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On the parallel between normality and extremal disconnectedness. (English) Zbl 1296.06006
J. Pure Appl. Algebra 218, No. 5, 784-803 (2014).

Among various characterisations of normality and extremal disconnectedness, respectively, there are conditions which are dual to each other. For instance, in the language of locales, a locale L is normal if and only if whenever $\mathfrak{c}(a) \vee \mathfrak{c}(b) = \top$, then there exist $u, v \in L$ such that $\mathfrak{c}(u) \wedge \mathfrak{c}(v) = \perp$ and $\mathfrak{c}(a) \vee \mathfrak{c}(u) = \top = \mathfrak{c}(b) \vee \mathfrak{c}(v)$. On the other hand, L is extremally disconnected if and only if whenever $\mathfrak{c}(a) \wedge \mathfrak{c}(b) = \perp$, then there exist $u, v \in L$ such that $\mathfrak{c}(u) \vee \mathfrak{c}(v) = \top$ and $\mathfrak{c}(a) \wedge \mathfrak{c}(u) = \perp = \mathfrak{c}(b) \wedge \mathfrak{c}(v)$. There are other dual characterisations in terms of C^* -embedding of closed sublocales in the case of normality, and open sublocales in the case of extremal disconnectedness. There are also dual characterisations in terms of insertion-type theorems. The authors study these phenomena in parallel with a view to determining what the sources of the various dualities are, and also determining if the pairs of proofs which are somewhat disparate in nature can be unified within a single proof. This they achieve via a clever technique of fixing a class of sublocales of a given locale, and then studying dual notions of (a) normality and extremal disconnectedness relative to the fixed class, (b) continuity relative to the fixed class, and (c) upper and lower semi-continuity relative to the fixed class.

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MSC:

- [06D22](#) Frames, locales
- [06D30](#) De Morgan algebras, Łukasiewicz algebras
- [26A15](#) Continuity and related questions (one real variable)
- [54C30](#) Real-valued functions on topological spaces
- [54D15](#) Higher separation axioms
- [54G05](#) Extremally disconnected spaces, F -spaces, etc.

Cited in **3** Documents

Keywords:

[frame](#); [sublocales](#); [normality](#); [extremal disconnectedness](#)

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