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Insertion and extension results for pointfree complete regularity. (English) Zbl 1284.06020

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Since the advent of lower and upper semicontinuity in locales (to the development of which the current authors – either jointly, solely, or in collaboration with other authors – have contributed most significantly), there have been insertion-type characterisations of higher separation axioms. An example is that a locale L is normal precisely if whenever $f \leq g$, where f is an upper semicontinuous real function on L and g lower semicontinuous, then a continuous real function on L can be inserted between them [*J. Picado*, Topology Appl. 153, No. 16, 3203–3218 (2006; Zbl 1104.06007)]. Corresponding to this theorem is an extension theorem saying that a closed sublocale of a normal locale is C-embedded [*J. Gutiérrez García*, J. Pure Appl. Algebra 213, No. 6, 1064–1074 (2009; Zbl 1187.06005)]. As is the case with other works of these authors (individual, joint, or with other collaborators), this paper is lucidly written. It starts with a well-motivated question. In [*J. Gutiérrez García* and *T. Kubiak*, Appl. Gen. Topol. 8, No. 2, 239–242 (2007; Zbl 1152.54020)] a function $f : X \to [0,1]$ is called compact-like if $f^{-1}[t,1]$ is compact for every $t \in (0,1]$. It is then proved that a topological space X is completely regular if and only if for any functions $f, g : X \to [0,1]$ with f compact-like, g lower semicontinuous, and $f \leq g$, there is a continuous function $h : X \to [0,1]$ such that $f \leq h \leq g$. Naturally, then, one may ask if such an insertion-type characterisation holds for locales. The main theme of the paper addresses this question. A full answer is not known, but a partial one covering a large class of locales is available. By first characterising regular and completely regular spaces in terms of compact sublocales (of the spaces viewed as locales), they define what they call c-regular and completely c-regular locales. Every (completely) regular locale is (completely) c-regular. In fact, for a wide class of locales (including the spatial ones and the zero-dimensional ones) (complete) regularity is equivalent to (complete) c-regularity, but it is not known if this holds for all locales. The insertion theorem (after defining in a natural way upper and lower compact-like real functions on a locale) is then proved. It says that if L is a completely c-regular locale and $f \leq g$, for some upper compact-like f and lower compact-like g, then a continuous real function on L can be inserted between f and g. The converse holds for locales in which compact sublocales are complemented. The accompanying extension theorem states that every closed sublocale of a completely c-regular locale is C-embedded.

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MSC:

06D22 Frames, locales
54C15 Retractions of topological spaces
54C30 Real-valued functions on topological spaces
54C45 C- and C^*-embedding

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frame; locale; sublocale; complete separation; compact-like real function; upper semicontinuous; lower semicontinuous; insertion theorem; C-embedding; complete regularity

Full Text: Euclid