CHAPTER II

TORSION THEORIES IN CATEGORIES WITH INITIAL AND TERMINAL OBJECTS

1. REFLECTIVE SUBCATEGORIES AND PREFACTORIZATION SYSTEMS

Let $\mathcal{A}$ be a category and e and m be two morphisms in $\mathcal{A}$.

The morphism e is said to be <u>orthogonal to</u> m if for every pair (u,v) of morphisms in $\mathcal{A}$ such that $v.e = m.u$ there exists a unique morphism d in $\mathcal{A}$ such that $d.e = u$ and $m.d = v$.

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "u"'] & \cdot \arrow[d, "v"] \arrow[dl, dashed, "d"'] \\ \cdot \arrow[r, "m"'] & \cdot \end{tikzcd}$$

When e is orthogonal to m we shall write $e{\downarrow}m$.

Given a class $\mathcal{H}$ of morphisms in $\mathcal{A}$, let

$$\mathcal{H}^{\uparrow} = \left\{ e \in \mathrm{Mor}\mathcal{A} \mid e{\downarrow}h, \text{ for every } h \in \mathcal{H} \right\}$$
$$\mathcal{H}^{\downarrow} = \left\{ m \in \mathrm{Mor}\mathcal{A} \mid h{\downarrow}m, \text{ for every } h \in \mathcal{H} \right\}.$$

The following proposition is similar to I.1.5 and has also a trivial proof.

PROPOSITION 1.1. *Let* $\mathcal{H}$, $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be classes of morphisms of* $\mathcal{A}$. *Then:*

(i) $\mathcal{H} \subset \mathcal{H}^{\uparrow\downarrow}$.

(ii) $\mathcal{H} \subset \mathcal{H}^{\downarrow\uparrow}$.

(iii) $\mathcal{H}_1 \subset \mathcal{H}_2 \Rightarrow (\mathcal{H}_2{}^{\uparrow} \subset \mathcal{H}_1{}^{\uparrow}$ *and* $\mathcal{H}_2{}^{\downarrow} \subset \mathcal{H}_1{}^{\downarrow})$.

(iv) $\mathcal{H}^{\uparrow\downarrow\uparrow} = \mathcal{H}^{\uparrow}$ *and* $\mathcal{H}^{\downarrow\uparrow\downarrow} = \mathcal{H}^{\downarrow}$. □

DEFINITION 1.2. A pair $(\mathcal{E},\mathcal{M})$ of classes of morphisms in $\mathcal{A}$ is called a prefactorization system of $\mathcal{A}$ if $\mathcal{E}^{\downarrow} = \mathcal{M}$ and $\mathcal{M}^{\uparrow} = \mathcal{E}$.

EXAMPLES. For any class $\mathcal{H}$ of morphisms in $\mathcal{A}$, $(\mathcal{H}^{\uparrow},\mathcal{H}^{\uparrow\downarrow})$ and $(\mathcal{H}^{\downarrow\uparrow},\mathcal{H}^{\downarrow})$ are prefactorization systems of $\mathcal{A}$.

Next we recall some properties of the class $\mathcal{M}$.

PROPOSITION 1.3. *If* $(\mathcal{E},\mathcal{M})$ *is a prefactorization system in* $\mathcal{A}$, *then:*

(i) $\mathcal{E} \cap \mathcal{M} = \mathrm{Iso}\mathcal{A}$.

(ii) $\mathcal{M}$ *is closed under composition.*

(iii) *If* $f.g \in \mathcal{M}$ *and* $f \in \mathcal{M} \cup \mathrm{Mono}\mathcal{A}$ *then* $g \in \mathcal{M}$.

(iv) *The pullback of a morphism in* $\mathcal{M}$ *along a morphism in* $\mathcal{A}$ *lies in* $\mathcal{M}$.

(v) $\mathcal{M}$ *is stable under limits.*

Proof: See e.g. ([12], I.1.1). □

The corresponding properties for the class $\mathcal{E}$ follow by duality.

The class of prefactorization systems with the order relation defined by

$$(\mathcal{E},\mathcal{M}) \leq (\mathcal{E}',\mathcal{M}') \text{ if } \mathcal{M} \subset \mathcal{M}' \text{ (or, which is equivalent, if } \mathcal{E} \supset \mathcal{E}')$$

is a (possibly large) complete lattice with smallest and greatest elements $(\mathrm{Mor}\mathcal{A},\mathrm{Iso}\mathcal{A})$ and $(\mathrm{Iso}\mathcal{A},\mathrm{Mor}\mathcal{A})$, respectively.

DEFINITION 1.4. A pair $(\mathcal{E},\mathcal{M})$ of classes of morphisms in $\mathcal{A}$ is a factorization system of $\mathcal{A}$ provided that:

(i) $(\mathcal{E},\mathcal{M})$ is a prefactorization system.

(ii) $\mathcal{A}$ is $(\mathcal{E},\mathcal{M})$-factorizable, i.e. every morphism f in $\mathcal{A}$ has a factorization $f = m.e$, with $e \in \mathcal{E}$ and $m \in \mathcal{M}$.

REMARKS 1.5. (i) Let $(\mathcal{E},\mathcal{M})$ be a factorization system. Then the $(\mathcal{E},\mathcal{M})$-factorizations are unique up to isomorphism, i.e. if

$$A \xrightarrow{e} B \xrightarrow{m} C \quad \text{and} \quad A \xrightarrow{e'} B' \xrightarrow{m'} C$$

are two $(\mathcal{E},\mathcal{M})$-factorizations of a morphism f then there exists an isomorphism $d: B \longrightarrow B'$ such that $d.e = e'$ and $m'.d = m$.

(ii) The condition (i) of 1.4 can be replaced by the conditions (cf. [21]):

(i_1) $\mathcal{E} \subset \mathcal{M}^{\uparrow}$.

(i_2) $\mathcal{E}$ and $\mathcal{M}$ are closed under composition.

EXAMPLES. (a) In any category $\mathcal{A}$, $(\mathrm{Iso}\mathcal{A},\mathrm{Mor}\mathcal{A})$ and $(\mathrm{Mor}\mathcal{A},\mathrm{Iso}\mathcal{A})$ are factorization systems.

(b) The categories $\mathcal{S}et$ and $\mathcal{G}rp$ of sets and groups, respectively, as well as the category $\mathcal{A}b$ of abelian groups, have the factorization system (Epi,Mono).

(c) (Epi,ExtMono) is a factorization system of the category $\mathcal{T}op$ of topological spaces.

(d) In $\mathcal{S}et$, the pair $(\mathcal{E},\mathcal{M})$ with

$$\mathcal{E} = \{ e: X \longrightarrow Y \mid X = \emptyset \Rightarrow Y = \emptyset \} \text{ and}$$
$$\mathcal{M} = \{ m: X \longrightarrow Y \mid X = \emptyset \text{ or } m \text{ is a bijection} \}$$

is a factorization system.

DEFINITIONS 1.6. (1) A morphism $h: A \longrightarrow A'$ in $\mathcal{A}$ is said to be <u>orthogonal to</u> an object B of $\mathcal{A}$, and we write $h \perp B$, if $\mathrm{Hom}(h,B)$ is a bijection, i.e. if for every morphism $f: A \longrightarrow B$, there exists a unique morphism $g: A' \longrightarrow B$ such that $g.h = f$.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[dr, "f"'] & A' \arrow[d, dashed, "g"] \\ & B \end{tikzcd}$$

(2) Given a class $\mathcal{H}$ of morphisms in $\mathcal{A}$, we denote by $\mathcal{H}^{\perp}$ the subcategory of $\mathcal{A}$ defined by:

$$\mathrm{Obj}\mathcal{H}^{\perp} = \{ B \in \mathrm{Obj}\mathcal{A} \mid h \perp B, \text{ for every } h \in \mathcal{H} \}.$$

(3) If $\mathcal{B}$ is a subcategory of $\mathcal{A}$, we denote by $\mathcal{B}^{\top}$ the following class:

$$\{h \in \mathrm{Mor}\mathcal{A} \mid h \perp B, \text{ for every } B \in \mathrm{Obj}\mathcal{B}\}.$$

REMARK 1.7. The definition of co-orthogonality, the subcategory ${}^{\perp}\mathcal{H}$ and the class ${}^{\top}\mathcal{B}$ are defined dually.

Given a functor $R:\mathcal{A} \longrightarrow \mathcal{B}$, the class of the morphisms f in $\mathcal{A}$ such that $Rf \in \mathrm{Iso}\mathcal{B}$ will be denoted by Σ_R.

LEMMA 1.8. *Let* $<F,G;\eta,\xi>:\mathcal{A} \rightleftarrows \mathcal{B}$ *be an adjunction. Then:*

(i) $\Sigma_F = (G(\mathrm{Mor}\mathcal{B}))^{\uparrow} = G(\mathcal{B})^{\top}$.

(ii) *If* $\mathcal{A}$ *has a terminal object* T *we have that* $\Sigma_F = \{G(B) \longrightarrow T \mid B \in \mathrm{Obj}\mathcal{B}\}^{\uparrow}$.

Proof: (i) We shall prove that $Fa \downarrow b$ if and only if $a \downarrow Gb$, for $a \in \mathrm{Mor}\mathcal{A}$ and $b \in \mathrm{Mor}\mathcal{B}$, since this implies $\Sigma_F = (G(\mathrm{Mor}\mathcal{B}))^{\uparrow}$, according to the following sequence of obvious equivalences:

$$\begin{aligned} a \in (G(\mathrm{Mor}\mathcal{B}))^{\uparrow} &\Leftrightarrow a \downarrow Gb, \text{ for every morphism } b \text{ of } \mathcal{B} \\ &\Leftrightarrow Fa \downarrow b, \text{ for every morphism } b \text{ of } \mathcal{B} \\ &\Leftrightarrow Fa \text{ is an isomorphism} \\ &\Leftrightarrow a \in \Sigma_F. \end{aligned}$$

Moreover, it suffices to show one of the implications because the other can be obtained by duality.

$Fa \downarrow b \Rightarrow a \downarrow Gb$: Let us consider two morphisms $u:A \longrightarrow G(B)$ and $v:A' \longrightarrow G(B')$ of $\mathcal{A}$ such that $v.a = Gb.u$. Since the following diagram commutes

$$\begin{tikzcd} F(A) \arrow[r, "Fa"] \arrow[d, "Fu"'] & F(A') \arrow[d, "Fv"] \\ FG(B) \arrow[r, "FGb"] \arrow[d, "\xi_B"'] & FG(B') \arrow[d, "\xi_{B'}"] \\ B \arrow[r, "b"'] & B' \end{tikzcd}$$

and $Fa \downarrow b$, there exists a unique morphism $d: F(A') \longrightarrow B$ such that $d.Fa = \xi_B.Fu$ and $b.d = \xi_{B'}.Fv$.

The morphism $Gd.\eta_{A'}$ is the unique morphism such that $(Gd.\eta_{A'}).a = u$ and $Gb.(Gd.\eta_{A'}) = v$:

- $Gd.\eta_{A'}.a = Gd.GFa.\eta_A = G(d.Fa).\eta_A = G(\xi_B.Fu).\eta_A = G\xi_B.GFu.\eta_A = G\xi_B.\eta_{G(B)}.u = u$

because $G\xi_B.\eta_{G(B)} = 1_{G(B)}$, by one of the triangular identities of the adjunction.

- In an analogous way we have that

$$Gb.Gd.\eta_{A'} = G(b.d).\eta_{A'} = G(\xi_{B'}.Fv).\eta_{A'} = G\xi_{B'}.GFv.\eta_{A'} = G\xi_{B'}.\eta_{G(B')}.v = v.$$

- To prove the unicity of $Gd.\eta_{A'}$, let us consider a morphism $d': A' \longrightarrow G(B)$ such that $d'.a = u$ and $Gb.d' = v$. We have the equalities

$$\xi_B.Fd'.Fa = \xi_B.F(d'.a) = \xi_B.Fu \text{ and}$$

$$b.\xi_B.Fd' = \xi_{B'}.FGb.Fd' = \xi_{B'}.F(Gb.d') = \xi_{B'}.Fv$$

whence $d = \xi_B.Fd'$, which is equivalent to $d' = Gd.\eta_{A'}$, as can be easily observed.

Now we show that $\underline{\Sigma_F \subset G(\mathcal{B})^{\top}}$. Let $h: A \longrightarrow A'$ be a morphism of $\mathcal{A}$ such that $Fh \in \mathrm{Iso}\,\mathcal{B}$ and $f: A \longrightarrow G(B)$ a morphism of $\mathcal{A}$ with $B \in \mathrm{Obj}\,\mathcal{B}$. Since $\eta_A: A \longrightarrow GF(A)$ is an universal morphism from A to G there exists a unique morphism $\bar{f}: F(A) \longrightarrow B$ such that $G\bar{f}.\eta_A = f$. Consider the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[dr, "\eta_A"] \arrow[dd, "f"'] & A' \arrow[dr, "\eta_{A'}"] & \\
& GF(A) \arrow[r, shift left, "GFh"] \arrow[dl, "G\bar{f}"] & GF(A') \arrow[l, shift left, "(GFh)^{-1}"] \\
G(B) & &
\end{tikzcd}$$

We have that $G\bar{f}.(GFh)^{-1}.\eta_{A'}.h = G\bar{f}.\eta_A = f$.

It remains to prove that $G\bar{f}.(GFh)^{-1}.\eta_{A'}$ is the unique morphism fulfilling this condition. Let us assume that $g: A' \longrightarrow G(B)$ satisfies $g.h = f$. Then $GFg.GFh = GFf$, i.e. $GFg = GFf.(GFh)^{-1}$, whence

$$g = 1_{G(B)}.g = G\xi_B.\eta_{G(B)}.g = G\xi_B.GFg.\eta_{A'} = G\xi_B.GFf.(GFh)^{-1}.\eta_{A'} = G(\xi_B.Ff).(GFh)^{-1}.\eta_{A'}.$$

But $\xi_B.Ff = \bar{f}$ because $G(\xi_B.Ff).\eta_A = f$, and so $g = G\bar{f}.(GFh)^{-1}.\eta_{A'}$.

Furthermore, $\underline{G(\mathfrak{B})^{\top} \subset (G(\mathrm{Mor}\mathfrak{B}))^{\uparrow}}$. Indeed, consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "u"'] & A' \arrow[d, "v"] \\ G(B) \arrow[r, "Gb"'] & G(B') \end{tikzcd}$$

where h is a morphism of $G(\mathfrak{B})^{\top}$ and b a morphism of $\mathfrak{B}$. Since $h \in G(\mathfrak{B})^{\top}$ there exists a unique morphism $d: A' \longrightarrow G(B)$ such that $d.h = u$. By the same reason v is the unique morphism such that $v.h = Gb.u$. But $Gb.d.h = Gb.u$ so $Gb.d = v$.

(ii) Suppose now that $\mathcal{A}$ has a terminal object T.

$\underline{G(\mathfrak{B})^{\top} \subset \{G(B) \longrightarrow T \mid B \in \mathrm{Obj}\mathfrak{B}\}^{\uparrow}}$: Let $h: A \longrightarrow A'$ be a morphism of $G(\mathfrak{B})^{\top}$ and $b: G(B) \longrightarrow T$ with $B \in \mathrm{Obj}\mathfrak{B}$. Given u and v such that $b.u = v.h$, there exists a unique morphism d such that $d.h = u$, because $h \in G(\mathfrak{B})^{\top}$. Obviously $b.d = v$, therefore $h \downarrow b$.

$\underline{\{G(B) \longrightarrow T \mid B \in \mathrm{Obj}\mathfrak{B}\}^{\uparrow} \subset (G(\mathrm{Mor}\mathfrak{B}))^{\uparrow}}$: Let $h: A \longrightarrow A'$ be a morphism of $\{G(B) \longrightarrow T \mid B \in \mathrm{Obj}\mathfrak{B}\}^{\uparrow}$. Consider morphisms $u: A \longrightarrow G(B)$, $f: B \longrightarrow B'$ and $v: A' \longrightarrow G(B')$ such that B and B' are objects of $\mathfrak{B}$ and $Gf.u = v.h$.

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "u"'] & A' \arrow[d, "v"] \arrow[dr] & \\ G(B) \arrow[r, "Gf"'] & G(B') \arrow[r] & T \end{tikzcd}$$

Since $h \in \{G(B) \longrightarrow T \mid B \in \mathrm{Obj}\mathfrak{B}\}^{\uparrow}$ then v is the unique morphism such that $v.h = Gf.u$ and there exists a unique morphism $d: A' \longrightarrow G(B)$ such that $d.h = u$. Thus $v = Gf.d$, because $Gf.d.h = Gf.u$. □

From now on, throughout this section, we shall consider a category $\mathcal{A}$ with terminal object T.

Let us denote by $\mathfrak{R}$ the conglomerate of reflective subcategories of $\mathcal{A}$, ordered by

inclusion, and by $\mathcal{P}$ the ordered conglomerate of prefactorization systems $(\mathcal{E},\mathcal{M})$ of $\mathcal{A}$ that fulfil the following condition:

(P) $\forall A \in \mathrm{Obj}\mathcal{A}$ the morphism $A \longrightarrow T$ has an $(\mathcal{E},\mathcal{M})$-factorization.

<u>LEMMA 1.9</u>. *If* $\mathcal{B} \in \mathcal{R}$ *then the prefactorization system* $((\mathrm{Mor}\mathcal{B})^{\uparrow},(\mathrm{Mor}\mathcal{B})^{\uparrow\downarrow})$ *belongs to* $\mathcal{P}$.

<u>Proof</u>: Given $A \in \mathrm{Obj}\mathcal{A}$ and $f: A \longrightarrow T$, consider the reflection morphism $r_A : A \longrightarrow R(A)$ of A in $\mathcal{B}$. Since T is terminal, there exists a unique morphism $\bar{f}: R(A) \longrightarrow T$ and $\bar{f}.r_A = f$. Moreover $r_A \in (\mathrm{Mor}\mathcal{B})^{\uparrow}$. Indeed, let $b \in \mathrm{Mor}\mathcal{B}$ and u and v such that $b.u = v.r_A$. By definition of r_A there exists a unique morphism d such that $d.r_A = u$. By the same reason v is the unique morphism such that $v.r_A = b.u$. Since $b.d.r_A = b.u$ then $b.d = v$.

$$\begin{tikzcd}
A \arrow[r, "r_A"] \arrow[d, "u"'] & R(A) \arrow[d, "v"] \arrow[dl, dashed, "d"'] \\
B \arrow[r, "b"'] & B'
\end{tikzcd}$$

Thus $r_A \downarrow b$ as we claimed.

Finally, since $\mathcal{B}$ is reflective, $T \in \mathrm{Obj}\mathcal{B}$ ([27], 36.13), then $\bar{f} \in \mathrm{Mor}\mathcal{B} \subset (\mathrm{Mor}\mathcal{B})^{\uparrow\downarrow}$. □

If $(\mathcal{E},\mathcal{M})$ is a prefactorization system we denote by $\mathcal{M}/T$ the subcategory of $\mathcal{A}$ whose class of objects is defined by $\{ A \in \mathrm{Obj}\mathcal{A} \mid A \longrightarrow T \in \mathcal{M} \}$.

<u>LEMMA 1.10</u>. *If* $(\mathcal{E},\mathcal{M}) \in \mathcal{P}$ *then* $\mathcal{M}/T \in \mathcal{R}$.

<u>Proof</u>: Let $A \in \mathrm{Obj}\mathcal{A}$, $f: A \longrightarrow T$, and $m.e$ be the $(\mathcal{E},\mathcal{M})$-factorization of f, with $e: A \longrightarrow B$ and $m: B \longrightarrow T$. By definition, $B \in \mathrm{Obj}(\mathcal{M}/T)$. Furthermore, e is a reflection morphism of A in $\mathcal{M}/T$. Indeed, assuming that $g: A \longrightarrow B'$ is a morphism of codomain in $\mathcal{M}/T$, we have that $m.e = m'.g$, where $m': B' \longrightarrow T$ is the morphism from B' to the terminal object. Since $e \downarrow m'$, there exists a unique morphism d such that $d.e = g$ and $m'.d = m$

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "g"'] & B \arrow[d, "m"] \arrow[dl, dashed, "d"] \\ B' \arrow[r, "m'"'] & T \end{tikzcd}$$

which is also the unique morphism such that $d.e = g$. □

REMARKS 1.11. (i) The reflection morphisms belong to $\mathcal{E}$ and so $\mathcal{M}/T$ is an $\mathcal{E}$-reflective subcategory of $\mathcal{A}$.

(ii) By lemma 1.8 $\Sigma_R = (\mathrm{Mor}\,\mathcal{M}/T)^{\uparrow}$, where R is the reflector of $\mathcal{A}$ in $\mathcal{M}/T$. Hence $\mathcal{E} \subset \Sigma_R$ because $\mathcal{E} \subset (\mathrm{Mor}\,\mathcal{M}/T)^{\uparrow}$.

Lemmas 1.9 and 1.10 define order-preserving maps

$$\begin{aligned} \Phi : \mathcal{R} &\longrightarrow \mathcal{P} \\ \mathcal{B} &\longmapsto ((\mathrm{Mor}\,\mathcal{B})^{\uparrow}, (\mathrm{Mor}\,\mathcal{B})^{\uparrow\downarrow}) \end{aligned}$$

and

$$\begin{aligned} \Psi : \mathcal{P} &\longrightarrow \mathcal{R} \\ (\mathcal{E}, \mathcal{M}) &\longmapsto \mathcal{M}/T. \end{aligned}$$

PROPOSITION 1.12. $\Psi\Phi = \mathrm{Id}_{\mathcal{R}}$ *and* $\Phi\Psi \leq \mathrm{Id}_{\mathcal{P}}$.

To prove this proposition we shall make use of the following results:

LEMMA 1.13. *Let* $\mathcal{B}$ *be a reflective subcategory of* $\mathcal{A}$ *with reflector* $R : \mathcal{A} \longrightarrow \mathcal{B}$. *Then* $(\mathrm{Mor}\,\mathcal{B})^{\uparrow} = \{ B \longrightarrow T \mid B \in \mathrm{Obj}\,\mathcal{B} \}^{\uparrow} = \mathcal{B}^{\top} = \Sigma_R$.

Proof: It is a specialization of 1.8. □

LEMMA 1.14. *Let* $\mathcal{B}$ *be a reflective subcategory of* $\mathcal{A}$ *and*

$$\{ r_A : A \longrightarrow R(A) \mid A \in \mathrm{Obj}\,\mathcal{A} \}$$

be the class of reflection morphisms.

If $\mathcal{H}$ *is such that* $\{ r_A \mid A \in \mathrm{Obj}\,\mathcal{A} \} \subset \mathcal{H} \subset \Sigma_R$ *then* $\mathcal{H}^{\downarrow}/T = \mathcal{H}^{\perp} = \mathcal{B}$.

Proof: See ([12], II.5.5). □

Proof of 1.12:

If $\mathcal{B} \in \mathcal{R}$ then $\Psi\Phi(\mathcal{B}) = \Psi((\mathrm{Mor}\,\mathcal{B})^{\uparrow},(\mathrm{Mor}\,\mathcal{B})^{\uparrow\downarrow})$

$$= \Psi(\Sigma_R,(\Sigma_R)^{\downarrow}), \text{ by lemma 1.13.}$$

Furthermore, taking $\mathcal{H} = \Sigma_R$ in 1.14 we obtain $\Psi\Phi(\mathcal{B}) = (\Sigma_R)^{\downarrow}/T = \mathcal{B}$.

If $(\mathcal{E},\mathcal{M}) \in \mathcal{P}$ then

$$\Phi\Psi(\mathcal{E},\mathcal{M}) = \Phi(\mathcal{M}/T) = ((\mathrm{Mor}(\mathcal{M}/T))^{\uparrow},(\mathrm{Mor}(\mathcal{M}/T))^{\uparrow\downarrow}).$$

Moreover, since $\mathcal{E} \subset (\mathrm{Mor}(\mathcal{M}/T))^{\uparrow}$, we can conclude that $\Phi\Psi(\mathcal{E},\mathcal{M}) \leq (\mathcal{E},\mathcal{M})$. □

Given a prefactorization system $(\mathcal{E},\mathcal{M}) \in \mathcal{P}$, we call reflective interior of $(\mathcal{E},\mathcal{M})$ to the image of $(\mathcal{E},\mathcal{M})$ by $\Phi\Psi$ and we denote it by $(\mathring{\mathcal{E}},\mathring{\mathcal{M}})$. The elements of $\mathcal{P}$ that coincide with their reflective interior are called reflective prefactorization systems and the corresponding conglomerate is denoted by $\mathring{\mathcal{P}}$. It is clear that the corestriction of Φ and the restriction of Ψ define a bijection between $\mathcal{R}$ and $\mathring{\mathcal{P}}$.

EXAMPLE. Consider the category $\mathcal{N}il$ of nilpotent groups, a set P of prime numbers and its complement P' in the set of all prime numbers. By $n \in P'$ we mean that n is a product of primes of P'. A nilpotent group G is called P-local if the map $x \longrightarrow x^n$, $x \in G$, is a bijection for every $n \in P'$. The subcategory $\mathcal{B}$ of P-local groups is reflective ([31], I.1). Let $(r_G : G \longrightarrow G_P)_{G \in \mathcal{N}il}$ be the family of reflection morphisms. An homomorphism $\varphi : G \longrightarrow H$ is called P'-isolated if the diagram

$$\begin{tikzcd} G \arrow[r, "\varphi"] \arrow[d, "r_G"'] & H \arrow[d, "r_H"] \\ G_P \arrow[r, "R(\varphi)"'] & H_P \end{tikzcd}$$

is a pullback. The morphism φ is called P-injective if $\mathrm{Ker}\,\varphi$ is a set of P'-torsion elements and is called P-surjective if, given $h \in H$, there exists a $n \in P'$ such that $h^n \in \mathrm{Im}\,\varphi$. If φ is both P-injective and P-surjective then it is called a P-bijection. According to ([31], I.1.9) each reflection morphism is a P-bijection. Then, by ([31], I.3.3), φ is a P-bijection if and only if $R(\varphi)$ is an isomorphism. The P-bijections are preserved by pullbacks. Indeed, consider the pullback

$$\begin{tikzcd} P \arrow[r, "\overline{\beta}"] \arrow[d, "\overline{\alpha}"'] & K \arrow[d, "\alpha"] \\ G \arrow[r, "\beta"'] & H \end{tikzcd}$$

i.e. P is the set

$$\{ (x,y) \in G \times K \mid \beta(x) = \alpha(y) \},$$

$\overline{\alpha}$ and $\overline{\beta}$ are the projections in G and K, respectively, and assume that α is a P-bijection. Since $\operatorname{Ker}\overline{\alpha} \cong \operatorname{Ker}\alpha$ then $\overline{\alpha}$ is P-injective. Finally, given $x \in G$, we have $\beta(x) \in H$ and so, there exists a $n \in P'$ such that $\beta(x)^n \in \operatorname{Im}\alpha$, i.e. there exist a $n \in P'$ and a $y \in K$ such that $\alpha(y) = \beta(x^n)$. Therefore, $(x^n, y) \in P$ and $\overline{\alpha}((x^n,y)) = x^n$ thus $x^n \in \operatorname{Im}\overline{\alpha}$. In conclusion, $\overline{\alpha}$ is P-surjective and since it is P-injective then it is a P-bijection.

The pair $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ where $\mathcal{E}$ is the family of P-bijections and $\mathcal{M}$ the family of P'-isolated morphisms is a factorization system of $\mathcal{N}$il:

(i) <u>$\mathcal{N}$il is $(\mathcal{E},\mathcal{M})$-factorizable</u>. Indeed, we obtain an $(\mathcal{E},\mathcal{M})$-factorization of φ by considering the diagram

$$\begin{tikzcd} G \arrow[dr, dashed, "e"] \arrow[drr, "\varphi"] \arrow[ddr, "r_G"'] & & \\ & K \arrow[r, "m"] \arrow[d, "\bar{r}"'] & H \arrow[d, "r_H"] \\ & G_P \arrow[r, "R(\varphi)"'] & H_P \end{tikzcd}$$

where the square is a pullback. Since r_H is a P-bijection because $R(r_H)$ is an isomorphism, then $\bar{r}$ is also a P-bijection and so e is a P-bijection. On the other hand, m is P'-isolated because $\bar{r}$ is isomorphic to r_K, since $\bar{r}$ is P-bijective and G_P is P-local ([31], p. 7).

(ii) <u>$\mathcal{E} \subset \mathcal{M}^{\uparrow}$</u>: In the following commutative diagram

$$\begin{tikzcd}
 & A \arrow[r, "e"] \arrow[dl, "r_A"'] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dr, "r_B"] & \\
A_P \arrow[dr, "R(u)"'] \arrow[ddrr, bend right, "R(e)"'] & C \arrow[r, "m"] \arrow[d, "r_C"'] & D \arrow[d, "r_D"] & B_P \\
 & C_P \arrow[r, "R(m)"'] & D_P &
\end{tikzcd}$$

the lower square is a pullback. Then there exists a unique morphism λ such that $m.\lambda = v$ and $r_C.\lambda = R(u).R(e)^{-1}.r_B$ which implies

$$r_C.\lambda.e = R(u).R(e)^{-1}.r_B.e = R(u).R(e)^{-1}.R(e).r_A = r_C.u.$$

Furthermore, $m.\lambda.e = v.e = m.u$, and so $\lambda.e = u$.

Let λ' be such that $m.\lambda' = v$ and $\lambda'.e = u$. Then $R(\lambda').R(e) = R(u)$, so $r_C.\lambda' = R(\lambda').r_B = R(u).R(e)^{-1}.r_B$. Hence $\lambda' = \lambda$.

(iii) Finally, $\mathcal{E}$ and $\mathcal{M}$ are classes closed under composition with isomorphisms.

Therefore, $\mathcal{F}$ is a factorization system.

But, as we already observed, the equality $(\mathrm{Mor}\,\mathcal{B})^{\uparrow} = \Sigma_R$ always holds. Hence, in this example, $(\mathrm{Mor}\,\mathcal{B})^{\uparrow}$ is the family of P-bijections and so $\mathcal{F}$ is reflective.

The following proposition characterizes the reflective prefactorization systems.

<u>PROPOSITION 1.15</u>. *Let* $(\mathcal{E},\mathcal{M}) \in \mathcal{P}$. *Then* $(\mathcal{E},\mathcal{M})$ *is reflective if and only if the following holds:*

$$(f.g \in \mathcal{E} \ \textit{and}\ f \in \mathcal{E} \Rightarrow g \in \mathcal{E}).$$

<u>Proof</u>: This proposition is an immediate consequence of the equivalence

$$g \in \overset{\circ}{\mathcal{E}} \Leftrightarrow \exists f \in \mathcal{E} : f.g \in \mathcal{E}$$

that we prove next. Let us assume that $g : A \longrightarrow B$ belongs to $\overset{\circ}{\mathcal{E}} = (\mathrm{Mor}(\mathcal{M}/T))^{\uparrow} = \Sigma_R$, where R is the reflector of $\mathcal{A}$ in $\mathcal{M}/T$. Since $Rg.r_A = r_B.g$, where Rg is an isomorphism and $r_A, r_B \in \mathcal{E}$ then $r_B.g \in \mathcal{E}$.

Conversely, if there exists a morphism $f \in \mathcal{S}$ such that $f.g \in \mathcal{S}$ then $f.g, f \in \Sigma_R$, because $\mathcal{S} \subset \overset{\circ}{\mathcal{S}} = \Sigma_R$ and therefore $g \in \overset{\circ}{\mathcal{S}}$. □

2. CATEGORIES OF FRACTIONS

Let Σ be a class of morphisms in a category $\mathcal{A}$.

DEFINITION 2.1. A category of fractions of $\mathcal{A}$ with respect to Σ is a pair $(P_\Sigma, \mathcal{A}(\Sigma^{-1}))$ where $\mathcal{A}(\Sigma^{-1})$ is a category and $P_\Sigma : \mathcal{A} \longrightarrow \mathcal{A}(\Sigma^{-1})$ is a functor satisfying the following conditions:

(i) For every $s \in \Sigma$, $P_\Sigma(s)$ is an isomorphism.

(ii) If $F : \mathcal{A} \longrightarrow \mathcal{A}'$ is a functor such that $F(s)$ is an isomorphism whenever $s \in \Sigma$, then there exists a unique functor $\overline{F} : \mathcal{A}(\Sigma^{-1}) \longrightarrow \mathcal{A}'$ such that $\overline{F}.P_\Sigma = F$.

Obviously, when the category $\mathcal{A}(\Sigma^{-1})$ exists (see ([40], 1.13) and ([41], 19.1.2)) it is unique up to isomorphism of categories. The class Σ_{P_Σ} is called the saturation of Σ and Σ is said to be saturated when $\Sigma_{P_\Sigma} = \Sigma$.

The theorem 1.13.10 of [40] guarantees that if $\mathcal{B}$ is a reflective subcategory of $\mathcal{A}$, with reflector R, then the category $\mathcal{A}(\Sigma_R{}^{-1})$ exists. If $(\mathcal{S}, \mathcal{M}) \in \mathcal{P}$ then the equivalence proved in 1.15 implies that $\overset{\circ}{\mathcal{S}} \subset \Sigma_F$ if and only if $\mathcal{S} \subset \Sigma_F$. Then, since $\mathcal{A}((\overset{\circ}{\mathcal{S}})^{-1}) = \mathcal{A}(\Sigma_R{}^{-1})$ satisfies (i) and (ii) of 2.1, the category $\mathcal{A}(\mathcal{S}^{-1})$ exists and is isomorphic to $\mathcal{A}((\overset{\circ}{\mathcal{S}})^{-1})$.

PROPOSITION 2.2. *Let $\mathcal{F} = (\mathcal{S}, \mathcal{M}) \in \mathcal{P}$. The category of fractions of $\mathcal{A}$ with respect to $\mathcal{S}$ is equivalent to $\mathcal{M}/T$, $\overset{\circ}{\mathcal{S}}$ is the saturation of $\mathcal{S}$, and so $\mathcal{S}$ is saturated exactly when $\mathcal{F}$ is reflective.*

Proof: If we consider the reflector R of $\mathcal{A}$ in $\mathcal{M}/T$ then, by definition of the category of fractions, there exists a unique functor $\overline{R} : \mathcal{A}(\mathcal{S}^{-1}) \longrightarrow \mathcal{M}/T$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{A} \arrow[r, "P_{\mathcal{S}}"] \arrow[dr, "R"'] & \mathcal{A}(\mathcal{S}^{-1}) \arrow[d, dashed, "\overline{R}"] \\ & \mathcal{M}/T \end{tikzcd}$$

We show that $\overline{R}$ is an equivalence. To prove this it suffices to show that there exist natural isomorphisms $\mathrm{Id}_{\mathcal{A}(\mathcal{E}^{-1})} \cong P_{\mathcal{E}}E\overline{R}$ and $\overline{R}P_{\mathcal{E}}E \cong \mathrm{Id}_{\mathcal{M}/T}$, where $E: \mathcal{M}/T \longrightarrow \mathcal{A}$ is the inclusion functor.

The isomorphism $\overline{R}P_{\mathcal{E}}E \cong \mathrm{Id}_{\mathcal{M}/T}$ is clear since the counity of the adjunction $R \dashv E$, $\xi: RE = \overline{R}P_{\mathcal{E}}E \xrightarrow{\cdot} \mathrm{Id}_{\mathcal{M}/T}$, is a natural isomorphism.

Let η be the unit of the adjunction $R \dashv E$. We saw in section 1 that $\eta_A = r_A \in \mathcal{E}$, for each $\mathcal{A}$-object A. Thus we can conclude that $P_{\mathcal{E}}\eta: P_{\mathcal{E}} \xrightarrow{\cdot} P_{\mathcal{E}}ER = P_{\mathcal{E}}E\overline{R}P_{\mathcal{E}}$ is a natural isomorphism. Consider the quasicategories ([27], 11.3) $[\mathcal{A}(\mathcal{E}^{-1}), \mathcal{A}(\mathcal{E}^{-1})]$ and $[\mathcal{A}, \mathcal{A}(\mathcal{E}^{-1})]$ of functors from $\mathcal{A}(\mathcal{E}^{-1})$ to $\mathcal{A}(\mathcal{E}^{-1})$ and functors from $\mathcal{A}$ to $\mathcal{A}(\mathcal{E}^{-1})$, respectively ([27], §15). The functor

$$\begin{array}{ccc} [\mathcal{A}(\mathcal{E}^{-1}), \mathcal{A}(\mathcal{E}^{-1})] & \longrightarrow & [\mathcal{A}, \mathcal{A}(\mathcal{E}^{-1})] \\ F & \longmapsto & FP_{\mathcal{E}} \\ (\theta: F \longrightarrow G) & \longmapsto & (\theta P_{\mathcal{E}}: FP_{\mathcal{E}} \longrightarrow GP_{\mathcal{E}}) \end{array}$$

is faithful and full (cf. [41], 19.1.4(d)). But $P_{\mathcal{E}}\eta$ is a natural isomorphism from $P_{\mathcal{E}}$ to $P_{\mathcal{E}}E\overline{R}P_{\mathcal{E}}$. Hence, $\mathrm{Id}_{\mathcal{A}} \cong P_{\mathcal{E}}E\overline{R}$ as we claimed. □

3. FINITELY WELL-COMPLETE CATEGORIES

DEFINITION 3.1. A category is said to be finitely well-complete if it admits finite limits and all intersections (even large ones) of strong monomorphisms.

We shall write an (FWC)-category meaning a finitely well-complete category.

EXAMPLES. (a) The categories $\mathcal{S}et$, $\mathcal{G}rp$ and $\mathcal{T}op$ are complete, cocomplete, wellpowered and cowellpowered ([27], [36]). Hence they are (FWC)-categories, as well as their duals.

(b) Any topological category over $\mathcal{S}et$ is (FWC) as well as its dual category. This is a specialization of the fact that a topological category $(\mathcal{A}, T: \mathcal{A} \longrightarrow \mathfrak{X})$ is (FWC) if $\mathfrak{X}$ is (FWC) (see [9], [1] VI.21).

(c) Any wellpowered abelian category with products (for example, the category of left modules over a ring) is a (FWC)-category.

(d) Consider the category $\mathcal{G}rf$ of directed graphs whose objects are the pairs (A,V), where A is a set and V a subset of A^2, and whose morphisms from (A,V) to (A',V') are

the maps $f: A \longrightarrow A'$ such that $(f(a_1),f(a_2)) \in V'$ whenever $(a_1,a_2) \in V$ i.e. morphisms are the maps compatible with the structure of the graph. Since the forgetful functor $U: \mathcal{G}rf \longrightarrow \mathcal{S}et$ creates limits and colimits then $\mathcal{G}rf$ is complete and cocomplete. Moreover $f:(A,V) \longrightarrow (A',V')$ is a monomorphism in $\mathcal{G}rf$ if and only if $f: A \longrightarrow A'$ is injective and given a graph (A',V'), for any set A and any map $f: A \longrightarrow A'$, the family of pairs (A,V) with $V \subset A^2$ and such that $f:(A,V) \longrightarrow (A',V')$ is compatible with the structure of graph is a set. Therefore, since $\mathcal{S}et$ is wellpowered, $\mathcal{G}rf$ is also wellpowered. Analogously we could conclude that $\mathcal{G}rf$ is cowellpowered. Hence, $\mathcal{G}rf$ and $\mathcal{G}rf^{op}$ are (FWC).

(e) Any algebraic category over $\mathcal{S}et$ is (FWC) because it is complete and wellpowered. Therefore, for instance, the categories of semigroups, monoids, rings, R-algebras, Boolean algebras, torsion-free abelian groups, C^*-algebras, compact abelian groups and Hausdorff compact spaces are (FWC). More generally, an algebraic category $(\mathcal{A}, U: \mathcal{A} \longrightarrow \mathfrak{X})$ is (FWC) if $\mathfrak{X}$ is complete and wellpowered or if $\mathfrak{X}$ is (FWC) and $\mathrm{StrMono}\mathfrak{X} = \mathrm{Mono}\mathfrak{X}$ (see [1]).

Any (FWC)-category has a terminal object that will be denoted by T.

<u>PROPOSITION 3.2</u>. *Let $\mathcal{N}$ be a class of morphisms of a category $\mathcal{A}$ satisfying the following conditions:*

(i) $\mathrm{Iso}\mathcal{A} \subset \mathcal{N} \subset \mathrm{Mono}\mathcal{A}$.

(ii) *$\mathcal{N}$ is closed under composition.*

(iii) *$\mathcal{A}$ admits all pullbacks of morphisms of $\mathcal{N}$ along any morphism of $\mathcal{A}$, and these pullbacks again belong to $\mathcal{N}$.*

(iv) *$\mathcal{A}$ admits all intersections of arbitrary families of morphisms of $\mathcal{N}$, and these intersections again belong to $\mathcal{N}$.*

Then $(\mathcal{N}^{\uparrow}, \mathcal{N})$ is a factorization system of $\mathcal{A}$.

<u>Proof</u>: See proof of the dual result in ([12], I.3.7). □

This proposition will be useful in the sequel.

<u>THEOREM 3.3</u>. *Let $<F,G,\eta,\xi>: \mathcal{A} \rightharpoonup \mathfrak{B}$ be an adjunction. If $\mathcal{A}$ is* (FWC) *then the prefactorization system $(\mathcal{E},\mathcal{M}) = ((G(\mathrm{Mor}\mathfrak{B}))^{\uparrow}, (G(\mathrm{Mor}\mathfrak{B}))^{\uparrow\downarrow})$ is a factorization system.*

<u>Proof</u>: We prove first that $\mathcal{A}$ is $(\mathcal{E},\mathcal{M})$-factorizable. Given $f: A \longrightarrow B$ in $\mathcal{A}$ consider the pullback $(P,(a,b))$ of (GFf, η_B). Then we have the following commutative diagram

$$\begin{tikzcd}
A \arrow[dr, dashed, "t"] \arrow[drr, "f"] \arrow[ddr, "\eta_A"'] & & \\
& P \arrow[r, "b"] \arrow[d, "a"] & B \arrow[d, "\eta_B"] \\
& GF(A) \arrow[r, "GFf"'] & GF(B)
\end{tikzcd}$$

where t is the unique morphism such that $b.t=f$ and $a.t=\eta_A$.

Since $GFf \in G(\mathrm{Mor}\mathcal{B}) \subset (G(\mathrm{Mor}\mathcal{B}))^{\uparrow\downarrow} = \mathcal{M}$, then $b \in \mathcal{M}$, by 1.3(iv). So it suffices to give an $(\mathcal{E},\mathcal{M})$-factorization of t.

Consider $\mathcal{N} = \mathcal{M} \cap \mathrm{StrMono}\mathcal{A}$. By 3.2 $(\mathcal{N}^{\uparrow},\mathcal{N})$ is a factorization system. If $t=n.u$ is the $(\mathcal{N}^{\uparrow},\mathcal{N})$-factorization of t, then $n \in \mathcal{M}$.

We show now that $u \in \mathcal{E}$. By lemma 1.8, this is equivalent to prove that $Fu \in \mathrm{Iso}\mathcal{B}$. Since $a.t=\eta_A$, then $\xi_{F(A)}.Fa.Ft=\xi_{F(A)}.F\eta_A=1$ and so Ft is a split monomorphism. But $Ft=Fn.Fu$, thus Fu is also a split monomorphism. Let us consider the pullback $(P',(a',b'))$ of $(GFu,\eta_{A'})$. We have a commutative diagram similar to the one above

$$\begin{tikzcd}
A \arrow[dr, dashed, "t'"] \arrow[drr, "u"] \arrow[ddr, "\eta_A"'] & & \\
& P' \arrow[r, "b'"] \arrow[d, "a'"] & A' \arrow[d, "\eta_{A'}"] \\
& GF(A) \arrow[r, "GFu"'] & GF(A')
\end{tikzcd}$$

where t' is the unique morphism such that $b'.t'=u$ and $a'.t'=\eta_A$.

The fact that Fu is a split monomorphism impiies, trivially, that GFu is also a split monomorphism and so a strong monomorphism. Hence $b' \in \mathrm{StrMono}\mathcal{A}$, since strong monomorphisms are preserved by pullbacks, and $b' \in \mathcal{N}$ because $GFu \in \mathcal{M}$. Since $u \in \mathcal{N}^{\uparrow}$ we have that $u \downarrow b'$ and so $b' \in \mathrm{Iso}\mathcal{A}$. From $GFu.a'.(b')^{-1}=\eta_{A'}$ we conclude that $\xi_{F(A')}.FGFu.Fa'.F(b')^{-1}=\xi_{F(A')}.F\eta_{A'}=1$, and so $Fu.\xi_{F(A)}.Fa'.F(b')^{-1}=1$. Thus, Fu is a split epimorphism and, since it is a monomorphism, it is an isomorphism, as it was required

to prove. □

As an immediate consequence of this theorem we have that if $\mathcal{A}$ is (FWC) and $\mathfrak{B}$ is a reflective subcategory of $\mathcal{A}$ then $\Phi\mathfrak{B}=((\mathrm{Mor}\mathfrak{B})^{\uparrow},(\mathrm{Mor}\mathfrak{B})^{\uparrow\downarrow}$ is a factorization system, and therefore the reflective interior of a prefactorization system in $\mathcal{P}$ is a factorization system of $\mathcal{A}$. In this case, the restrictions of Φ and Ψ referred in section 1 define a one-to-one correspondence between reflective subcategories of $\mathcal{A}$ and reflective factorization systems of $\mathcal{A}$.

For future reference, next we summarize the fact just described:

COROLLARY 3.4. *If $\mathcal{A}$ is* (FWC) *and $\mathfrak{B}$ is a reflective subcategory of $\mathcal{A}$ then the prefactorization system $\Phi\mathfrak{B}$ is a factorization system.* □

A functor is called conservative if it reflects isomorphisms.

The theorem of B. Day [14], that we state next, is also a consequence of 3.3.

THEOREM 3.5. *If $\mathcal{A}$ is* (FWC) *then for each left adjoint functor $F:\mathcal{A}\longrightarrow\mathcal{C}$ there exist a reflector $R:\mathcal{A}\longrightarrow\mathfrak{B}$ and a conservative left adjoint functor $Q:\mathfrak{B}\longrightarrow\mathcal{C}$ such that $F\cong QR$.*

Proof: Let us assume that $G:\mathcal{C}\longrightarrow\mathcal{A}$ is the right adjoint of F. By 3.3 the prefactorization system $(\mathcal{E},\mathcal{M})=((G(\mathrm{Mor}\mathcal{C}))^{\uparrow},(G(\mathrm{Mor}\mathcal{C}))^{\uparrow\downarrow})$ is a factorization system so it belongs to $\mathcal{P}$. Therefore $\mathcal{M}/T=\Psi(\mathcal{E},\mathcal{M})$ is a reflective subcategory of $\mathcal{A}$. We denote it by $\mathfrak{B}$, by $R:\mathcal{A}\longrightarrow\mathfrak{B}$ the reflector and by $E:\mathfrak{B}\longrightarrow\mathcal{A}$ the inclusion functor. By lemma 1.8, $\mathcal{E}^{\perp}=G(\mathcal{C})^{T\perp}$ and so $G(\mathcal{C})\subset\mathcal{E}^{\perp}$. But $\mathcal{E}^{\perp}=\mathfrak{B}$, by lemma 1.14, thus $G(C)\in\mathfrak{B}$, for every $C\in\mathcal{C}$. This implies that the functor G is factorizable as $G=EP$ along some functor $P:\mathcal{C}\longrightarrow\mathfrak{B}$. It is easy to verify that FE is left adjoint of P since F is left adjoint of $G=EP$ and E is an inclusion functor. Let $Q=FE$. It is clear that $QR\cong F$. To conclude the proof we show that Q is conservative. Let $f\in\mathrm{Mor}\mathfrak{B}$ such that Qf is an isomorphism, i.e. $f\in\Sigma_Q$. Furthermore, $\Sigma_Q=\Sigma_F\cap\mathrm{Mor}\mathfrak{B}=\mathcal{E}\cap\mathrm{Mor}\mathfrak{B}$. By 1.3(iii) it is obvious that $\mathrm{Mor}\mathfrak{B}\subset\mathcal{M}$, so $\Sigma_Q=\mathcal{E}\cap\mathcal{M}\cap\mathrm{Mor}\mathfrak{B}=\mathrm{Iso}\mathfrak{B}$, whence $f\in\mathrm{Iso}\mathfrak{B}$, as required. □

This theorem has the following interpretation:

Consider the quasicategory whose objects are the (FWC)-categories and whose morphisms are the left adjoint functors. Denote by $\mathcal{A}\mathrm{dj}$ the quasicategory that we get as the quotient of that quasicategory by the congruence "natural isomorphism of functors", i.e. the conglomerate of objects of $\mathcal{A}\mathrm{dj}$ is the conglomerate of (FWC)-categories and the conglomerate of morphisms is the conglomerate of the classes of natural isomorphism of left adjoint functors. If we denote the class of reflectors by $\mathcal{R}\mathrm{ef}$ and the class of left adjoint conservative functors by $\mathcal{C}\mathrm{ons}$ we may conclude that $(\mathcal{R}\mathrm{ef},\mathcal{C}\mathrm{ons})$ is a factorization system in $\mathcal{A}\mathrm{dj}$. Indeed, theorem 3.5 says that $\mathcal{A}\mathrm{dj}$ is $(\mathcal{R}\mathrm{ef},\mathcal{C}\mathrm{ons})$-factorizable. It remains to prove that $\mathcal{R}\mathrm{ef}=\mathcal{C}\mathrm{ons}^{\uparrow}$ and $\mathcal{C}\mathrm{ons}=\mathcal{R}\mathrm{ef}^{\downarrow}$ but, since $\mathcal{R}\mathrm{ef}$ and $\mathcal{C}\mathrm{ons}$ are closed under composition with isomorphisms, it suffices to check that $\mathcal{R}\mathrm{ef}\subset\mathcal{C}\mathrm{ons}^{\uparrow}$. So consider the following commutative diagram in $\mathcal{A}\mathrm{dj}$,

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "R"] \arrow[d, "U"'] & \mathcal{B} \arrow[d, "V"] \\
\mathcal{C} \arrow[r, "S"'] & \mathcal{D}
\end{tikzcd}$$

where $R\in\mathcal{R}\mathrm{ef}$ and $S\in\mathcal{C}\mathrm{ons}$. Let us assume that $<R,E;\eta,\xi>$ and $<U,G;\alpha,\beta>$ are adjunctions, $V\dashv K$ and $S\dashv N$. We have to prove that there is a left adjoint functor D, unique up to isomorphism, such that $DR\cong U$ and $SD\cong V$. Since $R\in\mathcal{R}\mathrm{ef}$, it suffices to show that there exists a left adjoint D such that $DR\cong U$. Let $D=UE$. It is obvious that the class of morphisms $(U\eta_A : U(A)\longrightarrow UER(A))_{A\in \mathrm{Obj}\,\mathcal{A}}$ defines a natural transformation from U to UER. But $SU\eta_A\cong VR\eta_A$, whenever the second morphism is an isomorphism, so $U\eta_A$ is also an isomorphism since $S\in\mathcal{C}\mathrm{ons}$. Thus, $U\cong UER=DR$. Before showing that D is left adjoint we prove that $\eta_{G(C)}$ is an isomorphism, for every $C\in\mathcal{C}$. Denote the morphism

$$ERG(C)\xrightarrow{\alpha_{ERG(C)}} GUERG(C)\xrightarrow{G((U\eta_{G(C)})^{-1})} GUG(C)\xrightarrow{G\beta_C} G(C)$$

by θ_C. We have that

$$\theta_C\cdot\eta_{G(C)}=G\beta_C\cdot G((U\eta_{G(C)})^{-1})\cdot\alpha_{ERG(C)}\cdot\eta_{G(C)}=G\beta_C\cdot G((U\eta_{G(C)})^{-1})\cdot GU\eta_{G(C)}\cdot\alpha_{G(C)}=$$
$$=G\beta_C\cdot\alpha_{G(C)}=1.$$

On the other hand, by definition of universal morphism, there exists a unique

morphism $\bar{f}$ such that $E\bar{f}.\eta_{G(C)}=\eta_{G(C)}$. But $\eta_{G(C)}.\theta_C.\eta_{G(C)}=\eta_{G(C)}$ and E is a full embedding hence $\eta_{G(C)}.\theta_C=1$.

Therefore, $\eta_G : G \xrightarrow{\cdot} ERG$ is a natural isomorphism. Consequently, $U \dashv ERG$ and so $D=UE \dashv RG$, because E is a full embedding.

We point out that the preceding theorem generalizes a result of H. Applegate and M. Tierney ([2], pp. 79-80).

4. REFLECTIVE SUBCATEGORIES AND COREFLECTIVE SUBCATEGORIES ASSOCIATED TO A FACTORIZATION SYSTEM

Let us assume that the category $\mathcal{A}$ has a terminal object T and an initial object I. Hence, we can also define two order-inverting maps Φ' and Ψ' between the conglomerate $\mathcal{P}'$ of prefactorization systems $(\mathcal{E},\mathcal{M})$ of $\mathcal{A}$ such that

(P') $\forall A \in \mathrm{Obj}\mathcal{A}$ the morphism $I \longrightarrow A$ has an $(\mathcal{E},\mathcal{M})$-factorization

and the conglomerate $\mathcal{R}'$ of coreflective subcategories of $\mathcal{A}$, as follows:

$$\begin{aligned} \Phi' : \mathcal{R}' &\longrightarrow \mathcal{P}' \\ \mathcal{C} &\longmapsto ((\mathrm{Mor}\mathcal{C})^{\downarrow\uparrow},(\mathrm{Mor}\mathcal{C})^{\downarrow}) \end{aligned}$$

$$\begin{aligned} \Psi' : \mathcal{P}' &\longrightarrow \mathcal{R}' \\ (\mathcal{E},\mathcal{M}) &\longmapsto I/\mathcal{E} \end{aligned}$$

where $I/\mathcal{E}$ is the category of $\mathcal{E}$-quotients of I.

In this case, each prefactorization system $\mathcal{F}=(\mathcal{E},\mathcal{M})$ has not only a reflective interior $\overset{\circ}{\mathcal{F}}=(\overset{\circ}{\mathcal{E}},\overset{\circ}{\mathcal{M}})$ but also a <u>coreflective closure</u> $\bar{\mathcal{F}}=(\bar{\mathcal{E}},\bar{\mathcal{M}})=\Phi'\Psi'(\mathcal{F})$.

If the categories $\mathcal{A}$ and $\mathcal{A}^{op}$ are (FWC) we can successively apply the operators Φ, Φ', Ψ and Ψ' to a factorization system (or, more generally, to a prefactorization system satisfying conditions (P) and (P')), getting several factorization systems.

PROPOSITION 4.1. *Assume that* $\mathcal{A}$ *and* $\mathcal{A}^{op}$ *are* (FWC). *Let* $\mathfrak{F}$ *and* $\mathfrak{G}$ *be two prefactorization systems of* $\mathcal{A}$ *fulfilling conditions* (*P*) *and* (*P'*). *Then:*

(i) $\overset{\circ}{\mathfrak{F}} \leq \mathfrak{F} \leq \overline{\mathfrak{F}}$.

(ii) $\overset{\circ\circ}{\mathfrak{F}} = \overset{\circ}{\mathfrak{F}}$ *and* $\overline{\overline{\mathfrak{F}}} = \overline{\mathfrak{F}}$.

(iii) $\mathfrak{F} \leq \mathfrak{G} \Rightarrow \overset{\circ}{\mathfrak{F}} \leq \overset{\circ}{\mathfrak{G}}$ *and* $\overline{\mathfrak{F}} \leq \overline{\mathfrak{G}}$.

(iv) $\overset{\circ}{\mathfrak{F}} \leq \overset{\circ\circ}{\mathfrak{F}} \leq \overset{\circ\overline{\circ}}{\mathfrak{F}} \leq \overline{\mathfrak{F}}$, $\overset{\circ}{\mathfrak{F}} \leq \overset{\circ\circ}{\mathfrak{F}} \leq \overset{\overline{\circ}\circ}{\mathfrak{F}} \leq \overline{\mathfrak{F}}$.

(v) $\overset{\circ\circ}{\mathfrak{F}} = \overset{\circ}{\mathfrak{F}}$ *and* $\overset{\overline{\circ}\overline{\circ}}{\mathfrak{F}} = \overline{\mathfrak{F}}$.

Proof: (i), (ii) and (iii) are obvious and (iv) is a consequence of them.

(v) The implications $\overset{\overline{\circ}}{\mathfrak{F}} \leq \overline{\mathfrak{F}} \Rightarrow \overset{\circ\overline{\circ}}{\mathfrak{F}} \leq \overset{\circ}{\overline{\mathfrak{F}}}$ and $\overset{\circ}{\overline{\mathfrak{F}}} \leq \overline{\mathfrak{F}} \Rightarrow \overset{\circ}{\mathfrak{F}} \leq \overset{\circ\overline{\circ}}{\mathfrak{F}}$ tell us that $\overset{\circ\overline{\circ}}{\mathfrak{F}} = \overset{\circ}{\overline{\mathfrak{F}}}$. The other equality can be obtained by duality. □

We conclude that, from a given factorization system $\mathfrak{F}$, by applying the operations "reflective interior" and "coreflective closure", we obtain at most seven distinct factorization systems. The number of distinct reflective subcategories is at most three, namely $\mathcal{B} \subset \mathcal{B}' \subset \mathcal{B}''$ corresponding to the reflective factorization systems

$$\overset{\circ}{\mathfrak{F}} \leq \overset{\circ\circ}{\mathfrak{F}} \leq \overset{\circ}{\overline{\mathfrak{F}}}$$

being the number of different coreflective subcategories also at most three, namely $\mathcal{C} \subset \mathcal{C}' \subset \mathcal{C}''$ corresponding to the coreflective factorization systems

$$\overline{\mathfrak{F}} \geq \overset{\overline{\circ}}{\mathfrak{F}} \geq \overset{\overline{\circ}}{\mathfrak{F}}.$$

In conclusion, we have the following situation:

$$
\begin{tikzcd}
 & & \mathcal{F}=(\mathcal{E},\mathcal{M}) \arrow[dll, "\Psi'"'] \arrow[drr, "\Psi"] & & \\
\mathcal{C}=\mathrm{I}/\mathcal{E} \arrow[d, "\Phi'"'] & & & & \mathcal{B}=\mathcal{M}/\mathrm{T} \arrow[d, "\Phi"] \\
\overline{\mathcal{F}} \arrow[d, "\Psi'"'] \arrow[drrrr, "\Psi"] & & & & \overset{\circ}{\mathcal{F}} \arrow[d, "\Psi"] \arrow[dlll, "\Psi'"'] \\
\mathrm{I}/\overline{\mathcal{E}}=\mathcal{C} \arrow[d, "\Phi'"'] & \mathcal{C}''=\mathrm{I}/\overset{\circ}{\mathcal{E}} \arrow[d, "\Phi'"] & & \mathcal{B}''=\overline{\mathcal{M}}/\mathrm{T} \arrow[d, "\Phi"] & \overset{\circ}{\mathcal{M}}/\mathrm{T}=\mathcal{B} \arrow[d, "\Phi"] \\
\overline{\mathcal{F}} & \overline{\overset{\circ}{\mathcal{F}}} \arrow[d, "\Psi'"'] \arrow[drrrr, "\Psi"] & & & \overset{\circ}{\overline{\mathcal{F}}} \arrow[d, "\Psi"] \arrow[dlll, "\Psi'"'] \\
 & \mathrm{I}/\overset{\circ}{\overline{\mathcal{E}}}=\mathcal{C}'' & \mathcal{C}'=\mathrm{I}/\overset{\circ}{\overline{\mathcal{E}}} \arrow[d, "\Phi'"'] & \mathcal{B}'=\overset{\circ}{\overline{\mathcal{M}}}/\mathrm{T} \arrow[d, "\Phi"] & \overset{\circ}{\overline{\mathcal{M}}}/\mathrm{T}=\mathcal{B}'' \\
 & & \overline{\overset{\circ}{\overline{\mathcal{F}}}} \arrow[d, "\Psi'"'] \arrow[drr, "\Psi"] & & \overset{\circ}{\overline{\overset{\circ}{\overline{\mathcal{F}}}}} \arrow[d, "\Psi"] \arrow[dll, "\Psi'"'] \\
 & & \mathrm{I}/\overline{\overset{\circ}{\overline{\mathcal{E}}}}=\mathcal{C}' & \mathrm{X}=? \quad \mathrm{Y}=? & \overset{\circ}{\overline{\overset{\circ}{\overline{\mathcal{M}}}}}/\mathrm{T}=\mathcal{B}'
\end{tikzcd}
$$

The chain ends here, because $\mathrm{X}=\mathcal{C}''$ and $\mathrm{Y}=\mathcal{B}''$. Actually, by 4.1 (v)

$$\Phi(\mathrm{Y})=\Phi\Psi(\overset{\circ}{\overline{\mathcal{F}}})=\overset{\circ}{\overline{\overset{\circ}{\overline{\mathcal{F}}}}}=\overset{\circ}{\overline{\mathcal{F}}}$$

so $Y = \Psi\Phi(Y) = \Psi(\overset{\circ}{\overline{\mathcal{F}}}) = \mathcal{B}''$. Dually $X = \mathcal{C}''$.

REMARK 4.2. The fact that $\mathcal{A}$ and $\mathcal{A}^{op}$ are (FWC) is assumed to guarantee that all prefactorization systems in 4.1(iv) are factorization systems and, consequently, to insure the existence of the reflective subcategories $\mathcal{B}'$ and $\mathcal{B}''$ and of the coreflective subcategories $\mathcal{C}'$ and $\mathcal{C}''$.

EXAMPLES 4.3. (a) In any category $\mathcal{A}$ with initial and terminal objects the factorization systems $\mathbb{0} = (\mathrm{Mor}\mathcal{A}, \mathrm{Iso}\mathcal{A})$ and $\mathbb{1} = (\mathrm{Iso}\mathcal{A}, \mathrm{Mor}\mathcal{A})$ are reflective and coreflective. Therefore, in these two examples, the seven corresponding factorization systems coincide.

(b) Let $\mathcal{A}_1$ be the category $\mathcal{G}\mathrm{rp}$ and $\mathcal{F}_1 = (\mathrm{Epi}\mathcal{A}_1, \mathrm{Mono}\mathcal{A}_1)$. We have that

$$\Psi\mathcal{F}_1 = \mathrm{Mono}/T = \{ G \in \mathcal{G}\mathrm{rp} \mid G \longrightarrow 0 \in \mathrm{Mono} \} = \{ G \in \mathcal{G}\mathrm{rp} \mid G \cong 0 \}.$$

Then $\overset{\circ}{\mathcal{F}}_1 = (\mathrm{Mor}, \mathrm{Iso}) < \mathcal{F}_1$.

On the other hand,

$$\Psi'\mathcal{F}_1 = I/\mathrm{Epi} = \{ G \in \mathcal{G}\mathrm{rp} \mid 0 \longrightarrow G \in \mathrm{Epi} \} = \{ G \in \mathcal{G}\mathrm{rp} \mid G \cong 0 \}$$

thus $\overline{\mathcal{F}}_1 = (\mathrm{Iso}, \mathrm{Mor}) > \mathcal{F}_1$. Hence,

$$\mathbb{0} = \overset{\circ}{\overline{\overset{\circ}{\mathcal{F}}}}_1 = \overline{\overset{\circ}{\mathcal{F}}}_1 = \overset{\circ}{\mathcal{F}}_1 < \mathcal{F}_1 < \overline{\mathcal{F}}_1 = \overset{\circ}{\overline{\mathcal{F}}}_1 = \overline{\overset{\circ}{\overline{\mathcal{F}}}}_1 = \mathbb{1}$$

and so $\mathcal{B} = \mathcal{B}' \subset \mathcal{B}''$ and $\mathcal{C} = \mathcal{C}' \subset \mathcal{C}''$.

(c) Let $\mathcal{A}_2$ be a category such that, $\mathrm{Hom}_{\mathcal{A}_2}(A, I) = \emptyset$, for every object A non-isomorphic to I. This occurs for example in $\mathcal{S}\mathrm{et}$, $\mathcal{T}\mathrm{op}$ and $\mathcal{G}\mathrm{rf}$.

Consider the factorization system $\mathcal{F}_2 = (\mathcal{E}_2, \mathcal{M}_2)$ with

$$\mathcal{E}_2 = \{ A \longrightarrow B \mid A \cong I \Rightarrow B \cong I \} \text{ and}$$
$$\mathcal{M}_2 = \{ A \longrightarrow B \mid A \cong I \} \cup \mathrm{Iso}\mathcal{A}_2.$$

Now

$$\Psi\mathcal{F}_2 = \{ A \in \mathrm{Obj}\mathcal{A}_2 \mid A \longrightarrow T \in \mathcal{M}_2 \} = \{ A \in \mathrm{Obj}\mathcal{A}_2 \mid A \cong I \text{ ou } A \cong T \}$$

so $\overset{\circ}{\mathcal{F}}_2 = \mathcal{F}_2$, as can be easily verified. On the other hand,

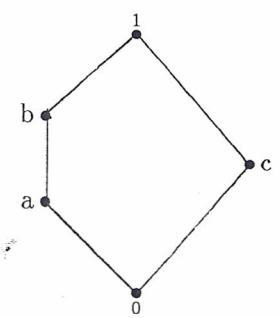
$$\Psi'\mathcal{F}_2 = \{ A \in \mathrm{Obj}\mathcal{A}_2 \mid I \longrightarrow A \in \mathcal{E}_2 \} = \{ A \in \mathrm{Obj}\mathcal{A}_2 \mid A \cong I \}$$

which implies $\overline{\mathcal{F}}_2 = (\mathrm{Iso}\mathcal{A}_2, \mathrm{Mor}\mathcal{A}_2)$. Therefore,

$$\mathbb{0} < \mathcal{F}_2 = \overset{\circ}{\mathcal{F}}_2 < \overset{\circ\circ}{\mathcal{F}}_2 = \overline{\mathcal{F}}_2 = \overset{\circ\cdot}{\mathcal{F}}_2 = \overline{\overset{\circ}{\mathcal{F}}}_2 = \overline{\mathcal{F}}_2 = \mathbb{1}.$$

In this case, $\mathcal{B} \subset \mathcal{B}' = \mathcal{B}''$ and $\mathcal{C} = \mathcal{C}' = \mathcal{C}''$.

(d) Consider the category $\mathcal{A}_3$ obtained from the lattice represented by the diagram



i.e.

$$\mathrm{Obj}\mathcal{A}_3 = \{0,a,b,c,1\} \text{ and } \#\mathrm{Mor}(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}, \quad \forall\, x,y \in \mathrm{Obj}\mathcal{A}_3.$$

It is trivially verified that $\mathcal{B} = \{c,1\}$ is a reflective subcategory of $\mathcal{A}_3$. Then $\mathcal{F}_3 = \Phi\mathcal{B}$ is a reflective factorization system since $\mathcal{A}_3$ is (FWC). We have that $\mathcal{F}_3 = (\mathcal{E}_3, \mathcal{M}_3)$ with

$$\mathcal{E}_3 = \{0 \longrightarrow 0, 0 \longrightarrow c, a \longrightarrow a, a \longrightarrow b, a \longrightarrow 1, b \longrightarrow b, b \longrightarrow 1, 1 \longrightarrow 1\},$$
$$\mathcal{M}_3 = \{0 \longrightarrow 0, 0 \longrightarrow a, 0 \longrightarrow b, c \longrightarrow c, c \longrightarrow 1, a \longrightarrow a, b \longrightarrow b, 1 \longrightarrow 1\}$$

and, therefore, $\mathcal{C} = 0/\mathcal{E}_3 = \{0,c\}$. On the other hand, $\overline{\mathcal{F}}_3 = \Phi'\mathcal{C} = (\overline{\mathcal{E}}_3, \overline{\mathcal{M}}_3)$ where

$$\overline{\mathcal{M}}_3 = (\mathrm{Mor}\mathcal{C})^{\downarrow} = \{0 \longrightarrow 0, 0 \longrightarrow a, 0 \longrightarrow b, c \longrightarrow c, c \longrightarrow 1, a \longrightarrow a, a \longrightarrow b, b \longrightarrow b, 1 \longrightarrow 1\} \supset \mathcal{M}_3.$$

Then $\mathcal{B}'' = \{c,1\} = \mathcal{B}$, hence $\mathcal{B} = \mathcal{B}' = \mathcal{B}''$. In conclusion, we have that

$$\mathbb{0} < \mathcal{F}_3 = \overset{\circ}{\mathcal{F}}_3 = \overline{\overset{\circ}{\mathcal{F}}}_3 = \overset{\circ\circ}{\mathcal{F}}_3 < \overline{\mathcal{F}}_3 = \overline{\overset{\circ}{\mathcal{F}}}_3 = \overline{\overset{\circ\circ}{\mathcal{F}}}_3 < \mathbb{1}, \quad \mathcal{B} = \mathcal{B}' = \mathcal{B}'' \text{ and } \mathcal{C} = \mathcal{C}' = \mathcal{C}''.$$

(e) Consider the category $\mathcal{T}op$ and $\mathcal{F} = (\mathrm{Epi}, \mathrm{ExtMono})$. Then

$$\Psi\mathcal{F} = \mathrm{ExtMono}/T = \{X \in \mathcal{T}op \mid X \longrightarrow T \in \mathrm{ExtMono}\} = \{X \in \mathcal{T}op \mid X \cong T\} \cup \{\emptyset\}.$$

Hence $\overset{\circ}{\mathcal{F}} = (\overset{\circ}{\mathcal{E}}, \overset{\circ}{\mathcal{M}})$, with

$$\overset{\circ}{\mathcal{E}} = \{ X \longrightarrow Y \mid X = \emptyset \Rightarrow Y = \emptyset \} \quad \text{and} \quad \overset{\circ}{\mathcal{M}} = \{ X \longrightarrow Y \mid X = \emptyset \} \cup \text{Iso}.$$

Thus $\overset{\circ}{\mathfrak{F}} < \mathfrak{F}$. On the other hand,

$$\Psi' \mathfrak{F} = I/\text{Epi} = \{ X \in \mathcal{T}op \mid I \longrightarrow X \in \text{Epi} \} = \{ \emptyset \}$$

and so $\overline{\mathfrak{F}} = (\text{Iso}, \text{Mor}) > \mathfrak{F}$.

Since $\overset{\circ}{\mathfrak{F}}$ is equal to the factorization system $\mathfrak{F}_2$ of (c) (with $\mathcal{A}_2 = \mathcal{T}op$), then

$$\mathbb{0} < \overset{\circ}{\mathfrak{F}} < \mathfrak{F} < \overset{\circ}{\overline{\mathfrak{F}}} = \overline{\overset{\circ}{\mathfrak{F}}} = \overline{\mathfrak{F}} = \overset{\circ}{\overline{\mathfrak{F}}} = \overline{\overset{\circ}{\mathfrak{F}}} = \mathbb{1},$$

$\mathcal{B} \subset \mathcal{B}' = \mathcal{B}''$ and $\mathcal{C} = \mathcal{C}' = \mathcal{C}''$.

(f) The existence of examples in which one of the inequalities in 4.1(iv) is strict allows us to get a factorization system in which the six factorization systems of 4.1(iv) are all distinct. For example, if we consider the categories $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ of the preceding examples, the corresponding duals $\mathcal{A}_4$, $\mathcal{A}_5$ and $\mathcal{A}_6$, the factorization systems $\mathfrak{F}_1$, $\mathfrak{F}_2$, $\mathfrak{F}_3$ and their duals $\mathfrak{F}_4$, $\mathfrak{F}_5$, $\mathfrak{F}_6$, constructing the product category $\mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times \mathcal{A}_4 \times \mathcal{A}_5 \times \mathcal{A}_6$ ([27], 4.8), then for the factorization system $\mathfrak{F} = \left(\prod_{i=1}^{6} \mathcal{E}_i, \prod_{i=1}^{6} \mathcal{M}_i \right)$ all the inequalities

$$\overset{\circ}{\mathfrak{F}} < \overset{\circ}{\overline{\overset{\circ}{\mathfrak{F}}}} \begin{array}{l} < \overset{\circ}{\overline{\mathfrak{F}}} < \\ < \overset{\circ}{\overline{\mathfrak{F}}} < \end{array} \overline{\overset{\circ}{\overline{\mathfrak{F}}}} < \overline{\mathfrak{F}}$$

are strict.

5. TORSION THEORIES

In the sequel, unless otherwise stated, $\mathcal{A}$ has an initial object I and a terminal object T.

The conglomerate $\mathcal{S}_{\mathcal{A}}$ of subcategories of $\mathcal{A}$ is partially ordered by inclusion and it is even a (possibly large) complete lattice. Consider the operators

$$\begin{aligned} \bullet^{\rightarrow} : \mathcal{S}_{\mathcal{A}} &\longrightarrow \mathcal{S}_{\mathcal{A}} \\ \mathcal{B} &\longmapsto \mathcal{B}^{\rightarrow} := \left\{ A \in \text{Obj}\,\mathcal{A} \mid \# \text{Hom}_{\mathcal{A}}(B,A) = 1, \ \forall B \in \text{Obj}\,\mathcal{B} \right\} \end{aligned}$$

and

$$\bullet^{\leftarrow}: \mathcal{S}_{\mathcal{A}} \longrightarrow \mathcal{S}_{\mathcal{A}}$$
$$\mathcal{B} \longmapsto \mathcal{B}^{\leftarrow} := \left\{ A \in \mathrm{Obj}\mathcal{A} \mid \#\mathrm{Hom}_{\mathcal{A}}(A,B) = 1,\ \forall\, B \in \mathrm{Obj}\mathcal{B} \right\},$$

that generalize the corresponding operators $\bullet^{\rightarrow}$ and $\bullet^{\leftarrow}$ defined in I.1.

Obviously, these operators satisfy similar conditions to the ones of I.1.5 and

$$\bullet^{\rightarrow}: \mathcal{S}_{\mathcal{A}}^{op} \longrightarrow \mathcal{S}_{\mathcal{A}} \quad \text{and} \quad \bullet^{\leftarrow}: \mathcal{S}_{\mathcal{A}} \longrightarrow \mathcal{S}_{\mathcal{A}}^{op}$$

are contravariant functors.

Again, the pair $(\bullet^{\rightarrow}, \bullet^{\leftarrow})$ is a Galois connection and the functors $T = \bullet^{\rightarrow\leftarrow}$ and $F = \bullet^{\leftarrow\rightarrow}$ are two closure operators in the complete lattice $\mathcal{S}_{\mathcal{A}}$.

The following proposition gives a relation between reflective subcategories and coreflectives subcategories which leads Cassidy, Hébert and Kelly to the foundations of a theory of torsion theories in categories with initial and terminal objects, that is, in a more general context than that of abelian categories, where it had its origin.

<u>PROPOSITION 5.1</u>. *Let* $\mathcal{F} = (\mathcal{E}, \mathcal{M})$ *be a prefactorization system of* $\mathcal{A}$, $\mathcal{B} = \mathcal{M}/T$ *and* $\mathcal{C} = I/\mathcal{E}$ *the corresponding subcategories. Then:*

(i) $\mathcal{C} \subset \mathcal{B}^{\leftarrow}$.

(ii) *If* $\mathcal{B}$ *is a reflective subcategory of* $\mathcal{A}$, *with reflector* R, *then* $\mathcal{B}^{\leftarrow} = \{ A \in \mathrm{Obj}\mathcal{A} \mid R(A) \cong R(I) \}$ *and if, in addition,* $\mathcal{F} = \Phi\mathcal{B}$ *then* $\mathcal{C} = \mathcal{B}^{\leftarrow}$.

(iii) *If* $\mathcal{F} \in \mathcal{P}$ *then* $\mathcal{C}'' = \mathcal{B}^{\leftarrow} = \{ A \in \mathrm{Obj}\mathcal{A} \mid R(A) \cong R(I) \}$.

<u>Proof</u>: (i) If C is an object of $\mathcal{C}$ then $e: I \longrightarrow C$ is a morphism of $\mathcal{E}$. For each object B of $\mathcal{B}$, the morphism $m: B \longrightarrow T$ beiongs to $\mathcal{M}$. Since $e \downarrow m$, it follows that $\#\mathrm{Hom}_{\mathcal{A}}(C,B) = 1$.

(ii) The fact that R is left adjoint implies that R(I) is initial in $\mathcal{B}$ and so that

$$\begin{aligned} A \in \mathcal{B}^{\leftarrow} &\Leftrightarrow \#\mathrm{Hom}_{\mathcal{A}}(A,B) = 1,\ \forall\, B \in \mathrm{Obj}\mathcal{B} \\ &\Leftrightarrow \#\mathrm{Hom}_{\mathcal{B}}(R(A),B) = 1,\ \forall\, B \in \mathrm{Obj}\mathcal{B} \\ &\Leftrightarrow R(A) \cong R(I) \end{aligned}$$

With regard to the inclusion $\mathcal{B}^{\leftarrow} \subset \mathcal{C}$ when $\mathcal{F} = ((\mathrm{Mor}\mathcal{B})^{\uparrow}, (\mathrm{Mor}\mathcal{B})^{\uparrow\downarrow})$, we have to show that if $C \in \mathcal{B}^{\leftarrow}$ then $e: I \longrightarrow C \in (\mathrm{Mor}\mathcal{B})^{\uparrow}$. So, let C be an object of $\mathcal{B}^{\leftarrow}$. If $b: B \longrightarrow B'$ is a morphism in $\mathcal{B}$ and $u: I \longrightarrow B$ and $v: C \longrightarrow B'$ are morphisms such that $v.e = b.u$ (we point out that u and v are unique such) it is clear that there exists a unique morphism d such that the following diagram is commutative

$$\begin{tikzcd} I \arrow[r, "e"] \arrow[d, "u"'] & C \arrow[d, "v"] \arrow[dl, dashed, "d"'] \\ B \arrow[r, "b"'] & B' \end{tikzcd}$$

since $\#\mathrm{Hom}_{\mathcal{A}}(C,B) = \#\mathrm{Hom}_{\mathcal{A}}(C,B') = 1$ and I is initial.

(iii) $\mathfrak{B}$ is reflective because $\mathfrak{F} \in \mathfrak{P}$, hence $\mathfrak{B}^{\leftarrow} = \{ A \in \mathrm{Obj}\mathcal{A} \mid R(A) \cong R(I) \}$, by (ii). But $\overset{\circ}{\mathfrak{F}}$ also belongs to $\mathfrak{P}$ and the corresponding subcategories $\overset{\circ}{\mathcal{M}}/T$ and $I/\overset{\circ}{\mathcal{E}}$ are, respectively, $\mathfrak{B}$ and $\mathfrak{C}''$. Applying again (ii) we get the equality $\mathfrak{C}'' = \mathfrak{B}^{\leftarrow}$. □

The characterization of torsion theories in abelian categories might suggest that one should define a torsion theory as a pair of subcategories $(\mathfrak{C},\mathfrak{B})$ such that $\mathfrak{C}^{\rightarrow} = \mathfrak{B}$ and $\mathfrak{B}^{\leftarrow} = \mathfrak{C}$. However, there would be no guarantee that $\mathfrak{B}$ is reflective and $\mathfrak{C}$ coreflective. Indeed, let us consider the category whose objects and non-identity morphisms are represented in the following diagram:

$$\begin{tikzcd} I \bullet \arrow[r, "p"] \arrow[d, "q"'] & \bullet P \arrow[d, "\bar{p}"] \\ Q \bullet \arrow[r, "\bar{q}"'] \arrow[ur, shift left, "f"] \arrow[ur, shift right, "g"'] & \bullet T \end{tikzcd}$$

Of course $\{I,Q\}^{\rightarrow} = \{Q,T\}$ and $\{Q,T\}^{\leftarrow} = \{I,Q\}$. However $\{I,Q\}$ is not coreflective because, as can be easily observed, it is impossible to define a coreflection morphism from P to $\{I,Q\}$. But $\{Q,T\}$ is reflective:

The morphisms $I \longrightarrow Q$, $P \longrightarrow T$, $Q \longrightarrow Q$, $T \longrightarrow T$ are reflection morphisms from $\mathcal{A}$ to $\{Q,T\}$.

So, even the reflectivity of the subcategory $\mathfrak{B}$ does not implies that $\mathfrak{B}^{\leftarrow}$ is coreflective. However, when the category is (FWC) then, by 3.4, $\Phi\mathfrak{B}$ is a factorization system if $\mathfrak{B}$ is reflective, thus, by 5.1(ii), $\mathfrak{B}^{\leftarrow}$ is coreflective. In this example (where, obviously, the category is not (FWC)) $\{I,P\}^{\rightarrow} = \{P,T\}$ and $\{P,T\}^{\leftarrow} = \{I,P\}$, $\{I,P\}$ is coreflective but $\{P,T\}$ is not reflective. Therefore, the dual category is not (FWC) also. It is easy to see directly that this category has neither products nor coproducts.

Another example is the category $\mathcal{G}rf$, which is (FWC) as well as its dual. Denoting by $\mathcal{R}efl$ the subcategory of graphs in which every vertex has a loop and by $\mathcal{O}loop$ the subcategory of graphs with exactly one loop, we have that $\mathcal{R}efl^{\rightarrow} = \mathcal{O}loop$ and $\mathcal{O}loop^{\leftarrow} = \mathcal{R}efl$. However, $\mathcal{R}efl$ is not coreflective (it is impossible to define morphisms from a graph with all loops to a graph with no loop) and, therefore, $\mathcal{O}loop$ is not reflective.

In the following definition, $\mathcal{A}$ denotes an arbitrary category and not necessarily a category with initial and terminal objects.

DEFINITION 5.2. A pair $(\mathcal{C},\mathcal{B})$ of subcategories of $\mathcal{A}$ is called a torsion theory of $\mathcal{A}$ provided that $\mathcal{B}$ is reflective, $\mathcal{C}$ is coreflective, $\mathcal{B}^{\leftarrow} = \mathcal{C}$, $\mathcal{C}^{\rightarrow} = \mathcal{B}$ and $\Phi\mathcal{B}$ and $\Phi'\mathcal{C}$ are factorization systems.

Then, we say that $\mathcal{C}$ is a torsion subcategory and its objects are torsion objects. Analogously, $\mathcal{B}$ is a torsion-free subcategory and its objects, torsion-free objects.

From now on, given a torsion theory $(\mathcal{C},\mathcal{B})$, we denote the reflection morphisms in $\mathcal{B}$ by $(r_A : A \longrightarrow R(A))_{A \in \mathrm{Obj}\,\mathcal{A}}$ and the coreflection morphisms in $\mathcal{C}$ by $(s_A : S(A) \longrightarrow A))_{A \in \mathrm{Obj}\,\mathcal{A}}$.

REMARK 5.3. According to remarks and examples preceding 5.2, if $\mathcal{A}$ and $\mathcal{A}^{op}$ are (FWC)-categories then $(\mathcal{C},\mathcal{B})$ is a torsion theory if and only if $\mathcal{B}$ is reflective, $\mathcal{C}^{\rightarrow} = \mathcal{B}$ and $\mathcal{B}^{\leftarrow} = \mathcal{C}$. Hence, in this case, any reflective subcategory $\mathcal{B}$ generates a torsion theory $(\mathcal{B}^{\leftarrow}, \mathcal{B}^{\leftarrow\rightarrow})$. This is the smallest torsion theory of $\mathcal{A}$ such that the objects of $\mathcal{B}$ are torsion-free objects. Dually, any coreflective subcategory $\mathcal{C}$ cogenerates the torsion theory $(\mathcal{C}^{\rightarrow\leftarrow}, \mathcal{C}^{\rightarrow})$ which is also the smallest one in the appropriate sense.

Now we prove that definition 5.2 is a generalization to any category of the one given by Cassidy, Hébert and Kelly in ([11], p. 307). In fact, if $\mathcal{A}$ has initial and terminal objects we have:

PROPOSITION 5.4. *A pair $(\mathcal{C},\mathcal{B})$ of subcategories of $\mathcal{A}$ is a torsion theory if and only if there is a reflective factorization system $\mathcal{F}$ such that $\overline{\mathcal{F}}$ is also a factorization system, $\mathcal{B} = \Psi\mathcal{F} = \Psi\overline{\mathcal{F}}$ and $\mathcal{C} = \Psi'\mathcal{F}$.*

Proof: If $(\mathcal{C},\mathcal{B})$ is a torsion theory of $\mathcal{A}$ then $\mathcal{F} = \Phi\mathcal{B}$ is a reflective factorization system. It is obvious that $\Psi\mathcal{F} = \mathcal{B}$. Assuming that $\mathcal{F} = (\mathcal{E},\mathcal{M})$, we denote the subcategory

$\Psi'\mathcal{F}=I/\mathcal{E}$ by $\mathcal{D}$. Then $\mathcal{D}=\mathcal{B}^{\leftarrow}$, by 5.1(ii) and so $\mathcal{D}=\mathcal{C}$, by hypothesis. Therefore $\mathcal{C}=\Psi'\mathcal{F}$, $\overline{\mathcal{F}}=\Phi'\mathcal{D}=\Phi'\mathcal{C}$ is a factorization system and $\Psi\overline{\mathcal{F}}=\overline{\mathcal{M}}/T=\mathcal{B}''$. Since $\mathcal{C}^{\rightarrow}=\mathcal{B}''$, by the dual of 5.1(iii), then $\mathcal{B}''=\mathcal{B}$, i.e. $\Psi\overline{\mathcal{F}}=\mathcal{B}$.

Conversely, $\mathcal{B}$ is reflective and $\mathcal{C}$ is coreflective because $\mathcal{B}=\Psi\mathcal{F}$ and $\mathcal{C}=\Psi'\mathcal{F}$ and it is clear that $\Phi\mathcal{B}$ and $\Phi'\mathcal{C}$ are factorization systems. Since $\mathcal{F}$ is reflective, from 5.1 (ii) we conclude that $\mathcal{C}=\mathcal{B}^{\leftarrow}$. Finally, by the dual of 5.1 (iii), $\mathcal{C}^{\rightarrow}=\mathcal{B}''=\Psi\overline{\mathcal{F}}=\mathcal{B}$. □

To state the equality $\mathcal{B}=\Psi\mathcal{F}=\Psi\overline{\mathcal{F}}$ is equivalent to assertion that the three subcategories $\mathcal{B}$, $\mathcal{B}'$ and $\mathcal{B}''$ coincide. Then, since $\mathcal{F}$ is reflective, the subcategories $\mathcal{C}$, $\mathcal{C}'$ and $\mathcal{C}''$ also coincide.

<u>EXAMPLES</u>. (a) For any category $\mathcal{A}$ with initial object I and terminal object T, the pairs $(\mathcal{A},\mathcal{T}er)$ and $(\mathcal{I}ni,\mathcal{A})$ are torsion theories, where $\mathcal{T}er$ is the subcategory of objects isomorphic to T and $\mathcal{I}ni$ the subcategory of objects isomorphic to I. These will be called the <u>trivial torsion theories</u>.

(b) An important question that naturally arises is that of knowing whether a category admits non-trivial torsion theories. For example, if we consider the category $\mathcal{A}_2$ of 4.3 (c) and the reflective factorization system $\mathcal{F}_2$, then $\overline{\mathcal{F}}_2$ is still a factorization system. But $\Psi\mathcal{F}_2=\{A\in \mathrm{Obj}\,\mathcal{A}_2 \mid A\cong I \text{ or } A\cong T\}$ and $\Psi\overline{\mathcal{F}}_2=\mathcal{A}_2$. Hence $\mathcal{F}_2$ does not give rise to any torsion theory. Let $\mathcal{A}_2=\mathcal{S}et$. This category has only four factorization systems, namely, (Mor,Iso), (Iso,Mor), $\mathcal{F}_2$ and (Epi,Mono) being only the first three reflective. Thus, we can conclude that in $\mathcal{S}et$ there exist only the trivial torsion theories.

(c) Let $\mathcal{A}$ be a preorder category defined by a complete lattice. Obviously, $\mathcal{A}$ and $\mathcal{A}^{op}$ are (FWC)-categories and a subcategory $\mathcal{B}$ of $\mathcal{A}$ is reflective if and only if it is closed under meets.

By 5.3, the torsion theories of $\mathcal{A}$ are the pairs $(\mathcal{B}^{\leftarrow},\mathcal{B})$ such that $\mathcal{B}$ is reflective and $\mathcal{B}^{\leftarrow\rightarrow}=\mathcal{B}$. It is trivially verified that, in this case, $\mathcal{B}=\mathcal{B}^{\leftarrow\rightarrow}$ implies the reflectivity of $\mathcal{B}$, so the torsion theories of $\mathcal{A}$ are the pairs $(\mathcal{B}^{\leftarrow},\mathcal{B})$ such that $\mathcal{B}^{\leftarrow\rightarrow}=\mathcal{B}$, i.e. $(\mathcal{C},\mathcal{B})$ is a torsion theory if and only if $\mathcal{C}^{\rightarrow}=\mathcal{B}$ and $\mathcal{B}^{\leftarrow}=\mathcal{C}$.

For example, in the category $\mathcal{A}_3$ of 4.3 (d) there exist only the torsion theories $(\{0,a\},\{a,b,1\})$, $(\{0,a,b\},\{b,1\})$, $(\{0\},\mathcal{A}_3)$, $(\mathcal{A}_3,\{1\})$ and $(\{0,c\},\{c,1\})$.

Let $\mathcal{B}$ be a reflective subcategory such that that $\mathcal{F}=\Phi\mathcal{B}$ belongs to $\mathcal{P}'$. Assume also that $\overline{\mathcal{F}}$ and $\overset{o}{\overline{\mathcal{F}}}$ are factorization systems.

We have then a reflective factorization system, $\overset{o}{\overline{\mathcal{F}}}$, such that $\overset{\overline{o}}{\overline{\mathcal{F}}}$ is also a

factorization system, because

$$\overline{\overset{\circ}{\overline{\mathcal{F}}}} = \overset{\circ}{\overline{\overline{\mathcal{F}}}} = \overset{\circ}{\overline{\mathcal{F}}} = \overline{\mathcal{F}}.$$

Since

$$\Psi\overset{\circ}{\overline{\mathcal{F}}} = \Psi\overline{\mathcal{F}} = \Psi\overset{\circ}{\overline{\mathcal{F}}} = \mathcal{B}', \quad \Psi\overset{\circ}{\overline{\overline{\mathcal{F}}}} = \Psi\overline{\mathcal{F}} = \mathcal{B}' \quad \text{and} \quad \Psi'\overset{\circ}{\overline{\mathcal{F}}} = \mathcal{C}' = \Psi'\overset{\circ}{\overline{\overline{\mathcal{F}}}} = \Psi'\overline{\mathcal{F}} = \mathcal{C}$$

then $(\mathcal{C},\mathcal{B}')$ is a torsion theory and $\mathcal{B} \subset \mathcal{B}'$.

In conclusion, a reflective subcategory $\mathcal{B}$ generates a torsion theory if $\mathcal{F} = \Phi\mathcal{B}$ belongs to $\mathcal{P}'$ and $\overline{\mathcal{F}}$ and $\overset{\circ}{\overline{\mathcal{F}}}$ are factorization systems.

If $\Phi\mathcal{B}$ is a factorization system we denote the object through which the morphism $I \longrightarrow T$ factors by the symbol $*$.

<u>PROPOSITION 5.5</u>. $(\mathcal{C},\mathcal{B})$ *is a torsion theory if and only if* $\mathcal{B}$ *is reflective,* $\mathcal{C}$ *is coreflective,* $\Phi\mathcal{B}$ *and* $\Phi'\mathcal{C}$ *are factorization systems,* $\mathcal{B} = \{A \in \mathrm{Obj}\,\mathcal{A} \mid S(A) \cong *\}$, *where* S *is the coreflector of* $\mathcal{A}$ *in* $\mathcal{C}$ *and* $\mathcal{C} = \{A \in \mathrm{Obj}\,\mathcal{A} \mid R(A) \cong *\}$, *where* R *is the reflector of* $\mathcal{A}$ *in* $\mathcal{B}$.

<u>Proof</u>: Let $(\mathcal{C},\mathcal{B})$ be a torsion theory. Then by definition $\mathcal{B}$ is reflective, $\mathcal{C}$ is coreflective and $\Phi\mathcal{B}$ and $\Phi'\mathcal{C}$ are factorization systems. Let $\mathcal{F} = \Phi\mathcal{B}$. By 5.1(ii) $\mathcal{B}^{\leftarrow} = \{A \in \mathrm{Obj}\,\mathcal{A} \mid R(A) \cong R(I)\} = \{A \in \mathrm{Obj}\,\mathcal{A} \mid R(A) \cong *\}$. Hence $\mathcal{C} = \{A \in \mathrm{Obj}\,\mathcal{A} \mid R(A) \cong *\}$ since $\mathcal{C} = \mathcal{B}^{\leftarrow}$ by hypothesis. Dually, if we consider the factorization system $\Phi'\mathcal{C}$, by the dual of 5.1 (ii), we conclude in a similar way that $\mathcal{B} = \{A \in \mathrm{Obj}\,\mathcal{A} \mid S(A) \cong *\}$.

Conversely, it suffices to prove that $\mathcal{B}^{\leftarrow} = \mathcal{C}$ and $\mathcal{C}^{\rightarrow} = \mathcal{B}$. Applying 5.1(ii) to the factorization system $\Phi\mathcal{B}$ we get $\mathcal{B}^{\leftarrow} = \{A \in \mathrm{Obj}\,\mathcal{A} \mid R(A) \cong *\}$ and, consequently, $\mathcal{B}^{\leftarrow} = \mathcal{C}$. Finally, applying the dual of 5.1 (ii) to $\Phi'\mathcal{C}$, we conclude that $\mathcal{C}^{\rightarrow} = \mathcal{B}$. □

We have, as an immediate consequence of this proposition, the following corollary:

<u>COROLLARY 5.6</u>. *If* $\mathcal{A}$ *and* $\mathcal{A}^{op}$ *are (FWC)-categories then the following assertions are equivalent:*

(*i*) $(\mathcal{C},\mathcal{B})$ *is a torsion theory.*

(ii) $\mathcal{B}$ *is reflective,* $\mathcal{C}$ *is coreflective,* $\mathcal{B} = \{A \in \mathrm{Obj}\,\mathcal{A} \mid S(A) \cong *\}$ *and* $\mathcal{C} = \{A \in \mathrm{Obj}\,\mathcal{A} \mid R(A) \cong *\}$. □

As an example of application of these statements we determine the torsion theories of

the category $\mathcal{G}$rf. Let $\mathcal{D}$isc be the subcategory of discrete graphs (without edges) and $\mathcal{PT}$er the subcategory of discrete graphs with one vertex and of the terminal graph.

PROPOSITION 5.7. *In the category* $\mathcal{G}$rf *there exist only the torsion theories* $(\mathcal{I}ni,\mathcal{G}rf)$, $(\mathcal{G}rf,\mathcal{T}er)$ *and* $(\mathcal{D}\mathrm{isc},\mathcal{PT}\mathrm{er})$.

Proof: We first remark that in $\mathcal{G}$rf the initial object is the empty graph, $\emptyset$, and the terminal object is the graph with one vertex and one loop. It is easy to see that these three pairs of subcategories verify the conditions of 5.3 and so they are torsion theories. Let us prove that there are any more torsion theories. If $(\mathcal{C},\mathcal{B})$ is a torsion theory of $\mathcal{G}$rf with $\mathcal{B}\neq\mathcal{T}$er and $\mathcal{B}\neq\mathcal{PT}$er then $\emptyset\in\mathcal{B}$. Indeed, in these conditions and being reflective, $\mathcal{B}$ is also distinct from $\mathcal{O}$loop and contains, at least, one of the following sets of graphs:

- the empty graph.
- a graph G with, at least, two vertices and a number of loops distinct from one.
- the graph P with one vertex and without edges and a graph G' of $\mathcal{O}$loop with, at least, two vertices.

In the first case there is nothing to prove.

In the second case let us study first the situation where G does not have loops. Let v_1 and v_2 be two vertices of G and consider the maps:

$$\begin{array}{l}\varphi_1 : G \longrightarrow G \\ \quad v \longmapsto v_1\end{array} \qquad \text{and} \qquad \begin{array}{l}\varphi_2 : G \longrightarrow G \\ \quad v \longmapsto v_2\ .\end{array}$$

As it is obvious, $\emptyset\longrightarrow G$ is the equalizer of φ_1 and φ_2, and so $\emptyset\in\mathcal{B}$. Let now G has more than one loop. We consider two vertices, v_1 and v_2, with a loop and by defining φ_1 and φ_2 in an analogous way we did in the previous case, we conclude in the same way also that $\emptyset\in\mathcal{B}$.

In the last case it suffices to consider the maps

$$\begin{array}{l}\varphi_1 : P \longrightarrow G' \\ \quad \bullet \longmapsto u_1\end{array} \qquad \text{and} \qquad \begin{array}{l}\varphi_2 : P \longrightarrow G' \\ \quad \bullet \longmapsto u_2\end{array}$$

where u_1 and u_2 are two vertices of G', and the conclusion follows.

The fact that $\emptyset$ has to belong to $\mathcal{B}$ immediately implies that the torsion theory $(\mathcal{C},\mathcal{B})$ is equal to $(\mathcal{I}\mathrm{ni},\mathcal{G}\mathrm{rf})$:

$$\emptyset \xrightarrow{e} \emptyset \xrightarrow{m} T$$

is the $\Phi\mathcal{B}$-factorization of $\emptyset\longrightarrow T$ since $e\in\mathcal{I}\mathrm{so}(\mathcal{G}\mathrm{rf})\subset(\mathrm{Mor}\,\mathcal{B})^{\uparrow}$ and $m\in\mathrm{Mor}\,\mathcal{B}\subset(\mathrm{Mor}\,\mathcal{B})^{\uparrow\downarrow}$. Therefore, in this case, $*=\emptyset$, $\mathcal{C}=\{G\in\mathrm{Obj}(\mathcal{G}\mathrm{rf})\mid R(G)\cong\emptyset\}=\{\emptyset\}$

by 5.6 and, consequently, $\mathcal{B} = \mathcal{G}rf$. □

PROPOSITION 5.8. *If* $(\mathcal{C},\mathcal{B})$ *is a torsion theory then:*

(i) $\mathcal{C} \cap \mathcal{B}$ *is the subcategory of objects isomorphic to* $*$.

(ii) $*$ *is initial in* $\mathcal{B}$ *and terminal in* $\mathcal{C}$.

(iii) *For every* $A \in \mathrm{Obj}\,\mathcal{A}$, $SR(A) \cong RS(A) \cong *$.

Proof: (i) Let X be an object of $\mathcal{C} \cap \mathcal{B}$. Then $I \longrightarrow X \in \mathcal{E}$ and $X \longrightarrow T \in \mathcal{M}$, where $(\mathcal{E},\mathcal{M}) = \Phi\mathcal{B}$, and, since the $(\mathcal{E},\mathcal{M})$-factorizations are unique up to isomorphism, we can conclude that $X \cong *$.

(ii) To prove the first part it suffices to observe that R, being a left adjoint, preserves colimits and that $R(I) \cong *$. The second part follows by duality.

(iii) For every $A \in \mathrm{Obj}\,\mathcal{A}$, $m : R(A) \longrightarrow T \in \mathcal{M}$ thus $S(m) : SR(A) \longrightarrow S(T) \cong *$ is an isomorphism because $\mathcal{M} = \Sigma_S$. Dually, $RS(A) \cong *$. □

6. TORSION THEORIES IN CATEGORIES WITHOUT ZERO OBJECT

Let us assume that the initial and terminal objects of $\mathcal{A}$ are not isomorphic, i.e. $\mathcal{A}$ does not have a zero object.

DEFINITION 6.1. A terminal object T is called strong terminal if $\mathrm{Hom}_{\mathcal{A}}(T,A) \neq \emptyset$, for every non-initial object A.

EXAMPLES. (a) $\mathcal{S}et$ and $\mathcal{T}op$ have strong terminal objects.

(b) The terminal object of $\mathcal{G}rf$ is not strong.

THEOREM 6.2. *Let* $\mathcal{A}$ *be a category with finite products, a strong terminal object which is a generator and no zero object. Then the torsion theories of* $\mathcal{A}$ *are exactly the trivial ones.*

Proof: Let $\mathcal{B}$ be a reflective subcategory of $\mathcal{A}$. Since $T \in \mathcal{B}$ then $\mathcal{T}er \subset \mathcal{B}$. If $\mathcal{B} \neq \mathcal{T}er$ then $I \in \mathcal{B}$. Before proving that we point out that if T is a strong terminal and $f,g \in \mathrm{Hom}_{\mathcal{A}}(T,Y)$ such that $f \neq g$ then the morphism $I \longrightarrow T$ is the equalizer of f and g.

Then, to prove that $I \in \mathfrak{B}$, it suffices to find a $\mathfrak{B}$-object Y and two different morphisms $f,g: T \longrightarrow Y$. Let X be an object of $\mathfrak{B}$ non-isomorphic to T. If X is not isomorphic to I, considering the morphisms $x: X \longrightarrow T$ and $t: T \longrightarrow X$ then $t.x \neq 1$. By definition of product there exists a unique morphism s such that the following diagram commutes

$$\begin{tikzcd} X & X \times X \arrow[l, "p_1"'] \arrow[r, "p_2"] & X \\ & X \arrow[ul, "1"] \arrow[u, dashed, "s"] \arrow[ur, "t.x"'] & \end{tikzcd}$$

whence $p_1 \neq p_2$. Since T is generator there exists a morphism $h: T \longrightarrow X \times X$ such that $p_1.h \neq p_2.h$. So, it suffices to take $Y = X$ and let $p_1.h$ and $p_2.h$ be the morphisms f and g.

Now let $(\mathfrak{C},\mathfrak{B})$ be a torsion theory of $\mathcal{A}$. If $\mathfrak{B} = \mathcal{T}er$ then, obviously, $(\mathfrak{C},\mathfrak{B}) = (\mathcal{A},\mathcal{T}er)$. If $\mathfrak{B} \neq \mathcal{T}er$ then $I \in \mathfrak{B}$, as we observed, so

$$I \longrightarrow I \longrightarrow T$$

is the $\Phi\mathfrak{B}$-factorization of $I \longrightarrow T$. Hence $\mathfrak{C} = \{A \in \mathrm{Obj}\,\mathcal{A} \mid R(A) \cong I\}$, by 5.5. But $\{A \in \mathrm{Obj}\,\mathcal{A} \mid R(A) \cong I\} = \mathcal{I}ni$. Actually, if $R(A) \cong I$ then

$$A \longrightarrow I \longrightarrow T$$

is the $\Phi\mathfrak{B}$-factorization of $A \longrightarrow T$. However the existence of a morphism from A to I implies that A is isomorphic to I: otherwise, since $\mathrm{Hom}_{\mathcal{A}}(T,A) \neq \emptyset$, T would be isomorphic to I, i.e. $\mathcal{A}$ would have zero object contradicting the assumptions. Therefore, $\mathfrak{C} = \mathcal{I}ni$. Then we immediately infer that $\mathfrak{B} = \mathcal{A}$ and the torsion theory is still trivial. □

<u>EXAMPLES</u>. (a) The categories $\mathcal{S}et$ and $\mathcal{T}op$ satisfy the conditions of this theorem.

(b) Any topological category over $\mathcal{S}et$ satisfy also these conditions. Thus the category $\mathcal{T}op$, the category $\mathcal{C}\mathcal{R}eg$ of completely regular topological spaces, the category $\mathcal{P}rox$ of proximity spaces as well as the categories of uniform spaces, convergence spaces, merotopic spaces, mensurable spaces and preorder sets [29], [9] are examples of categories which admit only the trivial torsion theories.

This theorem enables us to conclude that in many categories without zero object the concept of torsion theory trivializes. For that reason, in the following we study the torsion theories in categories with zero object. It is in this context that the most interesting results

arise and where the connection with results in abelian categories is closer. However, the existence of zero object does not insure that the categories have non-trivial torsion theories. For example, the category of partial maps, where the objects are the sets and the morphisms $f: X \longrightarrow Y$ are the maps from some subset $D(f)$ of X to Y, has only the trivial torsion theories, though it has a zero object.

7. TORSION THEORIES IN CATEGORIES WITH ZERO OBJECT

Throughout this section we denote by $\mathcal{A}$ a category with zero object that will be denoted by 0 as well as the morphism $A \longrightarrow 0 \longrightarrow B$ by $0: A \longrightarrow B$. Given a morphism f, we use, respectively, $\ker f$ and $\operatorname{Ker} f$ for the kernel morphism of f and the corresponding object, i.e. $\operatorname{Ker} f \xrightarrow{m} X$ with $m = \ker f$.

<u>DEFINITION 7.1</u>. A subcategory $\mathcal{B}$ of $\mathcal{A}$ is said to be <u>closed under extensions</u> if for any strong epimorphism $f: A \longrightarrow B$ such that $B \in \operatorname{Obj}\mathcal{B}$ and $\operatorname{Ker} f$ exists and belongs to $\mathcal{B}$ then $A \in \operatorname{Obj}\mathcal{B}$.

<u>PROPOSITION 7.2</u>. *Given a subcategory $\mathcal{B}$ of $\mathcal{A}$, $\mathcal{B}^{\rightarrow}$ is closed under monosources and extensions.*

<u>Proof</u>: Obviously $\mathcal{B}^{\rightarrow} = \{A \in \operatorname{Obj}\mathcal{A} \mid \operatorname{Hom}_{\mathcal{A}}(B,A) = 0, \text{ for all } B \in \operatorname{Obj}\mathcal{B}\}$. Let us assume that $(A_i)_{i \in I}$ is a family of objects of $\mathcal{B}^{\rightarrow}$ and $(f_i: A \longrightarrow A_i)_{i \in I}$ a monosource. If $f: B \longrightarrow A$, with $B \in \operatorname{Obj}\mathcal{B}$, then $f_i.f = 0 = f_i.0$, for every $i \in I$. Then $f = 0$ and so $A \in \mathcal{B}^{\rightarrow}$.

In order to show the closure for extensions, let $f: A \longrightarrow B$ be a strong epimorphism, $B \in \mathcal{B}^{\rightarrow}$, and $m: K \longrightarrow A$ be the kernel of f with $K \in \mathcal{B}^{\rightarrow}$. If $g: C \longrightarrow A$, with $C \in \operatorname{Obj}\mathcal{B}$, then there exists a unique morphism t such that $m.t = g$, since $f.g = 0$.

$$\begin{tikzcd}
K \arrow[r, "m"] & A \arrow[r, "f"] & B \\
C \arrow[u, dashed, "t"] \arrow[ur, "g"'] & &
\end{tikzcd}$$

It is now obvious that $t=0$, and so $g=0$. □

Notice that a subcategory of $\mathcal{A}$ closed under monosources is closed under all products that exist in $\mathcal{A}$.

LEMMA 7.3. *Let* $\mathcal{A}^{op}$ *be* (FWC). *For a subcategory* $\mathcal{B}$ *of* $\mathcal{A}$ *the following assertions are equivalent:*

(i) $\mathcal{B}$ *is closed under monosources.*

(ii) $\mathcal{B}$ *is reflective and closed under subobjects.*

Proof:

(i) ⇒ (ii): It is enough to show that $\mathcal{B}$ is reflective. Let $A \in \mathrm{Obj}\,\mathcal{A}$ and consider the family

$$\mathcal{M} = \bigcup_{B \in \mathrm{Obj}\,\mathcal{B}} \mathrm{Hom}_{\mathcal{A}}(A,B) = \{ g : A \longrightarrow B_g \in \mathrm{Mor}\,\mathcal{A} \mid B_g \in \mathrm{Obj}\,\mathcal{B} \}.$$

Let $(e_i)_{i \in I}$ be the family of strong epimorphisms with domain A through which all morphisms of $\mathcal{M}$ factors, $g = f_i^g . e_i$.

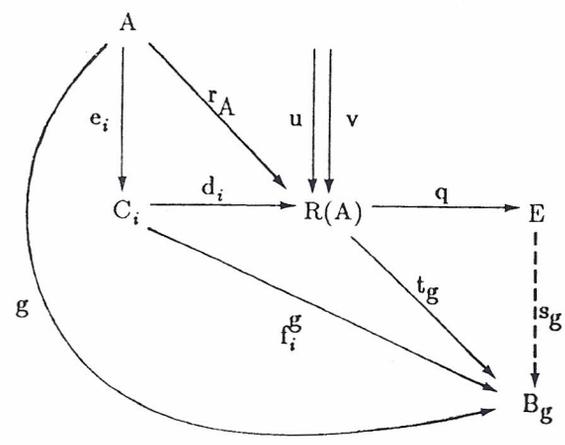
Consider the cointersection $r_A : A \longrightarrow R(A)$ of the family $(e_i)_{i \in I}$, which is a strong epimorphism. By definition of cointersection there exists, for each $g \in \mathcal{M}$, a unique morphism t_g such that in the diagram

$$\begin{tikzcd}
A \arrow[d, "e_i"'] \arrow[dr, "r_A"] \arrow[ddr, bend right=60, "g"'] & & \\
C_i \arrow[r, "d_i"] \arrow[dr, "f_i^g"'] & R(A) \arrow[d, dashed, "t_g"] & \\
& B_g &
\end{tikzcd}$$

the lower triangle commutes and, consequently, $g = t_g . r_A$.

We show next that the family $(t_g)_{g \in \mathcal{M}}$ is a monosource. If $t_g . u = t_g . v$, for every $g \in \mathcal{M}$, let $q : R(A) \longrightarrow E$ be the coequalizer of u and v. Then there exists, for each $g \in \mathcal{M}$,

a unique s_g such that $s_g.q=t_g$ and so $g=s_g.q.r_A$, for every $g\in\mathcal{M}$.



Since q and r_A are strong epimorphisms then $q.r_A$ is also a strong epimorphism and so there exists a $j\in I$ such that $q.r_A=e_j$. From $d_j.q.r_A=d_j.e_j=r_A$ we conclude that $d_j.q=1$, so q is an isomorphism, i.e. $u=v$, as it was required.

By (i), we conclude that $R(A)\in\mathcal{B}$ and also that r_A is the reflection morphism from A to $\mathcal{B}$. Actually, we proved even a stronger result: $\mathcal{B}$ is StrEpi-reflective.

<u>(ii) ⇒ (i)</u>: Given a monosource $(f_i:A\longrightarrow B_i)_{i\in I}$, with $B_i\in \mathrm{Obj}\,\mathcal{B}$, we show that $A\in\mathrm{Obj}\,\mathcal{B}$. Let us assume that $r_A:A\longrightarrow R(A)$ is the reflection morphism from A to $\mathcal{B}$. To prove that $A\in\mathrm{Obj}\,\mathcal{B}$ it suffices to show that r_A is a monomorphism since, by hypothesis, $\mathcal{B}$ is closed under subobjects. For each $i\in I$, there exists a unique morphism g_i such that $g_i.r_A=f_i$. Then $r_A.u=r_A.v$ implies that $f_i.u=f_i.v$ for each $i\in I$ and, consequently, $u=v$. □

Statements 7.2 and 7.3 allow us to simplify 5.3 and to conclude immediately that:

<u>COROLLARY 7.4</u>. *If $\mathcal{A}^{op}$ is* (FWC) *then $\mathcal{B}^{\rightarrow}$ is reflective for every subcategory $\mathcal{B}$. If, furthermore, $\mathcal{A}$ is* (FWC) *then $(\mathcal{C},\mathcal{B})$ is a torsion theory if and only if $\mathcal{C}=\mathcal{B}^{\leftarrow}$ and $\mathcal{B}=\mathcal{C}^{\rightarrow}$. In this case, any subcategory $\mathcal{D}$ generates the torsion theory $(\mathcal{D}^{\leftarrow},\mathcal{D}^{\leftarrow\rightarrow})$ and cogenerates the torsion theory $(\mathcal{D}^{\rightarrow\leftarrow},\mathcal{D}^{\rightarrow})$.* □

<u>PROPOSITION 7.5</u>. *Let $\mathcal{B}$ be a reflective subcategory of $\mathcal{A}$, $(r_A:A\longrightarrow R(A))_{A\in\mathrm{Obj}\,\mathcal{A}}$ being the family of reflection morphisms, $\mathcal{F}=\Phi\mathcal{B}$ its associated prefactorization system and $\mathcal{C}=I/\mathcal{S}$. Then:*

(i) $\mathcal{C}=\{A\in \mathrm{Obj}\mathcal{A} \mid R(A)\cong 0\}=\{A\in \mathrm{Obj}\mathcal{A} \mid r_A=0\}=\mathcal{B}^{\leftarrow}$.

(ii) *If every morphism* $r_A : A\longrightarrow R(A)$ *has kernel* $s_A : S(A)\longrightarrow A$ *then* $A\in\mathcal{C}$ *if and only if* $S(A)=A$, *and the morphism* s_A *is the coreflection from* A *to* $\mathcal{C}$ *whenever* $S(A)\in\mathcal{C}$.

Proof: (i) follows immediately from 5.1.

(ii) From (i) we know that $A\in\mathcal{C}$ if and only if $r_A=0$, which is equivalent to $S(A)=A$.

Finally, assuming that $S(A)\in\mathcal{C}$, consider the following commutative diagram

$$\begin{tikzcd}
S(A) \arrow[r, "s_A"] & A \arrow[r, "r_A"] & R(A) \\
 & C \arrow[u, "f"] \arrow[r, "r_C"'] & R(C) \arrow[u, "R(f)"']
\end{tikzcd}$$

where C is an object of $\mathcal{C}$ and f an arbitrary morphism.

Since $r_A.f=R(f).r_C=0$, there is a unique morphism t such that $s_A.t=f$. Thus, s_A is the coreflection. □

If $s_A : S(A)\longrightarrow A$ is the kernel of r_A we say that this kernel is idempotent if $S^2(A)=S(A)$, considered as subobjects of A.

PROPOSITION 7.6. *The following assertions are equivalent:*

(i) *The kernel of every* r_A *exists and it is idempotent.*

(ii) $\mathcal{C}$ *is coreflective and each coreflection morphism* s_A *is the kernel of* r_A.

Proof:

(i) ⇒ (ii): If $S^2(A)=S(A)$, for each $A\in\mathrm{Obj}\mathcal{A}$, then $S(A)\in\mathcal{C}$, by the preceding proposition, and so s_A is the coreflection morphism.

(ii) ⇒ (i): is clear. □

DEFINITION 7.7. We say that a reflector R is normal provided that it satisfies the equivalent conditions of 7.6.

A torsion theory $(\mathcal{C},\mathcal{B})$ is called <u>left normal</u> if the reflector in $\mathcal{B}$ is normal.

Dually, a torsion theory $(\mathcal{C},\mathcal{B})$ is <u>right normal</u> if the coreflector is normal, i.e. the reflection morphism for each object is the cokernel of the corresponding coreflection morphism in $\mathcal{C}$.

A torsion theory is called <u>normal</u> if it is both left and right normal.

<u>EXAMPLES</u>. (a) It is clear that every torsion theory of an abelian category is normal.

(b) In the category $\mathcal{G}rp$, any torsion theory is normal. In fact, let $(\mathcal{C},\mathcal{B})$ be a torsion theory of $\mathcal{G}rp$. By the dual of 7.2 and 7.3 ((i) $\Rightarrow$ (ii)), $\mathcal{C}$ is StrMono-coreflective. In $\mathcal{G}rp$ a strong monomorphism is not necessarily a kernel but in this case every coreflection morphism $s_G : S(G) \longrightarrow G$ is a kernel. Indeed, s_G is a monomorphism so $S(G)$ is isomorphic to a subgroup of G. But, by definition of coreflection morphism, any endomorphism f of G can be restricted to an endomorphism $S(f)$ of $S(G)$:

$$\begin{tikzcd}
S(G) \arrow[r, "s_G"] & G \\
S(G) \arrow[u, dashed, "S(f)"] \arrow[r, "s_G"'] & G \arrow[u, "f"']
\end{tikzcd}$$

Thus, considering all endomorphisms f_g of G, for $g \in G$, defined by $f_g(x) = gxg^{-1}$ for every $x \in G$, we can conclude that $S(G)$ is isomorphic to a normal subgroup of G. Hence s_G is a kernel.

Let $t_G : G \longrightarrow T(G)$ be the cokernel of s_G and consider the commutative diagram

$$\begin{tikzcd}
S(G) \arrow[rr, "0"] \arrow[dr, dashed, "f"] \arrow[dd, "s_G"'] & & ST(G) \arrow[dd, "s_{T(G)}"] \\
& P \arrow[ur, "y"] \arrow[dl, "x"] & \\
G \arrow[rr, "t_G"'] & & T(G)
\end{tikzcd}$$

where (x,y) is the pullback of $(t_G, s_{T(G)})$, which implies the existence of a unique morphism f such that $x.f = s_G$ and $y.f = 0$.

Since s_G is a kernel, then $s_G = \ker t_G$ and so it is easy to prove that $f = \ker y$. But in $\mathcal{G}rp$, pullbacks of cokernels along kernels are still cokernels thus y is a cokernel and, therefore, $y = \operatorname{coker} f$. Then $P \in \mathcal{C}$ because $\mathcal{C} = \mathcal{B}^{\leftarrow}$ is closed under coextensions. But x is a monomorphism since it is a pullback of a monomorphism. Hence, P is a subobject of G such that $S(P) = P$ and $S(G) \leq P \leq G$, for the usual order relation in the set of subobjects of G. Since $S^2(G) \cong S(G)$, then $P = S(G)$ as subobjects of G, i.e. f is an isomorphism. The fact that f is an isomorphism and y an epimorphism implies that $ST(G) \cong 0$ and, consequently, $t_{T(G)}$ is an isomorphism, i.e. $T(G)$ is idempotent. Finally, by the dual of 7.6, we can deduce that $r_G = \operatorname{coker} s_G$. Furthermore, it is clear that $s_G = \ker r_G$ and so the torsion theory is normal.

REMARK 7.8. Observing the arguments used in example (b) we conclude that in a finitely complete and finitely cocomplete category with zero object such that the cokernels are preserved by pullbacks along kernels, a torsion theory is normal if the coreflection into the torsion subcategory are kernels. By the dual of 7.2 and by the proof of the dual of 7.3 ((i) $\Rightarrow$ (ii)) any torsion subcategory of a (FWC)-category is StrMono-coreflective. Therefore, we may conclude that, in finitely well-complete and finitely cocomplete categories with zero object such that the cokernels are preserved by pullbacks along kernels and all the strong monomorphisms are kernels, every torsion theory is normal. By the duals of ([11], 5.1 (iii)) and ([11], remark 5.2) we may replace the hypothesis that the categories are (FWC) by the condition of existence of finite limits because, according to these results, any coreflective subcategory of a finitely cocomplete category, whose coreflection morphisms are monomorphisms, is a StrMono-coreflective subcategory.

For instance, in any finitely cocomplete regular additive category such that the monomorphisms are kernels, the torsion theories are normal.

In ([11], 8.2) the authors say that they do not know any example of a non-normal torsion theory. However S. Mantovani in ([37], p. 43) presents an example of a non-normal torsion theory in the category $\mathcal{P}$tHaus of pointed Hausdorff topological spaces:

$(\mathcal{C}, \mathcal{C}^{\rightarrow})$ where $\mathcal{C}$ is the subcategory of connected spaces.

The following proposition characterizes the torsion-free subcategories.

PROPOSITION 7.9. *Assume that $\mathcal{A}$ and $\mathcal{A}^{op}$ are* (FWC) *and that any torsion theory of $\mathcal{A}$ is left normal. A subcategory of $\mathcal{A}$ is a torsion-free subcategory if and only if it is a reflective subcategory closed under subobjects and extensions whose reflector is normal.*

Proof: We show that a subcategory $\mathfrak{B}$ is torsion-free if it satisfies these conditions since the converse is obvious. For that we consider the reflective factorization system $\mathfrak{F}=\Phi\mathfrak{B}$ and let $\mathfrak{C}=\Psi'\mathfrak{F}$. As we have already observed, $\mathfrak{C}=\mathfrak{B}^{\leftarrow}$ and $\overline{\mathfrak{F}}$ is a factorization system since $\mathcal{A}^{op}$ is (FWC), and then, by 5.1(iii), $\mathfrak{C}^{\rightarrow}=\mathfrak{B}$". To conclude the proof it remains to show that $\mathfrak{C}^{\rightarrow}=\mathfrak{B}$, and for that, it suffices to prove that $\mathfrak{C}^{\rightarrow}\subset\mathfrak{B}$. Let $A\in\mathfrak{C}^{\rightarrow}$ and $s_A:S(A)\longrightarrow A$ be the coreflection of A in $\mathfrak{C}$. Since $S(A)\in\mathfrak{C}$ it is clear that $s_A=0$, thus $S(A)\cong 0$ since s_A is a monomorphism. Then $S(A)\in\mathfrak{B}$ since $\mathfrak{B}$ is closed under subobjects. By the dual of 3.2 it follows that $(\mathrm{StrEpi}\mathcal{A},\mathrm{Mono}\mathcal{A})$ is a factorization system of $\mathcal{A}$. Let

$$A\xrightarrow{e}X\xrightarrow{m}R(A)$$

be the $(\mathrm{StrEpi}\mathcal{A},\mathrm{Mono}\mathcal{A})$-factorization of r_A. The object $R(A)$ belongs to $\mathfrak{B}$ so $X\in\mathfrak{B}$ and $\mathrm{Ker}\,e=S(A)\in\mathfrak{B}$. Hence, $A\in\mathfrak{B}$ because $\mathfrak{B}$ is closed under extensions. □

EXAMPLE. The category of abelian groups is a reflective subcategory of $\mathfrak{Grp}$ with reflection morphisms

$$r_G:G\longrightarrow G/[G,G],$$

where $[G,G]$ denotes the commutator subgroup of G. However, the reflector is not normal since $\mathrm{Ker}\,r_G=[G,G]$ is not always idempotent; it suffices to consider in a non-abelian soluble group as, for instance, the symmetric group S_3. Therefore $\mathcal{A}b$ is not a torsion-free subcategory of $\mathfrak{Grp}$ and since $\mathcal{A}b^{\leftarrow}$ is the subcategory $\mathfrak{Perf}$ of perfect groups then the inclusion $\mathcal{A}b\subset\mathfrak{Perf}^{\rightarrow}$ is strict and $(\mathfrak{Perf},\mathfrak{Perf}^{\rightarrow})$ is the torsion theory generated by $\mathcal{A}b$.

PROPOSITION 7.10. (i) *Every torsion-free subcategory of the category $\mathcal{A}$ is closed under monosources and extensions.*

(ii) *If $\mathcal{A}$ is a complete, cocomplete, wellpowered and cowellpowered category such that every strong epimorphism is a cokernel and pushouts of strong monomorphisms along cokernels are kernels, then a subcategory of $\mathcal{A}$ is a torsion-free subcategory if and only if it is reflective and closed under extensions and subobjects.*

Proof: (i) It is obvious.

(ii) Let us assume that $\mathfrak{B}$ is a reflective subcategory closed under extensions and subobjects. By 7.3, $\mathfrak{B}$ is closed under monosources and so under products. Let $\mathfrak{C}=\mathfrak{B}^{\leftarrow}$. To prove that $(\mathfrak{C},\mathfrak{B})$ is a torsion theory it suffices to show that $\mathfrak{C}^{\rightarrow}\subset\mathfrak{B}$. First we prove that, for each $X\in\mathrm{Obj}\,\mathcal{A}$, there exist objects $X_C\in\mathfrak{C}$ and $X_B\in\mathfrak{B}$ and morphisms $p:X_C\longrightarrow X$ and $e:X\longrightarrow X_B$ such that $p=\ker e$ with $e\in\mathrm{StrEpi}\mathcal{A}$.

Let Q be the family of quotient objects of X that belong to $\mathfrak{B}$. Since $\mathcal{A}$ is cowellpowered then Q is a set, $\{(X_i,e_i) : i \in I\}$. Consider the product

$$\left(\prod_{i \in I} X_i, (p_i)_{i \in I}\right)$$

which belongs to $\mathfrak{B}$. By the universal property of product there exists a unique morphism

$$f: X \longrightarrow \prod_{i \in I} X_i$$

such that $p_i.f = e_i$, for each $i \in I$. From the dual of 3.2 it follows immediately that $(\mathrm{StrEpi}\mathcal{A}, \mathrm{Mono}\mathcal{A})$ is a factorization system. This conclusion could also be inferred from the facts that $\mathcal{A}$ is finitely complete and finitely cocomplete, has zero object and every strong epimorphism is a cokernel (cf. III.2.4). Let

$$X \xrightarrow{e} X_B \xrightarrow{m} \prod_{i \in I} X_i$$

be the corresponding factorization of f and $p: X_C \longrightarrow X$ be the kernel of e. Then $X_B \in \mathfrak{B}$. We prove now that $X_C \in \mathcal{C} = \mathfrak{B}^{\leftarrow}$. Let g be a morphism from X_C to an arbitrary object B of $\mathfrak{B}$ and let us consider the following diagram

$$\begin{tikzcd}
X_C \arrow[rr, "p"] \arrow[dr, "g"] \arrow[dd, "e'"'] & & X \arrow[dd, "\overline{e'}"] \arrow[ddr, "e"] & \\
& B & & \\
X' \arrow[uur, "m'"] \arrow[rr, "\overline{p}"'] \arrow[dr] & & X'' \arrow[dr, dashed, "h"] & \\
& 0 \arrow[rr] & & X_B
\end{tikzcd}$$

where $m'.e'$ is the $(\mathrm{StrEpi}\mathcal{A}, \mathrm{Mono}\mathcal{A})$-factorization of g, $(X'', \overline{p}, \overline{e'})$ is the pushout of (p, e') and h is the unique morphism such that $h.\overline{p} = 0$ and $h.\overline{e'} = e$. It is obvious that h is a strong epimorphism. Our goal is to show that $g = 0$. To prove this it suffices to see that $X' \cong 0$. Since $\mathfrak{B}$ is closed under subobjects then $X' \in \mathfrak{B}$. The morphism e is a strong epimorphism, then a cokernel and, since $p = \ker e$, we have that $e = \mathrm{coker}\, p$. It is an easy exercise to verify that then $h = \mathrm{coker}\, \overline{p}$. But $\overline{p}$ is a kernel, since pushouts of strong monomorphisms along cokernels are kernels, so $\overline{p} = \ker h$. Since $\mathfrak{B}$ is closed under extensions we conclude that $X'' \in \mathfrak{B}$, hence $X'' \in Q$, i.e. there exists a $j \in I$ such that $(X'', \overline{e'}) = (X_j, e_j)$. Consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, "e"] \arrow[dd, "e_j"'] & X_B \arrow[d, "m"] \\
 & \prod_{i \in I} X_i \arrow[d, "p_j"] \\
X_j \arrow[r, "1"'] & X_j
\end{tikzcd}$$

Since e is a strong epimorphism, then there exists a unique morphism $d: X_B \longrightarrow X_j$ such that $d.e = e_j$ and $p_j.m = d$. But $h.e_j = e$, hence $d.h.e_j = d.e = e_j$ and $h.d.e = h.e_j = e$ which implies that $d.h = 1$ and $h.d = 1$, i.e. h is an isomorphism. Since $\bar{p}: X' \longrightarrow X''$ is the kernel of h then $X' \cong 0$.

Now we conclude that $\mathcal{C}^{\rightarrow} \subset \mathcal{B}$. Let $X \in \mathcal{C}^{\rightarrow}$. There exist objects $X_C \in \mathcal{C}$ and $X_B \in \mathcal{B}$ and morphisms $p: X_C \longrightarrow X$ and $e: X \longrightarrow X_B$ such that $p = \ker e$ with $e \in \mathrm{StrEpi}\mathcal{A}$. Since $X \in \mathcal{C}^{\rightarrow}$ then $p = 0$. Thus e is an isomorphism and so $X \in \mathcal{B}$, as we claimed.

The converse is an immediate consequence of 7.2 and 7.3. □

By lemma 7.3 we can conclude that for categories satisfying the conditions of 7.10 a subcategory is torsion-free if and only if it is closed under subobjects, extensions and products. So, we have in these categories a characterization similar to (I.1.12).

<u>EXAMPLE</u>. In $\mathcal{G}rp$, $\mathcal{B}$ is a torsion-free subcategory if and only if it is closed under subobjects, extensions and products.

8. CONNECTEDNESSES AND REFLECTIONS

The theory of connectednesses / disconnectednesses in Topology [3] and in other contexts such as the category of non-directed graphs [22], and the theory of radicals / semisimple classes of rings [19] have intriguing similarities with the theory of torsion subcategories / torsion-free subcategories.

The attempt to understand these similarities lead us to the questions we shall deal with in this chapter. Their inclusion here is justified by the analogies between factorizations of morphisms / reflections / torsion theories and factorizations of sources / prereflections / / connectedness theories, that culminate in theorem 8.11 and their consequences, allowing us to establish close relations between two definitions of torsion theory apparently distinct.

In this section we shall still assume that the category $\mathcal{A}$ has a zero object.

Let $\mathcal{E}$ be a class of morphisms of $\mathcal{A}$ containing the isomorphisms and closed under composition with them. From now on we consider factorizations of sources instead of morphisms.

DEFINITIONS 8.1. (1) The pair $(e,(m_i)_{i\in I})$ consisting of a morphism $e:A\longrightarrow B$ and a class $(m_i)_{i\in I}$ of morphisms $m_i:B\longrightarrow A_i$ is called an $\mathcal{E}$-factorization of the $\mathcal{A}$-source $(A,(f_i:A\longrightarrow A_i)_{i\in I})$ if $e\in\mathcal{E}$ and $f_i=m_i.e$ for every $i\in I$.

(2) An $\mathcal{E}$-factorization $(e,(m_i)_{i\in I})$ of a source $(A,(f_i)_{i\in I})$ is called orthogonal if, given morphisms $p\in\mathcal{E}$ and u and a source $(v_i)_{i\in I}$ such that $m_i.u=v_i.p$, for every $i\in I$, there exists a unique morphism d such that the diagram

$$\begin{tikzcd} \bullet \arrow[r, "p"] \arrow[d, "u"'] & \bullet \arrow[d, "v_i"] \\ \bullet \arrow[r, "m_i"'] \arrow[ur, dashed, "d"] & \bullet \end{tikzcd}$$

is commutative.

(3) A category is called $\mathcal{E}$-cocomplete if it admits cointersections of morphisms of $\mathcal{E}$ and these cointersections belong to $\mathcal{E}$ and the pushouts of morphisms of $\mathcal{E}$ along any morphism exist and belong to $\mathcal{E}$.

For additional information about factorizations of sources see e.g. [12] and [28].

EXAMPLE. Any category with a (FWC) dual is a StrEpi-cocomplete category.

The following proposition (cf. [44]) will be useful in the sequel.

PROPOSITION 8.2. (i) *Every source in* $\mathcal{A}$ *has an orthogonal* $\mathcal{E}$*-factorization if and only if* $\mathcal{A}$ *is* $\mathcal{E}$*-cocomplete and* $\mathcal{E}$ *is closed under composition.*

(ii) *If* $\mathcal{A}$ *is* $\mathcal{E}$*-cocomplete then* $\mathcal{E} \subset \mathrm{Epi}\,\mathcal{A}$. □

From now on we shall assume that $\mathcal{A}$ is $\mathcal{E}$-cocomplete, being $\mathcal{E}$ closed under composition and containing $A \longrightarrow 0$, for each $A \in \mathcal{A}$.

Let H be an endofunctor of $\mathcal{A}$.

DEFINITIONS 8.3. (1) An $\mathcal{E}$-prereflection is a natural transformation $\gamma : \mathrm{Id}_{\mathcal{A}} \overset{\cdot}{\longrightarrow} H$ such that $\gamma_A \in \mathcal{E}$, for every $A \in \mathrm{Obj}\,\mathcal{A}$.

(2) An $\mathcal{E}$-prereflection is called an $\mathcal{E}$-reflection if $\gamma_H : H \overset{\cdot}{\longrightarrow} H^2$ is a natural isomorphism.

Let γ be an $\mathcal{E}$-prereflection.

We denote by $\mathcal{K}_\gamma$ and $\mathcal{I}_\gamma$ the subcategories defined by:

$$\mathcal{K}_\gamma = \{ A \in \mathrm{Obj}\,\mathcal{A} \mid \gamma_A = 0 \}$$

$$\mathcal{I}_\gamma = \{ A \in \mathrm{Obj}\,\mathcal{A} \mid \gamma_A \text{ is an isomorphism} \}.$$

EXAMPLES 8.4. (a) If $\mathcal{A}$ is an abelian category and R a radical [43], the morphisms $(\gamma_M : M \longrightarrow M/R(M))_{M \in \mathrm{Obj}\,\mathcal{A}}$ define an Epi-reflection because $R(M/R(M)) = 0$. In this case

$$K_\gamma = \{ M \in \mathrm{Obj}\,\mathcal{A} \mid M/R(M) \cong 0 \} = \{ M \in \mathrm{Obj}\,\mathcal{A} \mid R(M) = M \}$$

and $$\mathcal{I}_\gamma = \{ M \in \mathrm{Obj}\,\mathcal{A} \mid M/R(M) \cong M \} = \{ M \in \mathrm{Obj}\,\mathcal{A} \mid R(M) = 0 \}.$$

(b) In $\mathcal{G}\mathrm{rp}$ the morphisms $(\gamma_G : G \longrightarrow G/[G,G])_{G \in \mathcal{G}\mathrm{rp}}$ define an Epi-reflection where

$$K_\gamma = \{ G \mid [G,G] = G \} \text{ and } I_\gamma = \{ G \mid [G,G] = 0 \}$$

that is the perfect groups and the abelian groups, respectively.

The following proposition which characterizes the $\mathcal{E}$-prereflections that are $\mathcal{E}$-reflections has a straightforward proof.

PROPOSITION 8.5. *If* $\gamma : \mathrm{Id}_A \xrightarrow{\cdot} H$ *is an* $\mathcal{E}$*-prereflection, then the following assertions are equivalent:*

(i) γ *is an* $\mathcal{E}$*-reflection.*

(ii) I_γ *is reflective with reflection morphisms* $(\gamma_A)_{A \in \mathrm{Obj}\mathcal{A}}$. □

Note that if γ is an $\mathcal{E}$-prereflection, I_γ can be reflective with reflection morphisms eventually different from $(\gamma_A)_{A \in \mathrm{Obj}\mathcal{A}}$ (cf. [46], remark 3 of corollary 3).

Given a subcategory $\mathcal{B}$ and an object A of $\mathcal{A}$, we consider the source $F_A^{\mathcal{B}} = (A,(f_i)_{i \in I})$ of all morphisms f_i, with domain A, such that $f_i.u = 0$ for every morphism u with codomain A and domain in $\mathcal{B}$. Let $\pi_A^{\mathcal{B}} : A \longrightarrow P^{\mathcal{B}}(A)$ be the $\mathcal{E}$-part of the orthogonal $\mathcal{E}$-factorization of $F_A^{\mathcal{B}}$. The correspondence between A and $P^{\mathcal{B}}(A)$ is functorial. Indeed, it is the function of objects of an endofunctor of $\mathcal{A}$ which assigns to each $\mathcal{A}$-morphism $f : A \longrightarrow A'$ the morphism $P^{\mathcal{B}}(f)$ defined as follows:

Given the sources $F_A^{\mathcal{B}} = (A,(f_i)_{i \in I})$ and $F_{A'}^{\mathcal{B}} = (A',(f'_j)_{j \in J})$ it is clear that for each $j \in J$ there exists a $i_j \in I$ such that $f'_j.f = f_{i_j}$. Denoting by $(\pi_A^{\mathcal{B}},(m_i)_{i \in I})$ and $(\pi_{A'}^{\mathcal{B}},(m'_j)_{j \in J}$ the orthogonal $\mathcal{E}$-factorizations of, respectively, $F_A^{\mathcal{B}}$ and $F_{A'}^{\mathcal{B}}$, the diagram

$$\begin{tikzcd}
A \arrow[r, "\pi_A^{\mathcal{B}}"] \arrow[d, "f"'] & P^{\mathcal{B}}(A) \arrow[dd, "m_{i_j}"] \\
A' \arrow[d, "\pi_{A'}^{\mathcal{B}}"'] \arrow[dr, "f'_j"] & \\
P^{\mathcal{B}}(A') \arrow[r, "m'_j"'] & A'_j
\end{tikzcd}$$

commutes, for every $j \in J$. Then, $P^{\mathcal{B}}(f) : P^{\mathcal{B}}(A) \longrightarrow P^{\mathcal{B}}(A')$ is the unique morphism such that $P^{\mathcal{B}}(f).\pi_A^{\mathcal{B}} = \pi_{A'}^{\mathcal{B}}.f$ and $m'_j.P^{\mathcal{B}}(f) = m_{i_j}$ for every $j \in J$.

As an immediate consequence, we obtain a natural transformation $\pi^{\mathcal{B}} : \mathrm{Id}_{\mathcal{A}} \xrightarrow{\cdot} P^{\mathcal{B}}$.

PROPOSITION 8.6. *If* $\mathcal{B}$ *is a subcategory of* $\mathcal{A}$ *then:*

(i) $\pi^{\mathcal{B}}$ *is an* $\mathcal{E}$*-prereflection.*

(ii) $\mathcal{B} \subset \mathcal{H}_{\pi^{\mathcal{B}}}$.

Proof: (i) It is trivial.

(ii) Let $B \in \mathrm{Obj}\mathcal{B}$ and $F_B^{\mathcal{B}} = (B,(f_i)_{i \in I})$. Since each f_i is a morphism such that $f_i.u = 0$ for every morphism u with codomain B and domain in $\mathcal{B}$, then $f_i = 0$, for every $i \in I$. So, we have, for each $i \in I$, the following commutative diagram

$$\begin{tikzcd} B \arrow[r] \arrow[d, "\pi_B^{\mathcal{B}}"'] \arrow[dr, "f_i"] & 0 \arrow[d] \\ P^{\mathcal{B}}(B) \arrow[r, "m_i"'] & A_i \end{tikzcd}$$

By definition of orthogonal $\mathcal{E}$-factorization we can conclude that $\pi_B^{\mathcal{B}}$ factors through the zero object, i.e. $B \in \mathcal{K}_{\pi^{\mathcal{B}}}$. □

We define $\rho_A^{\mathcal{B}} : A \longrightarrow R^{\mathcal{B}}(A)$, for each $A \in \mathrm{Obj}\mathcal{A}$, as the $\mathcal{E}$-morphism of the orthogonal $\mathcal{E}$-factorization of the source of all morphisms with domain A and codomain in $\mathcal{B}$. In a similar way, it is possible to define a functor $R^{\mathcal{B}}$ and a natural transformation $\rho^{\mathcal{B}} : \mathrm{Id}_{\mathcal{A}} \xrightarrow{\cdot} R^{\mathcal{B}}$.

PROPOSITION 8.7. *If $\mathcal{B}$ is a subcategory of $\mathcal{A}$ then:*

(i) *$\rho^{\mathcal{B}}$ is an $\mathcal{E}$-reflection.*

(ii) $\mathcal{B} \subset \mathcal{I}_{\rho^{\mathcal{B}}}$.

Proof: (i) The conclusion follows by proposition 8.5, observing that $\mathcal{E} \subset \mathrm{Epi}\mathcal{A}$.

(ii) For each $B \in \mathrm{Obj}\mathcal{B}$ let $(\rho_B^{\mathcal{B}},(m_i)_{i \in I})$ be the orthogonal $\mathcal{E}$-factorization of the source $(B,(f_i)_{i \in I})$ of all morphisms with domain B and codomain in $\mathcal{B}$. Since $f_j = 1$, for some $j \in I$, then $\rho_B^{\mathcal{B}}$ is a split monomorphism. Moreover, $\rho_B^{\mathcal{B}} \in \mathcal{E} \subset \mathrm{Epi}\mathcal{A}$, and so it is an isomorphism. □

In the conglomerate of the $\mathcal{E}$-prereflections, we define a preorder by:

$\gamma \leq \delta$ if there exists a natural transformation σ such that $\sigma.\gamma = \delta$.

Considering the quasicategory $(\mathrm{Id}_{\mathcal{A}} \downarrow [\mathcal{A},\mathcal{A}])$ (cf. [36]), where $[\mathcal{A},\mathcal{A}]$ denotes the quasicategory of endofunctors of $\mathcal{A}$, the $\mathcal{E}$-prereflections γ and δ are isomorphic (as objects of this quasicategory) if and only if $\gamma \leq \delta$ and $\delta \leq \gamma$. It is obvious that $\gamma \leq \delta$ implies

$\mathcal{K}_\gamma \subset \mathcal{K}_\delta$ and $\mathcal{I}_\delta \subset \mathcal{I}_\gamma$.

LEMMA 8.8.

(i) $\pi^{\mathcal{B}} \cong \min\{\gamma \mid \mathcal{B} \subset \mathcal{K}_\gamma\}$.

(ii) $\rho^{\mathcal{B}} \cong \max\{\gamma \mid \mathcal{B} \subset \mathcal{I}_\gamma\}$.

Proof: (i) Let us denote the $\min\{\gamma \mid \mathcal{B} \subset \mathcal{K}_\gamma\}$ by m. We have that $m \leq \pi^{\mathcal{B}}$, by 8.6(ii). Consider $\gamma : \mathrm{Id}_{\mathcal{A}} \longrightarrow H$ such that $\mathcal{B} \subset \mathcal{K}_\gamma$. For each $A \in \mathrm{Obj}\,\mathcal{A}$ and each morphism u with codomain A and domain B in $\mathcal{B}$, $\gamma_A . u = H(u).\gamma_B = 0$ since $\gamma_B = 0$. Let $(\pi^{\mathcal{B}}_A, (m_i)_{i \in I})$ be the orthogonal $\mathcal{E}$-factorization of $F^{\mathcal{B}}_A$. Then there exists a $i_A \in I$ such that $\gamma_A = m_{i_A} . \pi^{\mathcal{B}}_A$. Therefore, defining $\sigma : P^{\mathcal{B}} \longrightarrow H$ by $\sigma_A = m_{i_A}$, for every $A \in \mathrm{Obj}\,\mathcal{A}$, it is obvious that $\sigma . \pi^{\mathcal{B}} = \gamma$, i.e. $\pi^{\mathcal{B}} \leq \gamma$. Hence, $\pi^{\mathcal{B}} \leq \gamma$, for every γ such that $\mathcal{B} \subset \mathcal{K}_\gamma$, and so $\pi^{\mathcal{B}} \leq m$. Consequently, $\pi^{\mathcal{B}} = m$.

(ii) Let us denote the $\max\{\gamma \mid \mathcal{B} \subset \mathcal{I}_\gamma\}$ by M. According to 8.7(ii), $\rho^{\mathcal{B}} \leq M$. Consider $\gamma : \mathrm{Id}_{\mathcal{A}} \longrightarrow H$ such that $\mathcal{B} \subset \mathcal{I}_\gamma$, $(A, (f_i : A \longrightarrow B_i)_{i \in I})$ the source of all morphisms with domain A and codomain in $\mathcal{B}$ and $(\rho^{\mathcal{B}}_A, (m_i)_{i \in I})$ its orthogonal $\mathcal{E}$-factorization. Since, γ_B is an isomorphism, for every $B \in \mathrm{Obj}\,\mathcal{B}$, we have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "\gamma_A"] \arrow[d, "\rho^{\mathcal{B}}_A"'] \arrow[dr, "f_i"] & H(A) \arrow[dr, "H(f_i)"] & \\
R^{\mathcal{B}}(A) \arrow[r, "m_i"'] & B_i & H(B_i) \arrow[l, "\gamma^{-1}_{B_i}"]
\end{tikzcd}$$

Hence, there exists a unique morphism d_A such that $d_A . \gamma_A = \rho^{\mathcal{B}}_A$ and $m_i . d_A = \gamma^{-1}_{B_i} . H(f_i)$, for every $i \in I$. Defining $\delta : H \dashrightarrow R^{\mathcal{B}}$ by $\delta_A = d_A$, for each $A \in \mathrm{Obj}\,\mathcal{A}$, it is clear that $\delta . \gamma = \rho^{\mathcal{B}}$, i.e. $\gamma \leq \rho^{\mathcal{B}}$. We conclude that $M = \rho^{\mathcal{B}}$ because $\gamma \leq \rho^{\mathcal{B}}$ for every γ such that $\mathcal{B} \subset I_\gamma$ and so $M \leq \rho^{\mathcal{B}}$. □

DEFINITION 8.9. An $\mathcal{E}$-prereflection γ is called an <u>$\mathcal{E}$-connection</u> if $\gamma \cong \pi^{\mathcal{B}}$ for some subcategory $\mathcal{B}$.

Given an $\mathcal{E}$-prereflection $\gamma : \mathrm{Id}_{\mathcal{A}} \dashrightarrow H$ let us denote the $\mathcal{E}$-prereflections $\pi^{\mathcal{K}_\gamma}$ and

$\rho^{\mathcal{I}_\gamma}$ by $\underline{\gamma}$ and $\overline{\gamma}$, respectively. Lemma 8.8 tells us that $\underline{\gamma} \leq \gamma \leq \overline{\gamma}$.

LEMMA 8.10. *Let* $\gamma : \mathrm{Id}_{\mathcal{A}} \xrightarrow{\cdot} \mathrm{H}$ *be an* $\mathcal{E}$*-prereflection. Then:*

(i) $\overline{\gamma} \cong \min \{ \delta \mid \gamma \leq \delta$ and δ is an $\mathcal{E}$-reflection$\}$.

(ii) $\underline{\gamma} \cong \max \{ \delta \mid \delta \leq \gamma$ and δ is an $\mathcal{E}$-connection$\}$.

Proof: (i) Let us denote the minimum by m. By 8.7 we have that $\mathrm{m} \leq \overline{\gamma}$, since $\overline{\gamma}$ is an $\mathcal{E}$-reflection.

Let $\delta : \mathrm{Id}_{\mathcal{A}} \xrightarrow{\cdot} \mathrm{L}$ be an $\mathcal{E}$-reflection such that $\gamma \leq \delta$. By 8.5, $\mathcal{I}_\delta$ is reflective with reflection morphisms

$$(\delta_{\mathrm{A}} : \mathrm{A} \longrightarrow \mathrm{L}(\mathrm{A}))_{\mathrm{A} \in \mathrm{Obj}\,\mathcal{A}}$$

thus $\mathrm{L}(\mathrm{A}) \in \mathcal{I}_\delta \subset \mathcal{I}_\gamma$, for every object A of $\mathcal{A}$. Hence, by definition of $\overline{\gamma}_{\mathrm{A}} = \rho_{\mathrm{A}}^{\mathcal{I}_\gamma}$, δ_{A} factors through $\overline{\gamma}_{\mathrm{A}}$. It is easy to see that this way we define a natural transformation $\mathrm{R}^{\mathcal{I}_\gamma} \xrightarrow{\cdot} \mathrm{L}$ which shows that $\overline{\gamma} \leq \delta$. So, $\overline{\gamma} \leq \mathrm{m}$.

(ii) Denoting the maximum by M it is clear that $\underline{\gamma} \leq \mathrm{M}$. To prove that $\mathrm{M} \leq \underline{\gamma}$, let $\delta \cong \pi^{\mathcal{B}}$ be an $\mathcal{E}$-connection such that $\delta \leq \gamma$. By 8.6(ii), $\mathcal{B} \subset \mathcal{K}_{\pi^{\mathcal{B}}} = \mathcal{K}_\delta$. On the other hand, $\delta \leq \gamma$ implies that $\mathcal{K}_\delta \subset \mathcal{K}_\gamma$, and so $\mathcal{B} \subset \mathcal{K}_\gamma$. If $\mathrm{F}_{\mathrm{A}}^{\mathcal{B}} = (\mathrm{A}, (\mathrm{f}_i)_{i \in \mathrm{I}})$, $\mathrm{F}_{\mathrm{A}}^{\mathcal{K}_\gamma} = (\mathrm{A}, (\mathrm{g}_j)_{j \in \mathrm{J}})$ and $(\pi_{\mathrm{A}}^{\mathcal{B}}, (\mathrm{m}_i)_{i \in \mathrm{I}})$ and $(\pi_{\mathrm{A}}^{\mathcal{K}_\gamma}, (\mathrm{n}_j)_{j \in \mathrm{J}})$ are the respective orthogonal $\mathcal{E}$-factorizations then there exists, for each $j \in \mathrm{J}$, an $i_j \in \mathrm{I}$ such that $\mathrm{g}_j = \mathrm{f}_{i_j}$, which implies, for each $j \in \mathrm{J}$, the commutativity of the following diagram

$$\begin{tikzcd}
\mathrm{A} \arrow[r, "\pi_{\mathrm{A}}^{\mathcal{B}}"] \arrow[d, "\pi_{\mathrm{A}}^{\mathrm{K}_\gamma}"'] \arrow[dr, "\mathrm{g}_j = \mathrm{f}_{i_j}"] & \mathrm{P}^{\mathcal{B}}(\mathrm{A}) \arrow[d, "\mathrm{m}_{i_j}"] \\
\mathrm{P}^{\mathrm{K}_\gamma}(\mathrm{A}) \arrow[r, "\mathrm{n}_j"'] & \mathrm{A}_j
\end{tikzcd}$$

Hence, one has a unique morphism $\alpha_{\mathrm{A}} : \mathrm{P}^{\mathcal{B}}(\mathrm{A}) \longrightarrow \mathrm{P}^{\mathcal{K}_\gamma}(\mathrm{A})$ such that $\alpha_{\mathrm{A}} . \pi_{\mathrm{A}}^{\mathcal{B}} = \pi_{\mathrm{A}}^{\mathcal{K}_\gamma}$ and $\mathrm{n}_j . \alpha_{\mathrm{A}} = \mathrm{m}_{i_j}$, for every $j \in \mathrm{J}$. It is obvious that the morphisms $(\alpha_{\mathrm{A}})_{\mathrm{A} \in \mathrm{Obj}\,\mathcal{A}}$ define a natural transformation $\alpha : \mathrm{P}^{\mathcal{B}} \xrightarrow{\cdot} \mathrm{P}^{\mathcal{K}_\gamma}$ such that $\alpha . \pi^{\mathcal{B}} = \pi^{\mathcal{K}_\gamma}$. Hence, $\delta \leq \underline{\gamma}$, and so $\mathrm{M} \leq \underline{\gamma}$. $\square$

THEOREM 8.11. (1) *Let* $\mathcal{B}$ *be a subcategory of* $\mathcal{A}$. *Then the following assertions are equivalent:*

(i) $\mathcal{B}=\mathcal{B}^{\leftarrow\rightarrow}$.

(ii) $\mathcal{B}=\mathcal{I}_\gamma$ *for some* $\mathcal{E}$*-connection* γ.

(2) *Let* $\mathcal{C}$ *be a subcategory of* $\mathcal{A}$. *Then the following assertions are equivalent:*

(iii) $\mathcal{C}=\mathcal{C}^{\rightarrow\leftarrow}$.

(iv) $\mathcal{C}=\mathcal{K}_\gamma$ *for some* $\mathcal{E}$*-reflection* γ.

Proof:

(i) $\Rightarrow$ (ii): First we prove that for any subcategory $\mathcal{C}$, $\mathcal{C}^{\rightarrow}=\mathcal{I}_{\pi\mathcal{C}}$. Let $A\in\mathcal{C}^{\rightarrow}$. Then, the family of all morphisms g with domain A such that $g.u=0$ for every morphism u with codomain A and domain in $\mathcal{C}$ is exactly the source of all $\mathcal{A}$-morphisms with domain A. From the definition of $\pi_A^{\mathcal{C}}$ it follows that this morphism is a split monomorphism. Since it is always an epimorphism, $A\in\mathcal{I}_{\pi\mathcal{C}}$. Conversely, let $A\in\mathcal{I}_{\pi\mathcal{C}}$ and $F_A^{\mathcal{C}}=(A,(f_i)_{i\in I})$ be the respective source whose orthogonal $\mathcal{E}$-factorization is $(\pi_A^{\mathcal{C}},(m_i)_{i\in I})$. If $C\in\mathcal{C}$ and $f:C\longrightarrow A$ then it is clear that $f_i.f=0$, for every $i\in I$. Hence, in the commutative diagram

$$\begin{tikzcd}
C \arrow[r, "e"] \arrow[d, "f"'] & 0 \arrow[dd] \\
A \arrow[d, "\pi_A^{\mathcal{C}}"'] \arrow[dr, "f_i"] & \\
P^{\mathcal{C}}(A) \arrow[r, "m_i"'] & A_i
\end{tikzcd}$$

there exists a unique morphism d such that $d.e=\pi_A^{\mathcal{C}}.f$. But $\pi_A^{\mathcal{C}}$ is an isomorphism, hence $f=0$.

Now it suffices to take $\mathcal{C}=\mathcal{B}^{\leftarrow}$.

(ii) $\Rightarrow$ (i): If γ is an $\mathcal{E}$-connection then $\gamma\cong\underline{\gamma}$, by the preceding lemma, implying that $\mathcal{B}=\mathcal{I}_\gamma=\mathcal{I}_{\underline{\gamma}}$. To prove that ((i) $\Rightarrow$ (ii)), we saw that $\mathcal{I}_{\pi\mathcal{K}_\gamma}=\mathcal{K}_{\gamma}^{\rightarrow}$, i.e. $\mathcal{I}_{\underline{\gamma}}=\mathcal{K}_{\gamma}^{\rightarrow}$. Thus $\mathcal{B}^{\leftarrow\rightarrow}=\mathcal{K}_{\gamma}^{\rightarrow\leftarrow\rightarrow}=\mathcal{K}_{\gamma}^{\rightarrow}=\mathcal{B}$.

(iii) $\Rightarrow$ (iv): It is easy to verify that the equality $\mathcal{B}^{\leftarrow}=\mathcal{K}_{\rho\mathcal{B}}$ holds for any subcategory $\mathcal{B}$. Then the proof is concluded if we take $\mathcal{B}=\mathcal{C}^{\rightarrow}$.

(iv) $\Rightarrow$ (iii): By the preceding lemma $\gamma \cong \overline{\gamma}$. This implies that $\mathcal{C} = \mathcal{K}_{\overline{\gamma}}$. From ((iii) $\Rightarrow$ (iv)) we conclude that $\mathcal{K}_{\overline{\gamma}} = \mathcal{I}_{\gamma}^{\leftarrow}$. As a consequence of this fact we have that $\mathcal{C}^{\rightarrow\leftarrow} = \mathcal{I}_{\gamma}^{\leftarrow\rightarrow\leftarrow} = \mathcal{I}_{\gamma}^{\leftarrow} = \mathcal{C}$, as it was required. □

COROLLARY 8.12. *If $\mathcal{A}$ and $\mathcal{A}^{op}$ are (FWC)-categories and $\mathcal{A}$ is $\mathcal{E}$-cocomplete then:*

(i) $\mathcal{B}$ *is a torsion-free subcategory if and only if* $\mathcal{B} = \mathcal{I}_{\gamma}$ *for some* $\mathcal{E}$*-connection* γ.

(ii) $\mathcal{C}$ *is a torsion subcategory if and only if* $\mathcal{C} = \mathcal{K}_{\gamma}$ *for some* $\mathcal{E}$*-reflection* γ.

□

COROLLARY 8.13. *If $\mathcal{A}$ and $\mathcal{A}^{op}$ are (FWC)-categories and $\mathcal{A}$ is $\mathcal{E}$-cocomplete then the following conditions are equivalent:*

(i) $(\mathcal{C},\mathcal{B})$ *is a torsion theory.*

(ii) $\mathcal{B} = \mathcal{I}_{\gamma}$ *for some* $\mathcal{E}$*-connection* γ *and* $\mathcal{C} = \mathcal{K}_{\overline{\gamma}}$.

(iii) $\mathcal{C} = \mathcal{K}_{\gamma}$ *for some* $\mathcal{E}$*-reflection* γ *and* $\mathcal{B} = \mathcal{I}_{\underline{\gamma}}$.

Proof:

(i) $\Rightarrow$ (ii): Since $\mathcal{B}^{\leftarrow\rightarrow} = \mathcal{B}$, then $\mathcal{B} = \mathcal{I}_{\gamma}$ for some $\mathcal{E}$-connection γ. On the other hand, $\mathcal{C} = \mathcal{B}^{\leftarrow} = \mathcal{I}_{\gamma}^{\leftarrow} = \mathcal{K}_{\underset{\rho}{\mathcal{I}}_{\gamma}} = \mathcal{K}_{\overline{\gamma}}$, according to the proof of 8.11.

(ii) $\Rightarrow$ (i), (i) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (i) can be proved in a similar way. □

In addition, we can conclude that if γ is both an $\mathcal{E}$-connection and an $\mathcal{E}$-reflection then $(\mathcal{K}_{\gamma},\mathcal{I}_{\gamma})$ is a torsion theory.

EXAMPLE. Consider a (FWC) abelian category $\mathcal{A}$ such that its dual category is also (FWC) and let γ be the Epi-reflection referred to in 8.4(a). Assuming that the radical R is idempotent then γ is also an Epi-connection since $\gamma \cong \pi^{\mathcal{K}_{\gamma}}$. Indeed, let $F_M^{\mathcal{K}_{\gamma}} = (M,(f_i)_{i \in I})$ for every $M \in \mathrm{Obj}\,\mathcal{A}$, i.e. $(f_i)_{i \in I}$ is the family of morphisms g with domain M such that $g.u = 0$ for every morphism u with codomain M and domain in $\mathcal{K}_{\gamma}$. The fact that R is idempotent implies that $R(M) \in \mathcal{K}_{\gamma}$ and so $R(M) \subset \mathrm{Ker}\, f_i$, for each $i \in I$. Conversely, if g is a morphism with domain M such that $R(M) \subset \mathrm{Ker}\, g$, then g belongs to the family $(f_i)_{i \in I}$ because, for each $u: N \longrightarrow M$ with $N \in \mathcal{K}_{\gamma}$, we have that $u(R(N)) \subset R(M)$ (cf. [43]), since $R(N) = N$ and R is a radical. Thus $u(N) \subset \mathrm{Ker}\, g$, i.e. $g.u = 0$. Then, $(f_i)_{i \in I}$ is the family

$$\{ g \in \mathrm{Mor}\,\mathcal{A} \mid \mathrm{dom}(g) = M \text{ and } R(M) \subset \mathrm{Ker}\, g \}.$$

Consequently, each f_i factors through the morphism γ_M in a unique way:

$$\begin{tikzcd}
0 \arrow[r] & R(M) \arrow[r, "k_M"] \arrow[dr] & M \arrow[r, "\gamma_M"] \arrow[dr, "f_i"'] & M/R(M) \arrow[r] \arrow[d, dashed, "m_i"] & 0 \\
 & & \operatorname{Ker} f_i \arrow[u] & M_i &
\end{tikzcd}$$

Let us show now that the factorization $(\gamma_M,(m_i)_{i\in I})$ of the source $F_M^{\mathcal{K}_\gamma}$ is an orthogonal Epi-factorization. Given $e:A\longrightarrow B\in \operatorname{Epi}\mathcal{A}$, $v:A\longrightarrow M/R(M)$ and $(u_i:B\longrightarrow M_i)_{i\in I}$ such that $m_i.v=u_i.e$, for every $i\in I$, since $\operatorname{Ker}\gamma_M=R(M)$, then there exists some $j\in I$ such that $f_j=\gamma_M$ and, consequently, $m_j=1$. Since e is an epimorphism, then u_j is the unique morphism for which the following diagram

$$\begin{tikzcd}
A \arrow[r, "e"] \arrow[d, "v"'] & B \arrow[d, "u_i"] \arrow[dl, dashed, "u_j"'] \\
M/R(M) \arrow[r, "m_i"'] & M_i
\end{tikzcd}$$

commutes for every $i\in I$.

The orthogonal Epi-factorization $(\gamma_M,(m_i)_{i\in I})$ of the source $F_M^{\mathcal{K}_\gamma}$ and its unicity, up to isomorphism, enables us to conclude that $\gamma\cong\pi^{\mathcal{K}_\gamma}$ and so that $(\mathcal{K}_\gamma,\mathcal{J}_\gamma)$ is a torsion theory of $\mathcal{A}$.

Cassidy, Hébert and Kelly ([11], p. 307) (proposition 5.4 in this chapter) and Barr ([4], 1.1) gave different answers to the question of knowing which properties should be preserved when extending the concept of torsion to non-abelian categories. Actually, Barr defined the torsion theory of a category $\mathcal{A}$ as an idempotent monad in $\mathcal{A}$, $\mathbb{T}=(T,\eta,\mu)$, such that T preserves regular monomorphisms and η_A is an epimorphism, for every object A of $\mathcal{A}$, the torsion-free objects being the objects A for which η_A is an isomorphism. Therefore, since the monad is idempotent, the subcategory of the torsion-free objects is isomorphic to the category of $\mathbb{T}$-algebras.

Corollary 8.13 allows us to relate these two concepts, being the one introduced in [11] referred to just as torsion theory.

PROPOSITION 8.14. *Assume that* $\mathcal{A}$ *and* $\mathcal{A}^{op}$ *are* (FWC)-*categories.*

(*i*) *Let* $(\mathcal{C},\mathcal{B})$ *be a torsion theory of* $\mathcal{A}$. *Then the idempotent monad* (T,η,μ) *where* $\eta:\mathrm{Id}_{\mathcal{A}}\longrightarrow\mathrm{T}$ *is a* StrEpi-*reflection such that* $\mathcal{C}=\mathcal{K}_\eta$ *and* $\mu_{\mathrm{A}}=(\eta_{\mathrm{TA}})^{-1}$, *for every object* A *of* $\mathcal{A}$, *is a torsion theory, according to Barr's definition, if* T *preserves regular monomorphisms.*

(ii) *Let* (T,η,μ) *be a torsion theory of* $\mathcal{A}$, *in the sense of Barr. Then* $(\mathcal{K}_\eta,\mathcal{I}_{\underline{\eta}})$ *is a torsion theory if* $\mathcal{A}$ *is* Epi-*cocomplete or* $\eta_{\mathrm{A}}\in\mathrm{StrEpi}\mathcal{A}$, *for every* $\mathrm{A}\in\mathrm{Obj}\mathcal{A}$.

Proof: (i) To conclude that we get Barr's definition it suffices to observe that, since $\mathcal{A}^{op}$ is (FWC), $\mathcal{A}$ is StrEpi-cocomplete and then to apply 8.13. We point out that Barr defines the torsion-free class as $\mathcal{I}_\eta$, which is contained in $\mathcal{I}_{\underline{\eta}}=\mathcal{B}$, but the reflectivity of $\mathcal{B}$ implies easily that $\mathcal{I}_\eta=\mathcal{I}_{\underline{\eta}}$.

(ii) Since (T,η,μ) is an idempotent monad, then η is an Epi-reflection. So, when $\mathcal{A}$ is Epi-cocomplete we can use 8.13 to conclude that $(\mathcal{K}_\eta,\mathcal{I}_{\underline{\eta}})$ is a torsion theory. If $\eta_{\mathrm{A}}\in\mathrm{StrEpi}\mathcal{A}$, we can still use 8.13 since we have the StrEpi-cocompleteness of $\mathcal{A}$. □