Gutiérrez García, Javier; Picado, Jorge

Rings of real functions in pointfree topology. (English) Zbl 1238.06008

Topology Appl. 158, No. 17, 2264-2278 (2011).

Let L be a frame and $\mathcal{S}(L)$ be its sublocale, lattice-ordered by reverse inclusion so that it is a frame. Denote by $F(L)$ the ring of continuous real-valued functions on $\mathcal{S}(L)$. In their article ["Localic real-valued functions: a general setting", J. Pure Appl. Algebra 231, 1064–1074 (2009; Zbl 1187.06005)], *T. Kubiak* and the authors well motivated why the ring $F(L)$ should be viewed as the localic version of the ring of all real-valued, not necessarily continuous, functions on L. In the paper under review, new descriptions of the algebraic operations of $F(L)$ are presented which greatly enhance a better understanding of the posets of localic versions of lower and upper semicontinuous functions on a frame. Scales are put to good use in this regard. The authors proceed to study order-completeness properties of $F(L)$, at each step appropriately inviting the reader to compare their results with those in [*B. Banaschewski* and *S. S. Hong* "Completeness properties of function rings in pointfree topology", Commentat. Math. Univ. Carol. 44, No. 2, 245–259 (2003; Zbl 1098.06006)]. To name a few, it is shown that $F(L)$ is order-complete if and only if $\mathcal{S}(L)$ is extremally disconnected, and $F(L)$ is σ-complete if and only if $\mathcal{S}(L)$ is basically disconnected. After thoroughly scrutinising the algebraic operations of the posets of lower and upper semicontinuous functions, the authors apply the material developed in the process to characterise idempotents in the ring $F(L)$. Among other characterisations, idempotents of $F(L)$ are precisely the characteristic functions of complemented sublocales of L, whilst idempotents of $C(L)$ – the subring consisting of continuous functions – are exactly the characteristic functions of complemented closed sublocales of L. Another application is on pointfree strict insertion of functions. Results from earlier work in this regard are amplified by the better understanding of the algebraic operations developed in this paper. The paper closes with a brief exploration of frames L for which every member of $F(\mathcal{S}^\alpha(L))$, where $\mathcal{S}^\alpha(L)$ is the α-dissolution of L, is continuous. They are precisely the α-soluble frames (see [*T. Plewe*, "Higher order dissolutions and Boolean coreflections of locales", J. Pure Appl. Algebra 154, No. 1–3, 273–293 (2000; Zbl 0966.18004)]), that is, those whose α-dissolution is Boolean. A truly handsome observation to conclude a well-written paper.

Reviewer: Themba Dube (Unisa)

MSC:

06D22 Frames, locales
06F25 Ordered algebraic structures
13J25 Commutative ordered rings
54C30 Real-valued functions on topological spaces

Keywords:

frame; pointfree topology; rings of real-valued functions; continuous real functions; lower semicontinuous; upper semicontinuous; strict insertion

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References:

[1] Ball, R. N.; Hager, A. W.: On the localic yosida representation of an Archimedean lattice ordered group with weak order unit, J. pure appl. Algebra 70, 17-43 (1991) · Zbl 0732.06009· doi:10.1016/0022-4049(91)90004-L

[2] Banaschewski, B.: The real numbers in pointfree topology, Textos mat. 12 (1997) · Zbl 0891.54009

[3] Banaschewski, B.: On the function ring functor in pointfree topology, Appl. categ. Structures 13, 305-328 (2005) · Zbl 1158.54308· doi:10.1007/s10485-005-5795-7

[4] B. Banaschewski, J. Guti\'errez Garc\'{\i}a, J. Picado, Extended real functions in Pointfree Topology, Preprint DMUC 10-53, December 2010, submitted for publication. · Zbl 1260.06010

[5] Banaschewski, B.; Hong, S. S.: Completeness properties of function rings in pointfree topology, Comment. math.

Univ. carolin. 44, 245-259 (2003) · Zbl 1098.06006· emis:journals/CMUC/cmuc0302/cmuc0302.htm

[6] Banaschewski, B.; Mulvey, C. J.: Stone-\v{c}ech compactification of locales II, J. pure appl. Algebra 33, 107-122 (1984) · Zbl 0549.54017· doi:10.1016/0022-4049(84)90001-X

[7] Dowker, C. H.: On countably paracompact spaces, Canad. J. Math. 3, 219-224 (1951) · Zbl 0042.41007· doi:10.4153/CJM-1951-026-2

[8] Dube, T.: Notes on pointfree disconnectivity with a ring-theoretic slant, Appl. categ. Structures 18, 55-72 (2010) · Zbl 1188.06005· doi:10.1007/s10485-008-9162-3

[9] Ferreira, M. J.; Garc\'{\i}a, J. Guti\'errez; Picado, J.: Completely normal frames and real-valued functions, Topology appl. 156, 2932-2941 (2009) · Zbl 1190.06008· doi:10.1016/j.topol.2008.12.042

[10] Gillman, L.; Jerison, M.: Rings of continuous functions, (1960) · Zbl 0093.30001

[11] Garc\'{\i}a, J. Guti\'errez; Kubiak, T.; Picado, J.: Pointfree forms of dowker's and michael's insertion theorems, J. pure appl. Algebra 213, 98-108 (2009) · Zbl 1154.06006· doi:10.1016/j.jpaa.2008.05.010

[12] Garc\'{\i}a, J. Guti\'errez; Kubiak, T.; Picado, J.: Localic real-valued functions: a general setting, J. pure appl. Algebra 213, 1064-1074 (2009) · Zbl 1187.06005· doi:10.1016/j.jpaa.2008.11.004

[13] Garc\'{\i}a, J. Guti\'errez; Picado, J.: On the algebraic representation of semicontinuity, J. pure appl. Algebra 210, 299-306 (2007) · Zbl 1117.06006· doi:10.1016/j.jpaa.2006.09.004

[14] Johnstone, P. T.: Stone spaces, (1982) · Zbl 0499.54001

[15] Li, Y. -M.; Wang, G. -J.: Localic kat\v{e}tov-tong insertion theorem and localic tietze extension theorem, Comment. math. Univ. carolin. 38, 801-814 (1997) · Zbl 0938.06008

[16] Michael, E.: Continuous selections I, Ann. of math. 63, 361-382 (1956) · Zbl 0071.15902· doi:10.2307/1969615

[17] Picado, J.; Pultr, A.: Locales mostly treated in a covariant way, Textos mat. 41 (2008) · Zbl 1154.06007

[18] Plewe, T.: Higher order dissolutions and Boolean coreflections of locales, J. pure appl. Algebra 154, 273-293 (2000) · Zbl 0966.18004· doi:10.1016/S0022-4049(99)00193-0

[19] Plewe, T.: Sublocale lattices, J. pure appl. Algebra 168, 309-326 (2002) · Zbl 1004.18003· doi:10.1016/S0022-4049(01)00100-1

[20] T. Wilson, The assembly tower and some categorical and algebraic aspects of frame theory, PhD Thesis, Carnegie Mellon University, 1994.