Gutiérrez García, Javier; **Mozo Carollo, Imanol**; **Picado, Jorge**
Normal semicontinuity and the Dedekind completion of pointfree function rings. (English)
Zbl 06590289
Algebra Univers. 75, No. 3, 301-330 (2016).

This paper about the Dedekind completion of the ℓ-ring $\mathrm{C}(L)$ of real functions on a frame L. In an earlier paper [Forum Math. 27, No. 5, 2551–2585 (2015; Zbl 1332.06028)], the authors constructed the Dedekind completion of $\mathrm{C}(L)$ (and its bounded part $\mathrm{C}^*(L)$) in terms of what they called partial continuous real functions on L. The present paper supplements this earlier one by presenting three different alternative views of the completion. The first is the point-free extension of *R. P. Dilworth*'s [Trans. Am. Math. Soc. 68, 427–438 (1950; Zbl 0037.20205)] construction of the Dedekind completion of the ℓ-ring (Dilworth viewed it as a lattice) $\mathrm{C}(X)$ of real-valued continuous functions on a topological space X. The second view exhibits the Dedekind completion of $\mathrm{C}^*(L)$, for L a completely regular frame, as a function ring. More precisely, the authors show that the Dedekind completion of $\mathrm{C}^*(L)$ is $\mathrm{C}^*(\mathfrak{B}L)$, where $\mathfrak{B}L$ denotes the Booleanization of L. Indeed this is a function ring because, as in the classical case, every C is (isomorphic to) a C*. After drawing the attention of the reader to the fact that, in general, the Dedekind completion of an arbitrary completely regular frame cannot be a function ring, they identify a class for which it always can. The class in question is that of weakly continuously bounded frames. They show that for such a frame L, the Dedekind completion of $\mathrm{C}(L)$ is $\mathrm{C}(\mathfrak{G}L)$, where $\mathfrak{G}L$ is the Gleason envelope of L. The latter can be realized as some closed quotient of the coproduct of L with the frame of ideals of the Booleanization of the Stone-Čech compactification of L. The final construction of the Dedekind completion of $\mathrm{C}(L)$ is in terms of Hausdorff continuous partial real functions on L. It is the point-free version of the approach in terms of interval-valued functions of the Dedekind completion of $C(X)$ [*R. Anguelov*, Quaest. Math. 27, No. 2, 153–169 (2004; Zbl 1062.54017)]. The paper is well written, and treats this subject very thoroughly. Classical results in this area appear as corollaries of the authors' point-free theorems. Furthermore, these point-free theorems cover a wider scope than their classical antecedents.

Reviewer: Themba Dube (Unisa)

References:

[1] Anguelov R.: Dedekind order completion of C(X) by Hausdorff continuous real functions. Quaest. Math. 27, 153–169 (2004)· doi:10.2989/16073600409486091

[2] Banaschewski B.: Compact regular frames and the Sikorski Theorem. Kyungpook Math. J. 28, 1–14 (1988)

[3] Banaschewski B.: The real numbers in pointfree topology. Textos de Matem\'atica, vol. 12, Departamento de Matem\'atica da Universidade de Coimbra (1997)

[4] Banaschewski B, Guti\'errez Garc\'{\i}a J, Picado J: Extended real functions in pointfree topology. J. Pure Appl.

Algebra 216, 905–922 (2012)· doi:10.1016/j.jpaa.2011.10.026

[5] Banaschewski B, Hager A.W: Essential completeness in categories of completely regular frames. Appl. Categ. Structures 21, 167–180 (2013)· doi:10.1007/s10485-011-9262-3

[6] Banaschewski B, Hong S.S: Completeness properties of function rings in pointfree topology. Comment. Math. Univ. Carolin. 44, 245–259 (2003)

[7] Banaschewski B, Pultr A.: Booleanization. Cahiers Topologie G\'eom. Diff\'erentielle Cat\'eg. 37, 41–60 (1996)

[8] Chen X.: Closed frame homomorphisms. PhD Thesis, McMaster University (1991)

[9] D\u{a}ne\c{t} N.: The Dedekind completion of C(X): An interval-valued functions approach. Quaest. Math. 34, 213–223 (2011)· doi:10.2989/16073606.2011.594236

[10] Dieudonn\'e J.: Review of [14]. Math. Reviews 22, 1190–1191 (1961)

[11] Dilworth R.P.: The normal completion of the lattice of continuous functions. Trans. Amer. Math. Soc. 68, 427–438 (1950)· doi:10.1090/S0002-9947-1950-0034822-9

[12] Dube T.: An algebraic view of weaker forms of realcompactness. Algebra Universalis 55, 187–202 (2006)· doi:10.1007/s00012-006-1946-4

[13] Dube T.: Realcompactness and certain types of subframes. Algebra Universalis 58, 181–202 (2008)· doi:10.1007/s00012-008-2058-0

[14] Gillman L., Jerison M.: Rings of continuous functions. D. Van Nostrand (1960)

[15] Guti\'errez Garc\'{\i}a J, Kubiak T.: General insertion and extension theorems for localic real functions. J. Pure Appl. Algebra 215, 1198–1204 (2011)· doi:10.1016/j.jpaa.2010.08.004

[16] Guti\'errez Garc\'{\i}a J, Kubiak T.: Lower and upper regularizations of frame semicontinuous real functions. Algebra Universalis 60, 169–184 (2009)· doi:10.1007/s00012-009-2102-8

[17] Guti\'errez Garc\'{\i}a J., Kubiak T., Picado J.: Pointfree forms of Dowker's and Michael's insertion theorems. J. Pure Appl. Algebra 213, 98–108 (2009)

[18] Guti\'errez Garc\'{\i}a J., Kubiak T., Picado J.: Localic real functions: A general setting. J. Pure Appl. Algebra 213, 1064–1074 (2009)

[19] Guti\'errez Garc\'{\i}a J, Picado J.: On the algebraic representation of semicontinuity. J. Pure Appl. Algebra 210, 299–306 (2007)· doi:10.1016/j.jpaa.2006.09.004

[20] Guti\'errez Garc\'{\i}a J, Picado J.: Rings of real functions in pointfree topology. Topology Appl. 158, 2264–2278 (2011)· doi:10.1016/j.topol.2011.05.040

[21] Guti\'errez Garc\'{\i}a J, Picado J: On the parallel between normality and extremal disconnectedness. J. Pure Appl. Algebra 218, 784–803 (2014)· doi:10.1016/j.jpaa.2013.10.002

[22] Horn A.: The normal completion of a subset of a complete lattice and lattices of continuous functions. Pacific J. Math. 3, 137–152 (1953)· doi:10.2140/pjm.1953.3.137

[23] Horne J.G.: Countable paracompactness and cb-spaces. Notices Amer. Math. Soc. 6, 629–630 (1959)

[24] Johnson D.G.: The completion of an archimedean f-ring. J. London Math. Soc. 40, 493–496 (1965)· doi:10.1112/jlms/s1-40.1.493

[25] Johnstone P.T.: The Gleason cover of a topos, I. J. Pure Appl. Algebra 22, 171–192 (1980)· doi:10.1016/0022-4049(80)90100-0

[26] Johnstone P.T.: Stone Spaces. Cambridge Studies in Advanced Mathematics, Vol. 3, Cambridge University Press (1982)

[27] Mack J.E.: On a class of countably paracompact spaces. Proc. Amer. Math. Soc. 16, 467–472 (1965)· doi:10.1090/S0002-9939-1965-0177388-1

[28] Mack J.E, Johnson D.G.: The Dedekind completion of ${C(\mathcal{X})}$ C (X) . Pacific J. Math. 20, 231–243 (1967)· doi:10.2140/pjm.1967.20.231

[29] Mozo Carollo I, Guti\'errez Garc\'{\i}a J, Picado J.: On the Dedekind completion of function rings. Forum Math. 27, 2551–2585 (2015)

[30] Nakano H.: \"Uber das System aller stetigen Funktionen auf einen topologischen Raum. Proc. Imp. Acad. Tokyo 17, 308–310 (1941) (German)

[31] Picado J., Pultr A.: Frames and locales: Topology without points. Frontiers in Mathematics, vol. 28, Springer, Basel (2012)

[32] Steinberg S.A.: Lattice-ordered rings and modules. Springer, New York (2010)

[33] Stone M.H.: Boundedness properties in function lattices. Canad. J. Math. 1, 176–186 (1949)· doi:10.4153/CJM-1949-016-5