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Extended real functions in pointfree topology. (English) Zbl 1260.06010

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The frame of reals $\mathfrak{L}(\mathbb{R})$ is (generally) defined to be the frame generated by the rational pairs (p, q) subject to some relations. It can alternatively be defined as the frame generated by the symbols $(p, -)$ and $(-, q)$, for p and q in $\mathbb{Q}$, subject to certain relations which include the requirement that

$$\bigvee_{p\in\mathbb{Q}} (p, -) = \bigvee_{p\in\mathbb{Q}} (-, p) = 1_{\mathfrak{L}(\mathbb{R})}.$$

If this requirement is dropped, then the resulting frame, denoted $\mathfrak{L}(\overline{\mathbb{R}})$, is called the frame of extended reals. By an extended continuous real function on a frame L the authors mean a frame homomorphism $\mathfrak{L}(\overline{\mathbb{R}}) \to L$. The set of all these is denoted by $\overline{\mathsf{C}}(L)$. The definition and nomenclature are justified by the fact that the spectrum of $\mathfrak{L}(\overline{\mathbb{R}})$ is homeomorphic to the extended reals $\overline{\mathbb{R}}$, and, for any $X \in \mathbf{Top}$, $\overline{\mathsf{C}}(\mathfrak{O}X) \simeq \mathbf{Top}(X, \overline{\mathbb{R}})$. Extended scales are defined and, analogously to the real-valued case of $\mathcal{R}L$ (or $\mathsf{C}(L)$), it is shown how extended scales define members of $\overline{\mathsf{C}}(L)$. The algebra in $\overline{\mathsf{C}}(L)$ – which constitutes the heart of the paper – starts with lattice operations. The authors then follow that with the more complicated question of addition and multiplication. In this case they introduce a concept which is a pointfree counterpart of the domain of reality of a function which maps a space into the extended reals. They define sum and product compatibility of two elements of $\overline{\mathsf{C}}(L)$. Informally, two elements are sum compatible if they can be "added", and similarly for product compatibility. Denote by ω the element of $\mathfrak{L}(\overline{\mathbb{R}})$ given by $\omega = \bigvee\{(p, q) \mid p, q \in Q\}$, where (p, q) designates the element $(p, -) \wedge (-, q)$. The set $\mathsf{D}(L)$ consists of those $f \in \overline{\mathsf{C}}(L)$ for which $f(\omega)$ is a dense element in L. This is the frame counterpart of the set of those extended functions $f\colon X \to \overline{\mathbb{R}}$ for which $f^{-1}[\mathbb{R}]$ is a dense subset of X. The set $\mathsf{D}(L)$ is a sublattice of $\overline{\mathsf{C}}(L)$. However, if $f, g \in \overline{\mathsf{C}}(L)$ are sum compatible, then their sum defined in $\overline{\mathsf{C}}(L)$ is not necessarily in $\mathsf{D}(L)$. Similarly for product compatible elements of $\overline{\mathsf{C}}(L)$. The sublattice $\mathsf{D}(L)$ has its own addition and multiplication (called partial addition and partial multiplication) defined for certain types of its elements. It turns out that for extremally disconnected L, $\mathsf{D}(L)$ and $\mathsf{C}(\mathfrak{B}L)$, where $\mathfrak{B}L$ denotes the Booleanization of L, are isomorphic as ℓ-rings. The paper culminates with a characterisation of quasi-F frames L in terms of the partial addition and partial multiplication in $\mathsf{D}(L)$.

Reviewer: Themba Dube (Unisa)

MSC:

06D22 Frames, locales
06F25 Ordered algebraic structures
54C30 Real-valued functions on topological spaces
54G05 Extremally disconnected spaces, F-spaces, etc.

Keywords:

real functions on a frame; extended real functions; extremally disconnected frame; pointfree topology

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References:

[1] Ball, R. N.; Walters-Wayland, J.: C- and C$\ast $-quotients in pointfree topology, Dissertationes mathematicae (Rozprawy mat.) 412, 62 (2002) · Zbl 1012.54025· doi:10.4064/dm412-0-1· http://journals.impan.gov.pl/dm/Inf/412-0-1.html

[2] Banaschewski, B.: The real numbers in pointfree topology, Textos de matem\'atica 12 (1997) · Zbl 0891.54009

[3] Banaschewski, B.: Uniform completion in pointfree topology, Trends in logic 20, 19-56 (2003) · Zbl 1034.06008

[4] Banaschewski, B.: On the function ring functor in pointfree topology, Appl. categ. Structures 13, 305-328 (2005) · Zbl 1158.54308· doi:10.1007/s10485-005-5795-7

[5] Banaschewski, B.: A new aspect of the cozero lattice in pointfree topology, Topology appl. 156, 2028-2038 (2009) · Zbl 1175.06004· doi:10.1016/j.topol.2009.03.041

[6] Banaschewski, B.; Hong, S. S.: Completeness properties of function rings in pointfree topology, Comment. math. Univ. carolinae 44, 245-259 (2003) · Zbl 1098.06006· emis:journals/CMUC/cmuc0302/cmuc0302.htm

[7] Banaschewski, B.; Pultr, A.: Booleanization, cahiers topologie g\'eom, Diff\'erentielle cat\'eg. 37, 41-60 (1996) · Zbl 0848.06010· numdam:CTGDC_1996___37_1_41_0

[8] Dube, T.; Matlabyane, M.: Notes concerning characterizations of quasi-F frames, Quaest. math. 32, 551-557 (2009) · Zbl 1188.06006· doi:10.2989/QM.2009.32.4.4.962

[9] Gillman, L.; Jerison, M.: Rings of continuous functions, (1960) · Zbl 0093.30001

[10] Garc\'{\i}a, J. Guti\'errez; Kubiak, T.; Picado, J.: Localic real functions: a general setting, J. pure appl. Algebra 213, 1064-1074 (2009) · Zbl 1187.06005· doi:10.1016/j.jpaa.2008.11.004

[11] Garc\'{\i}a, J. Guti\'errez; Picado, J.: On the algebraic representation of semicontinuity, J. pure appl. Algebra 210, 299-306 (2007) · Zbl 1117.06006· doi:10.1016/j.jpaa.2006.09.004

[12] Garc\'{\i}a, J. Guti\'errez; Picado, J.: Rings of real functions in pointfree topology, Topology appl. 158, 2264-2278 (2011) · Zbl 1238.06008

[13] Johnstone, P. T.: Stone spaces, (1982) · Zbl 0499.54001

[14] Li, Y. -M.; Wang, G. -J.: Localic kat\v{e}tov–tong insertion theorem and localic tietze extension theorem, Comment. math. Univ. carolinae 38, 801-814 (1997) · Zbl 0938.06008

[15] Picado, J.; Pultr, A.: Locales mostly treated in a covariant way, Textos de matem\'atica 41 (2008) · Zbl 1154.06007

[16] Vickers, S.: Toposes pour LES vraiment nuls, Theory and formal methods 1996, third imperial college department of computing workshop on theory and formal methods, 1-12 (1996)

