Erné, Marcel; **Picado, Jorge**; **Pultr, Aleš**

Adjoint maps between implicative semilattices and continuity of localic maps. (English)

Zbl 07501177
Algebra Univers. 83, No. 2, Paper No. 13, 23 p. (2022)

There exists the concept of *adjoint pair of maps* between partially ordered sets (posets), i.e., given posets A, B, and maps $f : A \to B$, $g : B \to A$, if $f(a) \leqslant b$ if and only if $a \leqslant g(b)$ for every $a \in A$ and every $b \in B$, then g is said to be the *right* or *upper adjoint* of f, and f is said to be the *left* or *lower adjoint* of g. An example of an adjoint pair is provided by any map $f : S \to T$ between sets, namely, f induces an adjoint pair of maps $f_{\to} : \mathcal{P}S \to \mathcal{P}T$ and $f^{\leftarrow} : \mathcal{P}T \to \mathcal{P}S$ between the powersets $\mathcal{P}S$ and $\mathcal{P}T$ defined by $f_{\to}(U) = \{f(u) \,|\, u \in U\}$ for every $U \in \mathcal{P}S$ and $f^{\leftarrow}(V) = \{s \in S \,|\, f(s) \in V\}$ for every $V \in \mathcal{P}T$.

The above example could be extended to two functors between the category **Top** of topological spaces and continuous maps, and the categories **Frm** of frames and frame homomorphisms (maps preserving arbitrary joins and finite meets), and **Loc** of locales (another word for frames) and locale morphisms or localic maps (right adjoints of frame homomorphisms), which are the cornerstones of point-free topology [*P. T. Johnstone*, Stone spaces. Cambridge: Cambridge University Press (1982; Zbl 0499.54001)]. More precisely, there exists a contravariant functor $\mathcal{O}^{\leftarrow} : \mathbf{Top} \to \mathbf{Frm}$ defined on a continuous map $f : S \to T$ by $\mathcal{O}^{\leftarrow} f : \mathcal{O}T \to \mathcal{O}S$, $\mathcal{O}^{\leftarrow} f(V) = f^{\leftarrow}(V)$, where $\mathcal{O}S$ and $\mathcal{O}T$ are the topologies on the spaces S and T, respectively. Moreover, there exists a functor $\mathcal{O}_{\to} : \mathbf{Top} \to \mathbf{Loc}$ defined on a continuous map $f : S \to T$ by $\mathcal{O}_{\to} f : \mathcal{O}S \to \mathcal{O}T$, $\mathcal{O}_{\to} f(U) = T \backslash cl(f_{\to}(S \backslash U))$, where $cl(-)$ stands for the topological closure.

The present paper considers localic maps as "continuous maps" and characterizes them by certain concrete continuity properties in the closure-theoretical sense (preimages of closed subobjects are closed, and the formation of preimages commutes with complementation). To be more general , the authors study the category of implicative semilattices, i.e., meet-semilattices with top elements, where the unary meet operations $\lambda_a = a \wedge -$ have right adjoints $\alpha_a = a \to -$. One should observe that frames and locales are just the complete implicative semilattices. In order to have **Frm** and **Loc** as full subcategories of two respective dual categories, whose objects are implicative semilattices, the authors consider as morphisms not the usual implicative homomorphisms (preserving finite meets and the binary operation $\to$), but the left adjoint top-preserving semilattice homomorphisms, briefly referred to as r-morphisms, and in the opposite direction their right adjoints, the so-called *localizations* [*G. Bezhanishvili* and *S. Ghilardi*, Ann. Pure Appl. Logic 147, No. 1–2, 84–100 (2007; Zbl 1123.03055)] or l-morphisms. In one word, r-morphisms have right adjoints and preserve finite meets, whereas l-morphisms have left adjoints that preserve finite meets.

The paper is well written, gives most of its required preliminaries (the omitted concepts can be found in one of the references at the end of the paper), and will be of interest to the researchers studying point-free topology.

Reviewer: Sergejs Solovjovs (Praha)

MSC:

06D20	Heyting algebras (lattice-theoretic aspects)
06A15	Galois correspondences, closure operators (in relation to ordered sets)
06D22	Frames, locales
18B99	Special categories
18F70	Frames and locales, pointfree topology, Stone duality
54C05	Continuous maps

Keywords:

adjoint pair of maps; frame; frame homomorphism; Heyting algebra; implicative semilattice; locale; localic homomorphism; nucleus; point-free topology; quasi-open map; residuated map; subframe; sublocale; zero-dimensional space

Full Text: DOI

References:

[1] Adámek, J.; Herrlich, H.; Strecker, GE, Abstract and Concrete Categories. The Joy of Cats (2009), Dover: Dover Publications Inc., Dover · Zbl 0695.18001

[2] Aull, CE; Thron, WJ, Separation axioms between \(T_0\) and \(T_1\), Indag. Math., 24, 26-37 (1963) · Zbl 0108.35402

[3] Baer, R., On closure operators, Arch. Math., 10, 261-266 (1959) · Zbl 0089.01903 · doi:10.1007/BF01240795

[4] Banaschewski, B., Another look at the localic Tychonoff theorem, Comment. Math. Univ. Carol., 29, 647-656 (1988) · Zbl 0667.54009

[5] Beazer, R.; Macnab, DS, Modal operators on Heyting algebras, Colloq. Math., XVI, 1-12 (1979) · Zbl 0436.06010 · doi:10.4064/cm-41-1-1-12

[6] Bezhanishvili, G.; Ghilardi, S., An algebraic approach to subframe logics. Intuitionistic case, Ann. Pure Appl. Log., 147, 84-100 (2007) · Zbl 1123.03055 · doi:10.1016/j.apal.2007.04.001

[7] Bezhanishvili, G., et al.: The variety of nuclear implicative semilattices is locally finite. Conference Talk, TACL 2017, Prague (2017)

[8] Blyth, TS; Janowitz, MF, Residuation Theory (1972), Oxford: Pergamon Press, Oxford · Zbl 0301.06001

[9] Borceux, F., Handbook of Categorical Algebra I (1994), Cambridge: Cambridge University Press, Cambridge· doi:10.1017/CBO9780511525865

[10] Dowker, CH; Papert, D., Quotient frames and subspaces, Proc. Lond. Math. Soc., s3-16, 275-296 (1966) · Zbl 0136.43405 · doi:10.1112/plms/s3-16.1.275

[11] Erné, M.; Bentley, LH, Lattice representations for categories of closure spaces, Categorical Topology (Proc. Toledo, 1983), 197-222 (1984), Berlin: Heldermann, Berlin

[12] Erné, M.; Herrlich, H.; Porst, H-E, The ABC of order and topology, Category Theory at Work, 57-83 (1991), Berlin: Heldermann, Berlin · Zbl 0735.18005

[13] Erné, M.; Denecke, K.; Erné, M.; Wismath, Sh, Adjunctions and Galois connections: origins, history and development, Galois Connections and Applications, 1-138 (2004), Dordrecht: Kluwer, Dordrecht · Zbl 1067.06003

[14] Erné, M.: General Stone duality. In: Clementino, M.M., et al. (eds.) Proceedings of the IV CITA, 2001. Topology and its Applications, vol. 137, pp. 125-158 (2004) · Zbl 1043.06008

[15] Erné, M.; Mynard, F.; Pearl, E., Closure, Beyond Topology, 163-238 (2009), Providence: American Mathematical Society, Providence · Zbl 1192.54001 · doi:10.1090/conm/486/09510

[16] Erné, M.: Nuclear ranges in implicative semilattices. Algebra Universalis (to appear)

[17] Erné, M.: Assemblies of implicative semilattices. Preprint

[18] Esakia, L.: Heyting Algebras, Duality Theory; edited by Bezhanishvili, G., and Holliday, W. Springer, Berlin (2019) · Zbl 1436.06001

[19] Gierz, G.; Hofmann, KH; Keimel, K.; Lawson, JD; Mislove, M.; Scott, DS, Continuous Lattices and Domains (2003), Oxford: Oxford University Press, Oxford · Zbl 1088.06001 · doi:10.1017/CBO9780511542725

[20] Hofmann, KH; Mislove, M.; Hoffmann, RE; Hofmann, KH, Free objects in the category of completely distributive lattices, Continuous Lattices and Their Applications, 129-151 (1985), New York: Marcel Dekker, New York

[21] Isbell, J., Atomless parts of spaces, Math. Scand., 31, 5-32 (1972) · Zbl 0246.54028 · doi:10.7146/math.scand.a-11409

[22] Johnstone, PT, Stone Spaces (1982), Cambridge: Cambridge University Press, Cambridge · Zbl 0499.54001

[23] Johnstone, PT, The point of pointless topology, Bull. Am. Math. Soc., 8, 41-53 (1983) · Zbl 0499.54002 · doi:10.1090/S0273-0979-1983-15080-2

[24] Johnstone, P.T.: Sketches of an Elephant: A Topos Theory Compendium, vol. 2. Oxford Logic Guides, vol. 44. Oxford Science Publications, Oxford (2002)

[25] Johnstone, PT, Complemented sublocales and open maps, Ann. Pure Appl. Log., 137, 240-255 (2006) · Zbl 1080.18002 · doi:10.1016/j.apal.2005.05.030

[26] Joyal, A.; Tierney, M., An Extension of the Galois Theory of Grothendieck (1984), Providence: AMS, Providence · Zbl 0541.18002

[27] Köhler, P., Brouwerian semilattices, Trans. Am. Math. Soc., 268, 103-126 (1981) · Zbl 0473.06003 · doi:10.2307/1998339

[28] Köhler, P.: Brouwerian semilattices: the lattice of total subalgebras. In: Universal Algebra and Applications, pp. 47-56. Banach Center Publications, vol. 9. PWN-Polish Scientific Publishers, Warsaw (1982)

[29] Mac Lane, S.; Moerdijk, I., Sheaves in Geometry and Logic (1994), Berlin: Springer, Berlin· doi:10.1007/978-1-4612-0927-0

[30] Macnab, D.S.: An algebraic study of modal operators on Heyting algebras with applications in topology and sheafification. Ph.D. thesis, University of Aberdeen (1976)

[31] Macnab, DS, Modal operators on Heyting algebras, Algebra Univers., 12, 5-29 (1981) · Zbl 0459.06005 · doi:10.1007/BF02483860

[32] Martínez, J.: Frobenius identities in frames. Preprint. Pennsylvania State University. http://citeseerx.ist.psu.edu

[33] Monteiro, A.; Ribeiro, H., L' opération de fermeture et ses invariants dans les systèmes partiellement ordonnés, Port. Math., 3, 171-183 (1942) · Zbl 0027.43202

[34] Nemitz, WC, Implicative semi-lattices, Trans. Am. Math. Soc., 117, 128-142 (1965) · Zbl 0128.24804 · doi:10.1090/S0002-9947-1965-0176944-9

[35] Picado, J.; Pultr, A., Sublocale sets and sublocale lattices, Arch. Math. (Brno), 42, 409-418 (2006) · Zbl 1164.06313

[36] Picado, J.; Pultr, A., Locales Treated Mostly in a Covariant Way (2008), Coimbra: University of Coimbra, Coimbra · Zbl 1154.06007

[37] Picado, J.; Pultr, A., Frames and Locales: Topology Without Points (2012), Basel: Springer, Basel · Zbl 1231.06018 · doi:10.1007/978-3-0348-0154-6

[38] Picado, J.; Pultr, A., On equalizers in the category of locales, Appl. Categ. Struct., 29, 267-283 (2021) · Zbl 1467.18023 · doi:10.1007/s10485-020-09616-8

[39] Picado, J.; Pultr, A.; Tozzi, A.; Pedicchio, MC; Tholen, W., Locales, Categorical Foundations: Special Topics in Order, Topology, Algebra and Sheaf Theory, 49-101 (2003), Cambridge: Cambridge University Press, Cambridge· doi:10.1017/CBO9781107340985.005

[40] Picado, J.; Pultr, A.; Tozzi, A., Ideals in Heyting semilattices and open homomorphisms, Quaest. Math., 30, 391-405 (2007) · Zbl 1143.06004 · doi:10.2989/16073600709486208

[41] Simmons, H., The lattice theoretic part of topological separation axioms, Proc. Edinb. Math. Soc., 21, 41-48 (1978) · Zbl 0396.54014 · doi:10.1017/S0013091500015868

[42] Simmons, H.: A framework for topology. In: Logic Colloquium 77. Studies in Logic and the Foundations of Mathematics, vol. 87, pp. 239-251. North-Holland, Amsterdam (1978) · Zbl 0493.06005

[43] Simmons, H.: The assembly of a frame. University of Manchester (2006). doi:10.1.1.192.9717

[44] Skula, L., On a reflective subcategory of the category of all topological spaces, Trans. Am. Math. Soc., 142, 37-41 (1969) · Zbl 0185.50401

[45] Varlet, J., Relative annihilators in semilattices, Bull. Austral. Math. Soc., 9, 169-185 (1973) · Zbl 0258.06009 · doi:10.1017/S0004972700043094