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Uniformities and a quantale structure on localic groups. (English) Zbl 07527843

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It is well known that classical concepts and facts of general topology can be naturally extended to the realm of locales, often achieving better insights into the phenomena. Likewise, the notion of a uniformity has been studied in the localic context via Tukey's system of covers or as system of entourages (see [A. Pultr, Commentat. Math. Univ. Carol. 25, 91–104 (1984); *ibid* 25, 105–120 (1984; [Zbl 0543.54023](#)); J. Picado, Commentat. Math. Univ. Carol. 36, No. 2, 357–370 (1995; [Zbl 0832.54025](#)); J. Picado, Appl. Categ. Struct. 8, No. 1–2, 351–366 (2000; [Zbl 0965.06012](#))]). Both these approaches are equivalent, even though the product of locales (and consequently, the entourages) are not a conservative extension of the classical products of spaces and neighbourhoods of diagonals (see [J. Picado and A. Pultr, Appl. Categ. Struct. 21, No. 1, 49–66 (2013; [Zbl 1277.06005](#))], also [J. Picado, Commentat. Math. Univ. Carol. 36, No. 2, 357–370 (1995; [Zbl 0832.54025](#))]).

The algebraic structure on a topological group (i.e., a group internal to the category **Top** of topological spaces and continuous maps) induce uniformities on the underlying topological space. Likewise the algebraic structure on a localic group (i.e., a group internal to the category **Loc** of locales and localic maps) induce uniformities on the underlying locale. However in the absence of non-trivial *points* in locales (irrespective of their size), the idea of shifting neighbourhoods of the group unit via the translation homeomorphisms do not work. Despite this difficulty, the paper proves there are again induced uniformities (both of Tukey or Weil type) and the two are equivalent. The major tool in the paper is that of a special type of involutive quantale (called *G-quantale*), which is borne out of the group structure of the locale. Such quantales are investigated and characterised paving the path to formulating properties of the enriched quantales from which the group structure can be reconstructed. The localic group homomorphisms then turn out to be in one to one correspondence with quantale homomorphisms.

Finally, the *G*-quantales are also ordered semigroups and hence can be used as a set of values for a generalised metric. The paper shows the natural group uniformities are metric uniformities of thus generalised metrics.

On the whole the paper is written in an excellent expository style which in reviewer's opinion makes a fascinating reading.

Reviewer: [Partha Ghosh \(Johannesburg\)](#)

MSC:

[06D22](#) Frames, locales
[06F07](#) Quantales
[18F70](#) Frames and locales, pointfree topology, Stone duality
[54H11](#) Topological groups (topological aspects)

Keywords:

[frame](#) [entourage](#); [involutive quantale](#); [locale](#); [localic group](#); [localic group uniformities](#); [ordered semigroup](#); [quantale](#); [frame](#)

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