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Frame presentations of compact hedgehogs and their properties. (English) Zbl 07681679
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A *compact hedgehog space* is $\Lambda J(\kappa)$, where κ is a cardinal number and

$$J(\kappa) = \{-\infty\} \cup \bigcup_{i\in\kappa} \big((\overline{\mathbb{R}} \setminus \{-\infty\}) \times \{i\}\big)$$

where $\overline{\mathbb{R}}$ is the extended real line with the compact topology generated by the subbasis:

$$\Big\{(r,+\infty] \times \{i\} : r \in \mathbb{Q}, i \in \kappa\Big\} \cup \Big\{J(\kappa) \setminus \big([r,+\infty] \times \{i\}\big) : r \in \mathbb{Q}, i \in \kappa\Big\}.$$

The purpose of the present paper is to study the compact topology of the hedgehog via frame presentations using generators and relations, starting from rationals, independently of any notion of real numbers. The following is a brief description of the results obtained in §2–7:

1. A brief survey of definitions and facts from pointfree topology necessary for the paper is made; for more details see [*J. Picado* and *A. Pultr*, Frames and locales. Topology without points. Berlin: Springer (2012; Zbl 1231.06018)], while for general topology see [*R. Engelking*, General topology. Rev. and compl. ed. Berlin: Heldermann Verlag (1989; Zbl 0684.54001)].
2. The compact hedgehog topological space via the metric topology and the Lawson topology induced by the linear order are first discussed. This leads to the point free description of the *frame* $\mathcal{L}(\mathsf{c}J(\kappa))$ *of the compact hedgehog with* κ *spines* using generators and relations. It is shown that $\mathcal{L}(\mathsf{c}J(\kappa))$ is a compact regular frame (Theorem 3.3), is metrisable if and only if $\kappa \leq \aleph_0$ (Proposition 3.5) such that each regular subframe of it is also metrisable (Corollary 3.6), finally leading to the the spectrum of $\mathcal{L}(\mathsf{c}J(\kappa))$ which is $\Lambda J(\kappa)$ (Proposition 3.9).
3. Introduces families $\mathsf{F}_\kappa(L)$, $\mathsf{LSC}_\kappa(L)$, $\mathsf{USC}_\kappa(L)$, $\mathsf{C}_\kappa(L) = \mathsf{LSC}_\kappa(L) \cap \mathsf{USC}_\kappa(L)$ of functions, lower semicontinuous functions, upper semicontinuous functions, continuous functions on a frame L taking values in a $\mathcal{L}(\mathsf{c}J(\kappa))$. It is shown that every κ-indexed disjoint family of extended real valued functions on a frame L (a family $\Big\langle \mathcal{L}(\overline{\mathbb{R}}) \xrightarrow{h} L : i \in I\Big\rangle$ is called a *disjoint family* if $i \neq j \Rightarrow h_i(\bigvee_{r\in\mathbb{Q}}(r,-)) \vee h_j(\bigvee_{r\in\mathbb{Q}}(r,-)) = L)$ corresponds to a function in $\mathsf{F}_\kappa(L)$ (Proposition 4.2) and concludes on characterising the members of $\mathsf{LSC}_\kappa(L)$, $\mathsf{USC}_\kappa(L)$, $\mathsf{C}_\kappa(L)$ in terms of κ-indexed disjoint families of extended real valued functions (Corollary 4.3). Finally this is utilised in characterising disjoint κ-indexed families of cozero elements of a frame (Corollary 4.4) and the characteristic function of a κ-family of sublocales (given mutually disjoint family $\mathcal{C}$ of sublocales, its *characteristic function* $\chi_{\mathcal{C}}$ is the function in $\mathsf{F}_\kappa(L)$ guaranteed to exist by Proposition 4.2).
4. Recall a mutually disjoint family $\langle a_i : i \in \kappa\rangle$ of elements of a frame L is *discrete* (respectively, *codiscrete*) if there exists a cover C of L such that for any $c \in C$, $c \wedge a_i = 0$ (respectively, $c \leq a_i$) for all i with at most one exception; further from [*J. Gutiérrez García* et al., J. Pure Appl. Algebra 223, No. 6, 2345–2370 (2019; Zbl 1471.06005)] (or [*J. Picado* and *A. Pultr*, Separation in point-free topology. Cham: Birkhäuser (2021; Zbl 1486.54001)]), a frame is κ*-collectionwise normal* if for every codiscrete κ-family $\langle a_i : i \in \kappa\rangle$ there exists a discrete family $\langle b_i : i \in \kappa\rangle$ such that $a_i \vee b_i = 1$ $(i \in I)$, a frame is *totally* κ*-collectionwise normal* if every closed sublocale is z^c_κ-embedded (a sublocale S of a frame L is z^c_κ*-embedded in* L if for every $\mathcal{L}(\mathsf{c}J(\kappa)) \xrightarrow{f} S$ there exists a $\mathcal{L}(\mathsf{c}J(\kappa)) \xrightarrow{g} L$ such that $\nu_S(\bigvee_{r\in\mathbb{Q}} g\left((r,-) \times \{i\}\right)) = \bigvee_{r\in\mathbb{Q}} f\left((r,-) \times \{i\}\right)$ for every $i \in I$, where for any sublocale S, $\nu_S \dashv \mathsf{j}_S$ with j_S the inclusion map for S) and *totally collectionwise normal* if it is totally κ-collectionwise normal for every κ. Firstly, as a correction to a result in [*J. Gutiérrez García* et al., J. Pure Appl. Algebra 223, No. 6, 2345–2370 (2019; Zbl 1471.06005)], it is shown that for describing collectionwise normality one could relax to disjoint families instead of discrete families (Proposition 5.2). A localic version of the *pasting lemma* is developed (Proposition 5.4) using which it is shown that every

totally κ-collectionwise normal frame is κ-collectionwise normal (Proposition 5.5) and hence normal (Corollary 5.6).

5. This section characterises totally κ-collectionwise normal frames as precisely those where every continuous hedgehog valued continuous function on closed sublocales have a continuous extension (Theorem 6.3) – a type of Tietze extension theorem for compact hedgehogs.
6. In this section, in continuation from [*J. Gutiérrez García* et al., Houston J. Math. 35, No. 2, 469–484 (2009; Zbl 1176.54015)], some Katetov-Tong type insertion results characterising normality is proved. In particular, it is shown at the end of this section that total κ-collectionwise normality is hereditary with respect to closed sublocales (Lemma 7.2) leading to characterisation of total κ-collectionwise normality (Theorem 7.3).

The purpose of the remaining four sections of the paper is to treat several variants of normality and total κ-collectionwise normality from a unified framework. Towards this end, an object function $\mathbb{F}$ on the category of frames is called a *sublocale selection* if $\mathbb{F}(L)$ is a class of complemented sublocales of L and $\mathbb{F}^*(L) = \{S^* : S \in \mathbb{F}(L)\}$. For instance, one could consider sublocale selection $\mathbb{F}_{\mathtt{c}}$, $\mathbb{F}_{\mathtt{reg}}$, $\mathbb{F}_{\mathtt{z}}$, $\mathbb{F}_{\delta\mathtt{reg}}$ for closed sublocales, regular closed sublocales, zero sublocales and δ-regular closed sublocales, respectively.

Given a sublocale selection $\mathbb{F}$, a locale L is called $\mathbb{F}$-*normal* if for any $S, T \in \mathbb{F}(L)$ there exist $A, B \in \mathbb{F}(L)$ such that $S \cap A = \mathsf{O} = T \cap V$ and $A \vee B = L$. The $\mathbb{F}_{\mathtt{c}}$-normality is standard normality, $\mathbb{F}_{\mathtt{reg}}$-normality is *mild normality* (see [*J. Gutiérrez García* and *J. Picado*, J. Pure Appl. Algebra 218, No. 5, 784–803 (2014; Zbl 1296.06006)]) and $\mathbb{F}_{\mathtt{z}}$-normality is possessed by any frame, while both $\mathbb{F}^*_{\mathtt{c}}$-normality or $\mathbb{F}^*_{\mathtt{reg}}$-normality is extremal disconnectedness and $\mathbb{F}^*_{\mathtt{z}}$-normality is being a F-frame.

A sublocale selection $\mathbb{F}$ is a *Katetov selection* if for sublocales S, S', T, T', $S \sqsubseteq_{\mathbb{F}} T, T' \Rightarrow S \sqsubseteq_{\mathbb{F}} T \vee T'$ and $S, S' \sqsubseteq_{\mathbb{F}} T \Rightarrow S \cap S' \sqsubseteq_{\mathbb{F}} T$, where:

$$S \sqsubseteq_{\mathbb{F}} T \Leftrightarrow (\exists U \in \mathbb{F}(L))(\exists V \in \mathbb{F}^*(L))(S \leq V \leq U \leq T).$$

A general Katetov-Tong type insertion result for $\mathbb{F}$-normality is produced in Theorem 8.7, $\mathbb{F}$-*zero sublocales* and z-embeddedness is treated in §9, connections between $\mathbb{F}$-normality and normality is considered in §10 and some general Tietze type extension theorems produced in §11.

On the whole, this thirty six page long paper makes a very good interesting reading.

Reviewer: Partha Ghosh (Johannesburg)

MSC:

06D22 Frames, locales
18F70 Frames and locales, pointfree topology, Stone duality
54D15 Higher separation axioms (completely regular, normal, perfectly or collectionwise normal, etc.)

Keywords:

hedgehog space; compact hedgehog; frame; locale; zero sublocale; z-and z^c_k-embedded sublocales; collectionwise normality; total collectionwise normality; compact hedgehog-valued frame homomorphism; insertion results; sublocale selection

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