{A}$ since $\mathcal{A}$ is abelian. Since in abelian categories any torsion theory is normal, as we already referred to in the previous chapter, $s_{B''} = \ker r'_{B''}$ and $s_A = \ker r'_A$. Thus, we have the following commutative diagram where $S(B'') \cong 0$:

$$\begin{tikzcd} S(B'') \arrow[r, "s_{B''}"] \arrow[d, "S(j)"'] & B'' \arrow[r, "r'_{B''}"] \arrow[d, "j"] & R'(B'') \arrow[d, "R'(j)"] \\ S(A) \arrow[r, "s_A"'] & A \arrow[r, "r'_A"'] & R'(A) \end{tikzcd}$$

But $S(A) \cap B'' \cong 0$, since $\mathfrak{B}$ and $\mathfrak{C}$ are hereditary. Consequently, the left square is a pullback. Then $0 = s_{B''} = \ker(r'_A.j) = \ker(R'(j).r'_{B''}) = \ker(R'(j))$ which implies that $R'(j)$ is a monomorphism of $\mathcal{A}$. Thus $R'(j) \in \mathrm{Mono}\mathfrak{B}'$. So, we proved that $\mathrm{Mono}\mathfrak{B}'$ is closed under pushouts in $\mathfrak{B}'$. Now, the hypothesis of 2.7 being fulfilled, $\mathfrak{B}'$ has the factorization system $(\mathrm{Epi}\mathfrak{B}', \mathrm{StrMono}\mathfrak{B}')$ and $\mathrm{StrMono}\mathfrak{B}' = \mathrm{RegMono}\mathfrak{B}'$. □

LEMMA 2.9. *Let* $(\mathfrak{C}, \mathfrak{B}')$ *be an hereditary torsion theory generated by a localizing subcategory* $\mathfrak{B}$ *of the abelian category* $\mathcal{A}$. *An object* A *of* $\mathfrak{B}'$ *belongs to* $\mathfrak{B}$ *if and only if every monomorphism* $i: A \longrightarrow B$ *with* $B \in \mathfrak{B}'$ *is a strong monomorphism of* $\mathfrak{B}'$.

Proof: If $A \in \mathfrak{B}$ and $i: A \longrightarrow B$ is a monomorphism with $B \in \mathfrak{B}'$, then i is a regular monomorphism in $\mathcal{A}$ and, therefore, $R(i)$ is a regular monomorphism in $\mathfrak{B}'$ because R is left exact. But $R(i) = r_B.i$ whence i is a strong monomorphism in $\mathfrak{B}'$, since $R(i)$ is a strong monomorphism in $\mathfrak{B}'$ (2.8).

Conversely, since R is left exact, we saw in the beginning of the proof of 2.5 that, for each object A of $\mathfrak{B}'$, r_A is a monomorphism hence a strong monomorphism in $\mathfrak{B}'$. On the other hand, $r_A \in \mathrm{Epi}\mathfrak{B}'$ because if $u, v: R(A) \longrightarrow C$ with $C \in \mathfrak{B}'$ are morphisms such that $u.r_A = v.r_A$, then we have that $r_C.u.r_A = r_C.v.r_A$, which implies that $r_C.u = r_C.v$ by the universal property of the reflector, and consequently $u = v$ since r_C is a monomorphism.

Thus we have shown that r_A is an isomorphism, i.e. that $A \in \mathfrak{B}$. □

DEFINITION 2.10. We say that the category $\mathcal{A}$ has enough injectives if every object of $\mathcal{A}$ is a subobject of an injective object.

EXAMPLE. Any Grothendieck category has enough injectives (see I.2.4).

THEOREM 2.11. *Consider the correspondence* Z *between localizing subcategories and hereditary torsion theories of the abelian category* $\mathcal{A}$. *Then:*

(i) Z *is injective.*

(ii) *If* $\mathcal{A}$ *has enough injectives then* Z *is bijective.*

Proof: (i) Let $\mathfrak{B}$ and $\mathfrak{D}$ be two localizing subcategories of $\mathcal{A}$ such that $Z(\mathfrak{B}) = (\mathfrak{C}, \mathfrak{B}') = Z(\mathfrak{D})$. From 2.9, it follows that $\mathfrak{B} = \mathfrak{D}$.

(ii) Given an hereditary torsion theory $(\mathfrak{C}, \mathfrak{B}')$, consider the subcategory $\mathfrak{B}$ whose objects are the $\mathfrak{B}'$-objects A for which any monomorphism $i: A \longrightarrow B$, with $B \in \mathfrak{B}'$, is a strong monomorphism of $\mathfrak{B}'$. To prove that $\mathfrak{B}$ is a reflective subcategory of $\mathcal{A}$ it suffices

to show that it is a reflective subcategory of $\mathcal{B}'$ (the composition of the reflectors $R:\mathcal{B}'\longrightarrow\mathcal{B}$ and $R':\mathcal{A}\longrightarrow\mathcal{B}'$ is then a reflector from $\mathcal{A}$ to $\mathcal{B}$ which we will also denote by R). Let $A\in\mathcal{B}'$, $j:A\longrightarrow J$ be a monomorphism with J injective and

$$A\xrightarrow{r_A}B\xrightarrow{i}R'(J)$$

be the $(\mathrm{Epi}\mathcal{B}',\mathrm{StrMono}\mathcal{B}')$-factorization of $r'_J.j$, denoting by $(r'_A)_{A\in\mathrm{Obj}\mathcal{A}}$ the reflection morphisms from $\mathcal{A}$ to $\mathcal{B}'$. If $(\mathcal{C},\mathcal{B}')$ is hereditary and j is a monomorphism then $R'(j)$ is a monomorphism (see proof of 2.8). Since $R'(j)=r'_J.j=i.r_A$, then r_A is a monomorphism. We claim that $B\in\mathcal{B}$. To prove that, let us consider a monomorphism $k:B\longrightarrow D$ with $D\in\mathcal{B}'$. Since J is injective, there exists a morphism $h:D\longrightarrow J$ such that $h.k.r_A=j$ whence $i.r_A=R'(j)=R'(h).k.r_A$ which implies the equality $i=R'(h).k$, because $r_A\in\mathrm{Epi}\mathcal{B}'$. But i, and so k, belongs to $\mathrm{StrMono}\mathcal{B}'$. Consequently $B\in\mathcal{B}$.

Next we show that r_A is a reflection morphism from A to $\mathcal{B}$. In fact, given $f:A\longrightarrow B'$, with $B'\in\mathcal{B}$, if we consider the following pushout in $\mathcal{B}'$

$$\begin{tikzcd} A \arrow[r, "r_A"] \arrow[d, "f"'] & B \arrow[d, "v"] \\ B' \arrow[r, "u"'] & E \end{tikzcd}$$

then, by 2.8, u is a monomorphism in $\mathcal{B}'$ since r_A is a monomorphism in $\mathcal{A}$ and so in $\mathcal{B}'$. Then $u\in\mathrm{StrMono}\mathcal{B}'$, according to the definition of $\mathcal{B}$, since $B'\in\mathcal{B}$. Consequently, there exists a unique morphism $\bar{f}:B\longrightarrow B'$ such that $\bar{f}.r_A=f$ and $u.\bar{f}=v$. Furthermore, $\bar{f}$ is the unique morphism such that $\bar{f}.r_A=f$ since $r_A\in\mathrm{Epi}\mathcal{B}'$. But r_A is monomorphic and $\mathcal{B}'$ is closed under subobjects hence $\mathcal{B}^{\#}=\mathcal{B}'$. Then proving that $\mathcal{B}$ is localizing we obtain $Z(\mathcal{B})=(\mathcal{C},\mathcal{B}')$, according to 2.5, and in this way we show that Z is surjective. So, let us prove that the reflector R from $\mathcal{A}$ to $\mathcal{B}$ is a localization. We first remark that $\mathcal{B}$ is closed in $\mathcal{B}'$ under $\mathrm{StrMono}\mathcal{B}'$-subobjects:

If $m:A\longrightarrow B$ is a strong monomorphism of $\mathcal{B}'$, $A\in\mathcal{B}'$ and $B\in\mathcal{B}$ then there exists a unique morphism $\bar{m}$ such that $\bar{m}.r_A=m$. Then r_A is a split monomorphism since belongs to $\mathrm{Epi}\mathcal{B}'$, and so an isomorphism. Hence $A\in\mathcal{B}$.

The arguments used to prove that r_A is a reflection morphism allow us to conclude that any morphism $A\longrightarrow B$ of $\mathrm{Mono}\mathcal{A}\cap\mathrm{Epi}\mathcal{B}'$, with $A\in\mathcal{B}'$ and $B\in\mathcal{B}$, is a reflection morphism from A to $\mathcal{B}$.

Assume that $k: A \longrightarrow D$ is a monomorphism with $A, D \in \mathfrak{B}'$ and let

$$A \xrightarrow{p} B \xrightarrow{j} R(D)$$

be the (Epi$\mathfrak{B}'$,StrMono$\mathfrak{B}'$)-factorization of $r_D.k$. It is obvious that $B \in \mathfrak{B}$ because $\mathfrak{B}$ is closed in $\mathfrak{B}'$ under StrMono$\mathfrak{B}'$-subobjects and then, as we have just seen, $B \cong R(A)$ and $p = i.r_A$, for some isomorphism i. Hence $j.i.r_A = j.p = r_D.k = R(k).r_A$, which implies that $R(k) = j.i \in$ StrMono$\mathfrak{B}'$. Thus, $R: \mathfrak{B}' \longrightarrow \mathfrak{B}$ maps monomorphisms into strong monomorphisms of $\mathfrak{B}'$. Since R' preserves monomorphisms, as we observed in 2.8, $R: \mathcal{A} \longrightarrow \mathfrak{B}$ preserves monomorphisms sending them into strong monomorphisms of $\mathfrak{B}'$, which are regular monomorphisms in $\mathfrak{B}'$, also by 2.8. Furthermore, this functor is additive (note that $\mathfrak{B}$, being a reflective subcategory of $\mathcal{A}$, is additive) being a left adjoint of an additive functor.

Since the functor R is additive then it suffices that it preserves kernels in order to be a localization.

Assuming that $m = \ker f$ in $\mathcal{A}$, we can write $f = n.q$ where $q = \operatorname{coker} m$ and n is a monomorphism.

$$\begin{tikzcd} A \arrow[r, "m"] & B \arrow[r, "f"] \arrow[dr, "q"'] & C \\ & & D \arrow[u, "n"'] \end{tikzcd}$$

Let us prove that $R(m) = \ker R(q)$:

Since R is a left adjoint then $R(q)$ is the cokernel of $R(m)$ in $\mathfrak{B}$. Let $e: R(B) \longrightarrow E$ be the cokernel of $R(m)$ in $\mathfrak{B}'$. Since $R(q).R(m) = 0$ then there exists a unique morphism t such that $t.e = R(q)$ implying the existence of a unique morphism $\bar{t}$ such that $\bar{t}.r_E = t$. On the other hand, $r_E.e.R(m) = 0$, thus there exists a unique morphism h such that $h.R(q) = r_E.e$. The situation is the following:

$$\begin{tikzcd} R(A) \arrow[r, "R(m)"] & R(B) \arrow[r, "R(q)"] \arrow[dr, "e"'] & R(D) \arrow[d, dashed, shift left, "h"] \\ & E \arrow[ur, dashed, "t"] \arrow[r, "r_E"'] & R(E) \arrow[u, dashed, shift left, "\bar{t}"] \end{tikzcd}$$

from which easily follows that $\bar{t}.h=1$ and $h.\bar{t}=1$.

But $R(m)$ is a regular monomorphism in $\mathcal{B}'$ (recall that R assigns regular monomorphisms of $\mathcal{B}'$ to monomorphisms) and $e=\operatorname{coker} R(m)$ in $\mathcal{B}'$ whence $R(m)=\ker e$ in $\mathcal{B}'$ and so in $\mathcal{A}$. Then $R(m)=\ker R(q)$, since r_E is a monomorphism and $\bar{t}$ is an isomorphism.

Finally, $R(m)=\ker R(f)$ since $R(n)$ is a monomorphism. □

3. SOME CLASSICAL CASES

Assume that $\mathcal{A}$ is finitely complete.

REMARK 3.1. The bijection referred to in 1.12, according to the results from which it is derived, is defined by the maps Φ and Ψ in the following way:

From a localizing subcategory $\mathcal{B}$, with reflector R, we get $\mathcal{E}$ as the class Σ_R of morphisms whose images by R are isomorphisms (which are precisely the morphisms which are orthogonal to every object B of $\mathcal{B}$) and $\mathcal{M}$ as the class of morphisms m for which the diagram

$$\begin{tikzcd} A \arrow[r, "r_A"] \arrow[d, "m"'] & R(A) \arrow[d, "R(m)"] \\ B \arrow[r, "r_B"'] & R(B) \end{tikzcd}$$

is a pullback. The $(\mathcal{E},\mathcal{M})$-factorization $m.e$ of a morphism f is obtained taking m as the pullback of $R(f)$ along r_B:

$$\begin{tikzcd}
A \arrow[dr, dashed, "e"] \arrow[drr, "f"] \arrow[ddr, "r_A"'] & & \\
& P \arrow[r, "m"] \arrow[d] & B \arrow[d, "r_B"] \\
& R(A) \arrow[r, "R(f)"'] & R(B)
\end{tikzcd}$$

Conversely, from a local factorization system, $(\mathcal{E},\mathcal{M})$, we get the localizing subcategory $\mathcal{B}=\mathcal{M}/T$, whose class of objects consist of the orthogonal objects with respect to every $e\in\mathcal{E}$. The reflection morphism from A to $\mathcal{B}$ is the $\mathcal{E}$-morphism of the $(\mathcal{E},\mathcal{M})$-factorization of $A\longrightarrow T$.

The following proposition [5] is a refinement of the fact that $\mathcal{B}=\mathcal{E}^{\perp}$.

<u>PROPOSITION 3.2</u>. *Under the conditions of* 3.1 *we have that* $\mathcal{B}=(\mathcal{E}\cap \mathrm{Mono}\mathcal{A})^{\perp}$.

<u>Proof</u>: Obviously, $\mathcal{B}=\mathcal{E}^{\perp}\subset(\mathcal{E}\cap\mathrm{Mono}\mathcal{A})^{\perp}$.

Conversely, let $B\in(\mathcal{E}\cap\mathrm{Mono}\mathcal{A})^{\perp}$. Then $B\in\mathcal{B}$ if and only if r_B is an isomorphism. Let us consider the diagram

$$\begin{tikzcd}
A \arrow[r, "j"'] & C \arrow[r, shift left, "u"] \arrow[r, shift right, "v"'] & B \arrow[r, "r_B"'] & R(B)
\end{tikzcd}$$

where (u,v) is the kernel pair of r_B and j is the equalizer of u and v. Since $R(r_B)$ is an isomorphism, the fact that R is left exact implies that $R(u)=R(v)$. Thus $R(j)$ is an isomorphism and, consequently, $j\in\mathcal{E}\cap\mathrm{Mono}\mathcal{A}$. Hence $j\perp B$, and therefore $u=v$ since $u.j=v.j$. Then r_B is a monomorphism and so $r_B\in\mathcal{E}\cap\mathrm{Mono}\mathcal{A}$. Consequently, $r_B\perp B$, which implies the existence of a unique morphism t such that $t.r_B=1$. Then $r_B.t.r_B=r_B$ and, since $r_B\perp R(B)$, we have that $r_B.t=1$. Thus r_B is an isomorphism. □

<u>DEFINITIONS 3.3</u>. (1) An <u>epi-cosource</u> $(e_i)_{i\in I}$ of morphisms of $\mathcal{A}$ is called <u>strong</u> if, given a monomorphism m and morphisms u and $(v_i)_{i\in I}$ such that $m.v_i=u.e_i$, for each $i\in I$, there exists a unique morphism d for which the diagram

$$\begin{tikzcd} \cdot \arrow[r, "e_i"] \arrow[d, "v_i"'] & \cdot \arrow[d, "u"] \arrow[dl, dashed, "d"'] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$

commutes.

(2) A class $\mathcal{G}$ of objects of $\mathcal{A}$ is called a <u>strong generator</u> if, for each $A \in \mathrm{Obj}\mathcal{A}$, the family of all morphisms with domain in $\mathcal{G}$ and codomain A is a strong epi-cosource.

When the strong generator $\mathcal{G}$ is a set then it is called a <u>small strong generator</u>.

In [5], a more precise description of $\mathcal{B}$ is given by the following proposition:

<u>PROPOSITION 3.4</u>. *Assume that* $\mathcal{A}$ *has a strong generator* $\mathcal{G}$. *Then, under the conditions of* 3.1, $\mathcal{B} = \{ e \in \mathcal{E} \cap \mathrm{Mono}\mathcal{A} \mid \mathrm{cod}(e) \in \mathcal{G} \}^{\perp}$.

<u>Proof</u>: Let us denote $\{ e \in \mathcal{E} \cap \mathrm{Mono}\mathcal{A} \mid \mathrm{cod}(e) \in \mathcal{G} \}$ by $\mathcal{T}$.

Let $B \in \mathcal{T}^{\perp}$ and consider the diagram of 3.2. We concluded that $j \in \mathcal{E} \cap \mathrm{Mono}\mathcal{A}$. For each $g : G_g \longrightarrow C$ with domain in $\mathcal{G}$, let us consider the pullback

$$\begin{tikzcd} P_g \arrow[r, "t_g"] \arrow[d, "s_g"'] & G_g \arrow[d, "g"] \\ A \arrow[r, "j"'] & C \end{tikzcd} \qquad (1)$$

in which $t_g \in \mathcal{T}$ since $\mathcal{E}$ and $\mathrm{Mono}\mathcal{A}$ are classes stable under pullbacks. It is obvious that $u.g.t_g = v.g.t_g$ and, since $B \in \mathcal{T}^{\perp}$, then $u.g = v.g$. This implies that $u = v$ because $\mathcal{G}$ is a strong generator. Therefore $r_B \in \mathcal{E} \cap \mathrm{Mono}\mathcal{A}$.

Next we consider, for each $h : G_h \longrightarrow R(B)$ with domain in $\mathcal{G}$, the pullback

$$\begin{tikzcd} P_h \arrow[r, "x_h"] \arrow[d, "y_h"'] & G_h \arrow[d, "h"] \\ B \arrow[r, "r_B"'] & R(B) \end{tikzcd}$$

Then $x_h \in \mathcal{T}$. There exists a unique morphism z_h such that $z_h.x_h = y_h$ because $B \in \mathcal{T}^{\perp}$. Then $r_B.z_h.x_h = h.x_h$, and so $r_B.z_h = h$ because $x_h \in \mathcal{E}$ and $R(B) \in \mathcal{B} = \mathcal{E}^{\perp}$. Therefore for each morphism h with domain in $\mathcal{G}$ and codomain $R(B)$, $h = r_B.z_h$ and r_B is a monomorphism, thus, by definition of strong generator, there exists a morphism d such that $r_B.d = 1$, i.e. r_B is a split epimorphism. Since, moreover it is a monomorphism, then r_B is an isomorphism. □

The class of all objects of $\mathcal{A}$ is always a strong generator and in this case the proposition 3.4 reduces to 3.2. Clearly the most interesting situation occurs when there exists a small strong generator. In this case the class of all localizing subcategories of $\mathcal{A}$ is a set and if $\mathcal{A}$ is complete this set is a complete lattice (cf. [5], 6.4 and 6.5).

<u>PROPOSITION 3.5</u>. *Under the assumptions of* 3.4, *a monomorphism* $j: A \longrightarrow C$ *belongs to* $\mathcal{E}$ *if and only if for every morphism* $g: G_g \longrightarrow C$ *with domain in* $\mathcal{G}$, *the pullback* t_g *of* j *along* g *belongs to* $\mathcal{T}$.

<u>Proof</u>: In the proof of 3.4 we concluded that t_g belongs to $\mathcal{T}$ whenever j belongs to $\mathcal{E}$.

Conversely, let us consider the image of the pullback (1) of 3.4 by R. Since $t_g \in \mathcal{E}$, then $R(t_g)$ is an isomorphism and $R(g) = R(j).R(s_g).R(t_g)^{-1}$. But j is a monomorphism as well as $R(j)$, because R is left exact. On the other hand, the proof that a left adjoint preserves strong epi-cosources is essentially a technical one and is omitted. Then, from the diagram

$$\begin{tikzcd}
R(G_g) \arrow[r, "R(g)"] \arrow[d, "R(s_g).R(t_g)^{-1}"'] & R(C) \arrow[d, "1"] \\
R(A) \arrow[r, "R(j)"'] & R(C)
\end{tikzcd}$$

where the family of morphisms $R(g)$ is a strong epi-cosource, for every morphism g with domain in $\mathcal{G}$ and codomain C, and $R(j)$ is a monomorphism, we conclude that $R(j)$ is a split epimorphism and, therefore, an isomorphism. Hence $j \in \mathcal{E}$. □

Now let us analyse these results when $\mathcal{A}$ is the category R-Mod of left modules over a ring R with identity. We will see that the sets $\mathcal{T}$ considered in 3.4 are the well-known Gabriel topologies, defined by Gabriel in [24].

PROPOSITION 3.6. *The ring* R, *considered as* R-*module, is a strong generator of the category* R-Mod.

Proof: Let us show first that, for each R-module M, the family $\mathrm{Hom}_{\text{R-Mod}}(R,M)$ is an epi-cosource. Indeed, if f and g are distinct parallel morphisms with domain M, i.e. if there exists a $m \in M$ such that $f(m) \neq g(m)$ then defining $h: R \longrightarrow M$ by $h(r) = rm$, for each $r \in R$, we conclude that $f.h \neq g.h$. Finally, we verify that this epi-cosource is strong.
It is obvious that

$$\mathrm{Hom}_{\text{R-Mod}}(R,M) = \{ \varphi_m : R \longrightarrow M \mid \varphi_m(r) = rm,\ r \in R;\ m \in M \}$$

Given a monomorphism $n: N \longrightarrow N'$ and morphisms $u: M \longrightarrow N'$ and $(v_m)_{m \in M}$ such that $n.v_m = u.\varphi_m$, for each $m \in M$, we can define an homomorphism of R-modules, $d: M \longrightarrow N$, by $d(m) = v_m(1)$, for each $m \in M$, which is easily seen to be the unique morphism for which the diagram

$$\begin{tikzcd}
R \arrow[r, "\varphi_m"] \arrow[d, "v_m"'] & M \arrow[d, "u"] \arrow[dl, dashed, "d"'] \\
N \arrow[r, "n"'] & N'
\end{tikzcd}$$

commutes. □

Since, for every monomorphism $f: M \longrightarrow R$, $f(M)$ is a left ideal of R, we can identify the set $\mathcal{T}$ of 3.4 with the set of left ideals I of $\mathcal{A}$ for which the inclusion $I \longrightarrow A$ belongs to $\mathcal{E}$.

DEFINITION 3.7. For a ring R, a family $\mathcal{F}$ of left ideals of R is called a Gabriel topology of R if it satisfies the following conditions:

(G1) $R \in \mathcal{F}$.

(G2) $(I \in \mathcal{F}$ and $r \in R) \Rightarrow (I:r) := \{ b \in R \mid br \in I \} \in \mathcal{F}$.

(G3) The left ideal J belongs to $\mathcal{F}$ whenever there exists some $I \in \mathcal{F}$ such that $(J:i) \in \mathcal{F}$, for every $i \in I$.

REMARK 3.8. (i) Conditions (G1) and (G3) imply that if I and J are left ideals of R such that $I \in \mathcal{T}$ and $I \subset J$ then $J \in \mathcal{T}$. In fact, for each $i \in I$, $(J:i) = R \in \mathcal{T}$.

(ii) From (G2) and (G3) we infer that $I \cap J \in \mathcal{T}$ whenever $I, J \in \mathcal{T}$ since

$$(I \cap J : j) = (I:j) \cap (J:j) = (I:j) \quad \text{for each } j \in J.$$

Therefore, a Gabriel topology is a filter for the ordered set of ideals of R.

PROPOSITION 3.9. *The set* $\mathcal{T}$ *defined in* 3.4 *is a Gabriel topology of* R.

Proof:

(G1) It is obvious that $R \in \mathcal{T}$ since $\mathrm{Iso}\mathcal{A} \subset \mathcal{E}$.

(G2) Let $r \in R$ and $I \in \mathcal{T}$, i.e. let $j : I \longrightarrow R$ be an embedding belonging to $\mathcal{E}$. If we consider the morphism $g : R \longrightarrow R$ defined by $g(x) = xr$, we saw in the proof of 3.4 that the pullback t_g of j along g lies in $\mathcal{T}$. But

$$P_g = \{ (i,s) \in I \times R \mid j(i) = g(s) \} = \{ (i,s) \in I \times R \mid i = sr \},$$

and so, by the previous identification, $t_g(P_g) = (I:r) \in \mathcal{T}$.

(G3) Let us suppose that the left ideal I belongs to $\mathcal{T}$ and that, for every $i \in I$, $(J:i) \in \mathcal{T}$.

If $J \subset I$ it suffices to verify that the diagram

$$\begin{tikzcd}
(J:i) \arrow[r, "\overline{m}"] \arrow[d, "\overline{\varphi}"'] & R \arrow[d, "\varphi_i"] \\
J \arrow[r, "m"'] & I
\end{tikzcd}$$

is a pullback, where m and $\overline{m}$ are the inclusions and φ_i and $\overline{\varphi}$ are the morphisms which send each element r of their domain to ri. By hypothesis $\overline{m} \in \mathcal{E}$ and so $m \in \mathcal{E}$ by 3.5. Since the inclusion from I to R belongs to $\mathcal{E}$ we can conclude that $J \in \mathcal{T}$, because $\mathcal{E}$ is closed under composition.

Finally, if $J \not\subset I$ we consider the left ideal $I+J$. Since $I \subset I+J$ then, by 3.8, $I+J \in \mathcal{T}$. If $x = i+j \in I+J$ we have that

$$(J : i+j) = \{ r \in R \mid ri+rj \in J \} = \{ r \in R \mid ri \in J \} = (J:i) \in \mathcal{T}.$$

But $J \subset I+J$ and so we reduce this situation to the one already considered. □

By the propositions 3.4 and 3.9 the function Θ which maps each localizing subcategory $\mathfrak{B}$ to the corresponding set $\mathfrak{T}$ is injective, has codomain in the set Top(R) of the Gabriel topologies of the ring R and reverses the order defined by inclusion. Therefore, the existence in R-Mod of a strong generator allows us to describe the subcategory $\mathfrak{B}$ by a set $\mathfrak{T}$ contained in $\mathcal{E}$, instead of using the equality $\mathcal{E}^{\perp}=\mathfrak{B}$. This is the suitable topology. In fact a classical result states that $\mathfrak{T}$ belongs to the codomain of Θ precisely when $\mathfrak{T}$ is a Gabriel topology of R, i.e. there exists a bijection between the set of the localizing subcategories of R-Mod and Top(R) (cf. [24] or [43]). These sets $\mathfrak{T}$ are much easier to deal with. This is clear for instance in [7] and [8], where all Gabriel topologies of valuation rings and of the respective quotient rings are described. The surjectivity of Θ can be checked directly by proving that, given a Gabriel topology $\mathfrak{T}$, the subcategory $\mathfrak{B}$ whose class of objects is the class $\{M \in \text{R-Mod} \mid t(M)=0\}$, where $t(M)=\{x \in M \mid An(x) \in \mathfrak{T}\}$ and An(x) denotes the left ideal $\{r \in R \mid rx=0\}$ of R, is a localizing subcategory of $\mathfrak{B}$ and that, moreover, $\Theta(\mathfrak{B})=\mathfrak{T}$.

This bijection as well as the one we referred to in 2.11, make clear the reason why the concept of hereditary torsion theory has been very useful in the study of localizations in rings and modules. In fact, the Gabriel topologies determine the rings and modules of quotients (cf. [43], chapters IX and X).

Another classical situation where we can apply these results is the category of presheafs $\mathcal{A}=[\mathcal{C}^{op},\mathcal{S}et]$, where $\mathcal{C}$ is a small category.

Consider the contravariant functors $\mathrm{Hom}_{\mathcal{C}}(-,C):\mathcal{C}^{op}\longrightarrow\mathcal{S}et$, associated to each object C of $\mathcal{C}$.

<u>PROPOSITION 3.10</u>. *The set* $\mathcal{G}=\{\mathrm{Hom}_{\mathcal{C}}(-,C) \mid C \in \mathrm{Obj}\mathcal{C}\}$ *is a strong generator of* $\mathcal{A}=[\mathcal{C}^{op},\mathcal{S}et]$.

<u>Proof</u>: For each $F \in \mathrm{Obj}\mathcal{A}$ and each $C \in \mathrm{Obj}\mathcal{C}$ consider

$$\mathrm{Hom}_{\mathcal{A}}(\mathrm{Hom}_{\mathcal{C}}(-,C),F)=\{\lambda^{C}{}_{i} : i \in I_C\}.$$

First, we prove that, for every $F \in \mathrm{Obj}\mathcal{A}$, the family

$$\mathcal{F}=\left\{\lambda^{C}{}_{i} : \mathrm{Hom}_{\mathcal{C}}(-,C) \xrightarrow{\cdot} F \mid C \in \mathrm{Obj}\mathcal{C},\ i \in I_C\right\}$$

is an epi-cosource. Assume that α and β are two natural transformations such that $\alpha.\lambda^{C}{}_{i}=\beta.\lambda^{C}{}_{i}$, for every $\lambda^{C}{}_{i} \in \mathcal{F}$. Then we have that $\alpha_B.(\lambda^{C}{}_{i})_B=\beta_B.(\lambda^{C}{}_{i})_B$, for each $\lambda^{C}{}_{i} \in \mathcal{F}$ and $B \in \mathrm{Obj}\mathcal{C}$. For each $C \in \mathrm{Obj}\mathcal{C}$ and each $x \in F(C)$, if we consider

$$\gamma^{C}{}_{x} : \mathrm{Hom}_{\mathcal{C}}(-,C) \xrightarrow{\cdot} F \quad \text{defined by} \quad (\gamma^{C}{}_{x})_B(f)=Ff(x),$$

for each $B \in \mathrm{Obj}\mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(B,C)$, it is obvious that ${\gamma^C}_x \in \mathcal{F}$. The equality $\alpha_C.({\gamma^C}_x)_C = \beta_C.({\gamma^C}_x)_C$ implies that

$$(\alpha_C.({\gamma^C}_x)_C)(1_C) = (\beta_C.({\gamma^C}_x)_C)(1_C),$$

i.e. $\alpha_C(x) = \beta_C(x)$ and, consequently, $\alpha = \beta$.

Finally, we prove that the epi-cosource $\mathcal{F}$ is strong. Indeed, consider a monomorphism α and morphisms ρ and $({\sigma^C}_i)_{C \in \mathrm{Obj}\mathcal{C},\, i \in I_C}$ for which the following diagram is commutative:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(-,C) \arrow[r, "{\lambda^C}_i"] \arrow[d, "{\sigma^C}_i"'] & F \arrow[d, "\rho"] \\
G \arrow[r, "\alpha"'] & H
\end{tikzcd}$$

For each $B \in \mathrm{Obj}\mathcal{C}$ we get the following commutative diagram in $\mathcal{S}et$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(B,C) \arrow[r, "({\lambda^C}_i)_B"] \arrow[d, "({\sigma^C}_i)_B"'] & F(B) \arrow[d, "\rho_B"] \\
G(B) \arrow[r, "\alpha_B"'] & H(B)
\end{tikzcd}$$

But α_B is a monomorphism, for every $B \in \mathrm{Obj}\mathcal{C}$, since the monomorphisms in $\mathcal{A}$ are the natural transformations whose component morphisms are monomorphisms in $\mathcal{S}et$ (cf. [25], I.2.1). Let us show that, for each $B \in \mathrm{Obj}\mathcal{C}$, the family

$$(({\lambda^C}_i)_B)_{C \in \mathrm{Obj}\mathcal{C},\, i \in I_C}$$

is an epi-cosource. Consider two morphisms f and g such that $f.({\lambda^C}_i)_B = g.({\lambda^C}_i)_B$, for each $C \in \mathrm{Obj}\mathcal{C}$ and $i \in I_C$. For each $x \in F(B)$ consider

$${\gamma^B}_x : \mathrm{Hom}_{\mathcal{C}}(-,B) \xrightarrow{\cdot} F \quad \text{defined by} \quad ({\gamma^B}_x)_A(h) = Fh(x),$$

for $A \in \mathrm{Obj}\mathcal{C}$ and $h \in \mathrm{Hom}_{\mathcal{C}}(A,B)$. Then, since

$$(f.({\gamma^B}_x)_B)(1_B) = (g.({\gamma^B}_x)_B)(1_B),$$

we conclude that $f(x)=g(x)$ and so that $f=g$.

But, in a category with coproducts in which every monomorphism is strong, any small epi-cosource is strong, as can be easily observed. Thus, for each $B \in \mathrm{Obj}\mathcal{C}$, the family $((\lambda^C{}_i)_B)_{C \in \mathrm{Obj}\mathcal{C},\, i \in I_C}$ is a strong epi-cosource and then we can conclude that there exists a unique morphism d such that $\alpha_B.d=\rho_B$ and $d.(\lambda^C{}_i)_B=(\sigma^C{}_i)_B$, for each $C \in \mathrm{Obj}\mathcal{C}$ and $i \in I_C$. It is easy to see that this way we get a natural transformation $\delta : F \dashrightarrow G$ such that $\alpha.\delta=\rho$ and $\delta.\lambda^C{}_i=\sigma^C{}_i$, for every $C \in \mathrm{Obj}\mathcal{C}$ and every $i \in I_C$. By the unicity of d it is obvious that δ is the unique natural transformation satisfying these conditions. Hence the epi-cosource $\mathcal{F}$ is strong, as we claimed. □

In this category, the set $\mathcal{T}$ of 3.4 is equal to $\bigcup_{C \in \mathrm{Obj}\mathcal{C}} \mathcal{V}(\mathrm{Hom}_{\mathcal{C}}(-,C))$, where

$$\mathcal{V}(\mathrm{Hom}_{\mathcal{C}}(-,C))=\{\, \alpha \in \mathcal{E} \cap \mathrm{Mono}\mathcal{A} \mid \mathrm{Cod}(\alpha)=\mathrm{Hom}_{\mathcal{C}}(-,C)\,\}.$$

A subobject of $\mathrm{Hom}_{\mathcal{C}}(-,C)$ is a natural transformation $\alpha : U \dashrightarrow \mathrm{Hom}_{\mathcal{C}}(-,C)$ such that, for every object B of $\mathcal{C}$, UB is a subset of $\mathrm{Hom}_{\mathcal{C}}(B,C)$. Since α is a natural transformation then, for objects B and B' of $\mathcal{B}$, the morphism $g.h \in UB'$ whenever $g \in UB$ and $h \in \mathrm{Hom}_{\mathcal{C}}(B',B)$. Therefore, each element of $\mathcal{V}(\mathrm{Hom}_{\mathcal{C}}(-,C))$ is precisely a family

$$\bigcup_{B \in \mathrm{Obj}\mathcal{C}} UB$$

of morphisms with codomain C such that $g.h \in UB'$ if $g \in UB$ and $h \in \mathrm{Hom}_{\mathcal{C}}(B',B)$, i.e. each element of $\mathcal{V}(\mathrm{Hom}_{\mathcal{C}}(-,C))$ is a sieve ([32], [41]).

According to the Yoneda embedding [27], $Y : \mathcal{C} \longrightarrow [\mathcal{C}^{op}, \mathcal{S}et]$, $\mathcal{C}$ is isomorphic to a full subcategory of $[\mathcal{C}^{op}, \mathcal{S}et]$ and we can identify $\mathrm{Hom}_{\mathcal{C}}(-,C)$ with the object C when there arises no ambiguity. It can be verified, giving a similar proof to that one of 3.9, that

$$\mathcal{T}=\bigcup_{C \in \mathrm{Obj}\mathcal{C}} \mathcal{V}(C)$$

is a Grothendieck topology ([32], [41]) in $\mathcal{C}$.

So, it is possible to describe $\mathcal{T}$ by a topology which is the "good" one in the sense that the function which maps the localizing subcategory $\mathcal{B}$ to $\mathcal{T}$ is not only injective (3.4), but it is even a bijection with the class of Grothendieck topologies in $\mathcal{C}$ (cf. [41]).

This example is a particular case of a topos. In toposes the existence of a classifier subobject allows the explicit description of the class $\mathcal{E}$ by the intersection $\mathcal{E} \cap \mathrm{Mono}$. In this case, the "good" topology is the Lawvere-Tierney topology [32]: in a topos $\mathbb{E}$ there exists a bijection between isomorphic classes of left exact idempotent monads in $\mathbb{E}$ and Lawvere-Tierney topologies in $\mathbb{E}$.

In ([4], 2.4) Barr proved that in a category $\mathcal{A}$ there exists a bijection between isomorphic classes of idempotent monads in $\mathcal{A}$ and reflective subcategories of $\mathcal{A}$, which leads to the conclusion that there is a bijection between isomorphic classes of left exact idempotent monads of $\mathcal{A}$ and localizing subcategories of $\mathcal{A}$. We saw that in abelian categories with enough injectives there exists a bijection between hereditary torsion theories and localizing subcategories and that in module categories over a ring R these are in a one-to-one correspondence with the Gabriel topologies of the ring R. This enables us to understand the way that leads Barr to the generalization of hereditary torsion theories to non-abelian categories given in [4], as well as the divergence between this way and the one followed by Cassidy, Hébert and Kelly which we presented in this dissertation.

REFERENCES

[1] ADÁMEK, J., H. HERRLICH and G. E. STRECKER : *Abstract and concrete categories*,
John Wiley & Sons, New York, 1990.

[2] APPLEGATE, H. and M. TIERNEY : *Iterated cotriples*,
Reports of the Midwest Category Seminar IV, Lecture Notes in Math. 137, pp. 56-99, Springer-Verlag, 1970.

[3] ARHANGEL'SKII, A. and R. WIEGANDT : *Connectednesses and disconnectednesses in Topology*,
General Topology and its Appl. 5 (1975), 9-33.

[4] BARR, M. : *Non-abelian torsion theories*,
Can. J. Math. 25 (1973), 1224-1237.

[5] BORCEUX, F. and G. M. KELLY : *On locales of localizations*,
J. Pure Appl. Algebra 46 (1987), 1-34.

[6] BOUSFIELD, A. K. : *Constructions of factorization systems in categories*,
J. Pure Appl. Algebra 9 (1977), 207-220.

[7] BRANDAL, W. and E. BARBUT : *Localizations of torsion theories*,
Pacific J. Math. 107 (1983), 27-37.

[8] BRANDAL, W. and E. BARBUT : *Torsion theories over commutative rings*,
J. Algebra 101 (1986), 136-150.

[9] BRÜMMER, G. : *Topological categories*,
Topology and its Appl. 18 (1984), 27-41.

[10] CAGLIARI, F. and S. MANTOVANI : *Localizations in universal topological categories*, Proc. Amer. Math. Soc. 103 (1988), 639-640.

[11] CASSIDY, C. , M. HÉBERT and G. M. KELLY : *Reflective subcategories, localizations and factorization systems*, J. Austral. Math. Soc. (Série A) 38 (1985), 287-329. *Corrigenda* 41 (1986), 286.

[12] CLEMENTINO, M. M. : *Subcategorias reflectivas e factorizações*, Universidade de Coimbra, 1987.

[13] COHN, P. : *Universal Algebra*, Evanston, New York, 1965.

[14] DAY, B. : *On adjoint-functor factorization*, Proc. Sydney Category Theory Seminar 1972/1973, Lecture Notes in Math. 420, pp. 1-19, Springer-Verlag, 1974.

[15] DICKSON, S. E. : *On torsion classes of abelian groups*, J. Math. Soc. Japan 17 (1965), 30-35.

[16] DICKSON, S. E. : *Decomposition of modules I*, Math. Zeitschr. 90 (1965), 9-13.

[17] DICKSON, S.E. : *A torsion theory for abelian categories*, Trans. Amer. Math. Soc. 121 (1966), 223-235.

[18] DICKSON, S. E. : *Decomposition of modules II*, Math. Zeitschr. 104 (1968), 349-357.

[19] DIVINSKY, N. : *Duality between radical and semisimple classes of associative rings*, Scripta Math. 29 (1973), 409-416.

[20] DLAB, V. : *The concept of a torsion module*, Amer. Math. Monthly 75 (1968), 973-976.

[21] FREYD, P. J. and G. M. KELLY : *Categories of continuous functors I*, J. Pure Appl.Algebra 2 (1972), 169-191. *Erratum ibid* 4 (1974), 121.

[22] FRIED, E. and R. WIEGANDT : *Connectednesses and disconnectednesses of graphs*, Algebra Univ. 5 (1975), 411-428.

[23] FUCHS, L. : *Infinite abelian groups*,
Vol. I, Academic Press, New York-London, 1970.

[24] GABRIEL, P. : *Des catégories abéliennes*,
Bull. Soc. Mat. France 90 (1962), 323-448.

[25] GRILLET, P. : *Regular categories*,
Lecture Notes in Math. 236, pp. 121-222, Springer-Verlag, 1971.

[26] HERRLICH, H. and G. E. STRECKER : *Coreflective subcategories*,
Trans. Amer. Math.Soc. 157 (1971), 205-226.

[27] HERRLICH, H. and G. E. STRECKER : *Category Theory*,
Allyn and Bacon, Boston, 1973.

[28] HERRLICH, H. , G. SALICRUP and R. VÁSQUEZ : *Dispersed factorization structures*,
Can. J. Math. 31 (1979), 1059-1071.

[29] HERRLICH, H. : *Categorical Topology* 1971-1981,
Proc. Fifth Prague Topol. Symp. 1981, pp. 279-383, Heldermann Verlag, Berlin, 1982.

[30] HILTON, P. and U. STAMMBACH : *A course in Homological Algebra*,
Springer-Verlag, New York-Heidelberg-Berlin, 1971.

[31] HILTON, P. , G. MISLIN and J. ROITBERG : *Localization of nilpotent groups and spaces*,
Mathematic Studies 15, North-Holland, Amsterdam, 1975.

[32] JOHNSTONE, P. : *Topos Theory*,
L.M.S. Monographs 10, Academic Press, New York, 1977.

[33] KELLY, G. M. and F. W. LAWVERE: *On the complete lattice of essential localizations*,
Bull. Soc. Mat. Belgique (Série A) 41 (1989), 289-319.

[34] KOROSTENSKI, M. and W. THOLEN : *Prelocalizations*,
Preprint.

[35] LEVY, L. : *Torsion-free and divisible modules over non-integral domains*,
Canad. J. Math. 15 (1963), 132-151.

[36] MACLANE, S. : *Categories for the working mathematician*,
Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[37] MANTOVANI, S. : *Connessione e totale riflessività*,
Ph. D. Thesis, University of Milan, 1988.

[38] MARNY, T. : *On epireflective subcategories of topological categories*,
General Topology and its Appl. 10 (1979), 175-181.

[39] MITCHELL, B. : *Theory of Categories*,
Academic Press, New York, 1964.

[40] POPESCU, N. and L. POPESCU : *Theory of Categories*,
Sijthoff & Noordhoff International Publishers, The Netherlands, 1979.

[41] SCHUBERT, H. : *Categories*,
Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[42] STENSTRÖM, B. : *Rings and modules of quotients*,
Lecture Notes in Math. 237, Springer-Verlag, Berlin-Heidelberg-New York, 1971.

[43] STENSTRÖM, B. : *Rings of quotients*,
Springer-Verlag, Berlin-Heidelberg-New York, 1975.

[44] THOLEN, W. : *Semi-topological functors I*,
J. Pure Appl. Algebra 15 (1979), 53-73.

[45] THOLEN, W. : *Factorizations, fibres and connectedness*,
Proc. Int. Conf. Categorical Topology (Toledo, 1983), pp. 549-566, Heldermann Verlag, Berlin, 1984.

[46] THOLEN, W. : *Prereflections and reflections*,
Comm. in Algebra 14 (1986), 717-740.